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A New Quadrature Rule Derived from Spline Interpolation with Error Analysis

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Abstract—We present a new quadrature rule based on the spline interpolation along with the error analysis. Moreover, some error estimates for the reminder when the integrand is either a Lipschitzian function, a function of bounded variation or a function whose derivative belongs to L^p are given. We also give some examples to show that, practically, the spline rule is better than the trapezoidal rule.

Keywords—Quadrature, Spline interpolation, Trapezoidal rule, Numerical integration, Error analysis.

I. Introduction

QUADRATURE is an old-fashioned word that refers to the numerical approximation of definite integrals. There are several reasons for carrying out numerical integration. The integrand $f(x)$ may be known only at certain points, such as the one obtained by sampling. The formula for integrand may be known, but it may be difficult or impossible to find an antiderivative which is an elementary function. An example of such an integrand is $f(x) = e^{-x^2}$, the antiderivative of which cannot be written in an elementary form. It may be possible to find an antiderivative symbolically, but it may be easier to compute a numerical approximation than to compute the antiderivative. That may be the case if the antiderivative is given as an infinite series or product, or if its evaluation requires a special function which is not available.

Numerical integration methods can generally be described as combining evaluations of the integrand to get an approximation to the integral. An important part of the analysis of any numerical integration method is to study the behavior of the approximation error as a function of the number of integrand evaluations. A method which yields a small error for a small number of evaluations is usually considered superior. Reducing the number of evaluations of the integrand reduces the number of arithmetic operations involved, and therefore reduces the total round-off error. Also, each evaluation takes time and the integrand may be arbitrarily complicated.

This paper is organized as follows. In section 2, we present a new three-point quadrature rule which is based on the spline interpolation while section 3 is devoted to error analysis. Section 4 provides an estimate for the remainder when f is a Lipschitzian function while the fifth section deals with an upper bound for the remainder in spline inequality for the class of functions of bounded variation. Section 6 gives an inequality for functions whose derivatives belong to L^p. In section 7, some examples are presented to show that the new quadrature rule is better than the trapezoidal rule. Finally, conclusions in section 8 close the paper.

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II. New Quadrature Rule

A large class of quadrature rules can be derived by constructing interpolating functions which are easy to integrate. Typically, these interpolating functions are polynomials. We know that cubic spline functions are popular spline functions, for a variety of reasons. They are smooth functions with which to fit data, and when used for interpolation, they do not have the oscillatory behavior which is the characteristic of high-degree polynomial interpolation. Let $\Delta = \{x_i | \ i = 0, 1, 2..., n\}$ be a fixed partition of the interval $[a, b]$ by knots $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, and $y_i = f(x_i)$; $i = 0, 1, 2, ..., n$, be $(n + 1)$ prescribed real number. In addition let $h_{i+1} := x_{i+1} - x_i$; $i = 0, 1, 2, ..., n-1$; then the cubic spline function $S_\Delta(x)$ which interpolates the values of the function f at the knots $x_0, x_1, ..., x_n \in \Delta$ and satisfies $S''_\Delta(a) = S''_\Delta(b) = 0$ is readily characterized by their moments, and these moments of interpolating cubic spline function can be calculated as the solution of a system of linear equations. We can obtain the following representation of the cubic spline function in terms of its moments [4]:

$$S_\Delta(x) = \alpha_i + \beta_i(x - x_i) + \gamma_i(x - x_i)^2 + \delta_i(x - x_i)^3, \quad (1)$$

for $x \in [x_i, x_{i+1}], i = 0, 1, 2, ..., n - 1$, where

$$\alpha_i = y_i \ , \qquad \gamma_i = \frac{M_{i+1} - M_i}{6h_{i+1}},$$

$$\beta_i = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{2M_i - M_{i+1}}{6} h_{i+1}$$

$$M_0 = M_n = 0, \quad M_i = S''_\Delta(x_i); \quad i = 0, 1, 2, ..., n,$$

and the moments M_i, for $i = 0, 1, 2, ..., n$ obtain the following system

$$\begin{pmatrix} 2 & 0 & 0 & & & & \\ \mu_1 & 2 & \lambda_1 & 0 & & & \\ 0 & \mu_2 & \lambda_2 & 0 & & & \\ & & . & . & . & & \\ & & & . & . & . & \\ & & & 0 & \mu_{n-1} & 2 & \lambda_{n-1} \\ & & & & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \\ M_2 \\ . \\ . \\ . \\ M_n \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ . \\ . \\ . \\ d_n \end{pmatrix}$$

such that for $i = 1, 2, 3, ..., n - 1$

$$\lambda_i = \frac{h_{i+1}}{h_i + h_{i+1}} \ , \qquad \mu_i = 1 - \lambda_i \ ,$$

$$d_i = \frac{6}{h_i + h_{i+1}}\left(\frac{y_{i+1}-y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}\right),$$

and $d_0 = d_n = 0$. By integrating on both side of (1) on $[x_i, x_{i+1}]$, we obtain the formula

$$\int_{x_i}^{x_{i+1}} S_\Delta(x)dx = \frac{h_{i+1}}{2}(y_i + y_{i+1}) - \frac{h_{i+1}^3}{24}(M_i + M_{i+1}) \quad (2)$$

for $i = 0, 1, 2, ..., n-1$.

Now, let us suppose that $\Delta = \{a, \frac{a+b}{2}, b\}$ is a uniform partition of the interval $[a, b]$. Then by solving the system

$$\begin{pmatrix} 2 & 0 & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} M_0 \\ M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \end{pmatrix}$$

with $d_0 = d_2 = 0$, $M_0 = M_2 = 0$ and $d_1 = \frac{3}{h^2}[f(a) - 2f(\frac{a+b}{2}) + f(b)]$, we obtain

$$2M_1 = \frac{3}{h^2}[f(a) - 2f(\frac{a+b}{2}) + f(b)].$$

On the other hand, from (2) we have

$$\begin{aligned}\int_a^b S_\Delta(x)\mathrm{d}x &= \int_a^{\frac{a+b}{2}} S_\Delta(x)\mathrm{d}x + \int_{\frac{a+b}{2}}^b S_\Delta(x)dx \\ &= \frac{h}{2}\left[f(a) + 2f(\frac{a+b}{2}) + f(b)\right] \\ &\quad - \frac{h^3}{24}[M_0 + 2M_1 + M_2] \\ &= \frac{h}{8}\left[3f(a) + 10f(\frac{a+b}{2}) + 3f(b)\right].\end{aligned}$$

Therefore

$$\int_a^b S_\Delta(x)\mathrm{d}x = \frac{b-a}{16}\left[3f(a) + 10f(\frac{a+b}{2}) + 3f(b)\right],$$

and

$$\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{16}\left[3f(a) + 10f(\frac{a+b}{2}) + 3f(b)\right],$$

which is the desired three-point quadrature rule that henceforth we call it the spline rule.

Let $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ be the nodes of the partition σ_n of the interval $[a, b]$. The spline quadrature formula of the integral $\int_a^b f(x)\mathrm{d}x$ on the partition σ_n has the form

$$\begin{aligned}\int_a^b f(x)\mathrm{d}x &= \sum_{i=0}^{n-1} \frac{h_i}{16}[3f(x_i) + 10f(\frac{x_i + x_{i+1}}{2}) \\ &+ 3f(x_{i+1})] + R(f, \sigma_n), \end{aligned} \quad (3)$$

where $h_i = x_{i+1} - x_i$ for $i = 0, 1, 2, ..., n-1$.

III. Error Analysis

In this section, we analyze the error of the spline rule, i.e.,

$$\varepsilon(f) = \int_a^b f(x)\mathrm{d}x - \frac{b-a}{16}\left[3f(a) + 10f(\frac{a+b}{2}) + 3f(b)\right].$$

The standard derivation of quadrature error formulas is based on determining the class of polynomials for which these formulas produce exact results. The next definition is used to facilitate the discussion of this derivation [6].

Definition 1: The degree of accuracy, or precision, of a quadrature formula is the largest positive integer n such that the formula is exact for x^k, for each $k = 0, 1, 2, ..., n$.

Remark 1: Definition 1 implies that the spline rule has the degree of precision one.

Definition 2: For quadrature rule $\mathcal{L}(f) \simeq \int_a^b f(x)\mathrm{d}x$ with error $\varepsilon(f) = \int_a^b f(x)\mathrm{d}x - \mathcal{L}(f)$, Peano's kernel of the degree n is defined by $K_n(t) = \frac{1}{n!}\varepsilon((x-t)_+^n)$, where

$$(x-t)_+^n = \begin{cases} (x-t)^n & if\ x \geq t \\ 0 & if\ x < t. \end{cases}$$

We have the following theorem from [5] which is known as Peano's kernel theorem:

Theorem 1: Let $\mathcal{L}(f) \simeq \int_a^b f(x)\mathrm{d}x$ be a quadrature rule such that for each polynomial p of the degree n, $\varepsilon(p) = 0$. If $f \in C^{n+1}[a, b]$, then

$$\varepsilon(f) = -\int_a^b f^{(n+1)}(t)K_n(t)\mathrm{d}t.$$

Now, let f and K_1 be integrable functions on the interval $[a, b]$ where $f \in C^2[a, b]$ and $K_1(t)$ is Peano's kernel of the degree 1 such that it doesn't change sign on $[a, b]$. Via theorem 1, we have

$$\varepsilon(f) = -\int_a^b f''(t)K_1(t)\mathrm{d}t. \quad (4)$$

Indeed, according to the mean value theorem for integrals, there exists a $\zeta \in (a, b)$ such that

$$\int_a^b f''(t)K_1(t)\mathrm{d}t = -f''(\zeta)\int_b^a K_1(t)\mathrm{d}t \quad (5)$$

On the other hand, since the spline rule is accurate for polynomials of the degree 1, a simple calculation shows that

$$\varepsilon(x^2) = -2\int_b^a K_1(t)\mathrm{d}t. \quad (6)$$

Therefore, from $4, 5$ and 6, one obtains

$$\varepsilon(x^2) = -\frac{f''(\zeta)}{2}\varepsilon(x^2) = -\frac{(b-a)^3}{192}f''(\zeta), \quad \zeta \in (a, b).$$

Therefore the following theorem has been proved:

Theorem 2: Let $f \in C^2[a, b]$. If Peano's kernel $K_1(t)$ doesn't change sign on $[a, b]$, then there exists a $\zeta \in (a, b)$ such that

$$\varepsilon(f) = -\frac{(b-a)^3}{192}f''(\zeta),$$

where $\varepsilon(f)$ is the spline rule error.

For a uniform partition of the interval $[a, b]$, the following corollary is obtained:

Corollary 1: Under the assumptions of Theorem 2, for a uniform partition σ_n of $[a, b]$, the remainder term $R(f, \sigma_n)$ in the composite spline rule (3) satisfies

$$R(f, \sigma_n) = -\frac{b-a}{192}h^2 f''(\zeta).$$

or

$$|R(f,\sigma_n)| \le \frac{b-a}{192} h^2 \|f''\|_\infty,$$

and this conclusion shows that the composite spline rule error decreases with the minimum rate of h^2.

IV. Spline Inequality for Lipschitzian Functions

In this section, we present an inequality of spline type for L-Lipschitzian functions and show that the error estimate in the spline rule is less than the one in Simpson's rule for L-Lipschitzian functions.

Definition 3: A real function $f : [a,b] \to \mathbb{R}$ is called an L-Lipschitzian function if there exist a constant $L > 0$ such that

$$|f(x) - f(y)| \le L|x-y| \quad \forall x, y \in [a,b].$$

Dragomir [1] proved the following inequality of Simpson's type for L-Lipschitzian functions:

Theorem 3: Let $f : [a,b] \to \mathbb{R}$ be an L-Lipschitzian function on $[a,b]$. Then

$$\left|\int_a^b f(x)\mathrm{d}x - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f(\frac{a+b}{2})\right]\right| \le \frac{5}{36}L(b-a)^2. \quad (7)$$

Also, we have the following result from [1].

Lemma 1: If $p : [a,b] \to \mathbb{R}$ is a Riemann integrable function on $[a,b]$ and $v : [a,b] \to \mathbb{R}$ is a L-Lipschitzian function on $[a,b]$, then

$$\left|\int_a^b p(x)\mathrm{d}v(x)\right| \le L\int_a^b p(x)\mathrm{d}x. \quad (8)$$

Let $s(x)$ be a function defined as

$$s(x) = \begin{cases} x - \frac{13a+3b}{16} & x \in [a, \frac{a+b}{2}) \\ x - \frac{3a+13b}{16} & x \in [\frac{a+b}{2}, b] \end{cases}$$

Using integration by parts for Riemann-Stieltjes integral, we obtain

$$\int_a^b s(x)\mathrm{d}f(x) = \frac{b-a}{16}\left[3f(a) + 10f(\frac{a+b}{2}) + 3f(b)\right] - \int_a^b f(x)\mathrm{d}x.$$

In fact,

$$\begin{aligned}\int_a^b s(x)\mathrm{d}f(x) &= \int_a^{\frac{a+b}{2}} \left(x - \frac{13a+3b}{16}\right)\mathrm{d}f(x) \\ &+ \int_{\frac{a+b}{2}}^b \left(x - \frac{3a+13b}{16}\right)\mathrm{d}f(x) \\ &= \left[\left(x - \frac{13a+3b}{16}\right)f(x)\right]_a^{\frac{a+b}{2}} \\ &+ \left[\left(x - \frac{3a+13b}{16}\right)f(x)\right]_{\frac{a+b}{2}}^b - \int_a^b f(x)\mathrm{d}x \\ &= \frac{b-a}{16}\left[3f(a) + 10f(\frac{a+b}{2}) + 3f(b)\right] \\ &- \int_a^b f(x)\mathrm{d}x. \end{aligned} \quad (9)$$

Now, let $f : [a,b] \to \mathbb{R}$ be an L-Lipschitzian function on $[a,b]$. Using lemma 1, one obtains

$$\left|\int_a^b s(x)\mathrm{d}f(x)\right| \le L\int_a^b |s(x)\mathrm{d}x|.$$

On the other hand,

$$\begin{aligned}\int_a^b |s(x)|\mathrm{d}x &= \int_a^{\frac{a+b}{2}} \left|x - \frac{13a+3b}{16}\right|\mathrm{d}x \\ &+ \int_{\frac{a+b}{2}}^b \left|x - \frac{3a+13b}{16}\right|\mathrm{d}x \\ &= \int_a^{\frac{13a+3b}{16}} \left(\frac{13a+3b}{16} - x\right)\mathrm{d}x \\ &+ \int_{\frac{13a+3b}{16}}^{\frac{a+b}{2}} \left(x - \frac{13a+3b}{16}\right)\mathrm{d}x \\ &+ \int_{\frac{a+b}{2}}^{\frac{3a+13b}{16}} \left(\frac{3a+13b}{16} - x\right)\mathrm{d}x \\ &+ \int_{\frac{3a+13b}{16}}^b \left(x - \frac{3a+13b}{16}\right)\mathrm{d}x \\ &= \frac{17}{128}(b-a)^2. \end{aligned}$$

Thus, we have proved the following theorem:

Theorem 4: Let $f : [a,b] \to \mathbb{R}$ be an L-Lipschitzian function on $[a,b]$. Then the following inequality holds:

$$\left|\int_a^b f(x)\mathrm{d}x - \frac{b-a}{16}\left[3f(a) + 10f(\frac{a+b}{2}) + 3f(b)\right]\right| \le \frac{17}{128}L(b-a)^2. \quad (10)$$

Corollary 2: Suppose that $f : [a,b] \to \mathbb{R}$ is a differentiable function such that $f' \in L^\infty[a,b]$, i.e.,

$$\|f'\|_\infty := \sup_{x\in[a,b]} |f'(x)| < \infty,$$

then

$$\left|\int_a^b f(x)\mathrm{d}x - \frac{b-a}{16}\left[3f(a) + 10f(\frac{a+b}{2}) + 3f(b)\right]\right| \le \frac{17}{128}\|f'\|_\infty(b-a)^2.$$

Remark 2: By comparing (7) and (10), we conclude that the error estimate in (10) is less than the one in (7).

Remark 3: We state that the constant $\frac{17}{128}$, which occurred in the inequality, is the best constant. Namely, for the 1-Lipschitzian function

$$f(x) = \begin{cases} -x + \frac{13a+3b}{16}, & x \in [a, \frac{13a+3b}{16}) \\ x - \frac{13a+3b}{16}, & x \in [\frac{13a+3b}{16}, \frac{a+b}{2}) \\ -x + \frac{3a+13b}{16}, & x \in [\frac{a+b}{2}, \frac{3a+13b}{16}) \\ x - \frac{3a+13b}{16}, & x \in [\frac{3a+13b}{16}, b) \end{cases}$$

we have $|R(f)| = \frac{17}{128}(b-a)^2$.

Via Theorem 4, we obtain the following estimation for the remainder $R(f, \sigma_n)$.

Corollary 3: Under the assumptions of Theorem 4, the remainder term $R(f, \sigma_n)$ in the composite spline quadrature formula (3) satisfies

$$|R(f, \sigma_n)| \leq \frac{17}{128} \cdot L \sum_{j=0}^{n-1} h_j^2, \tag{11}$$

If σ_n is a uniform partition of $[a, b]$, we have the following corollary:

Corollary 4: Under the assumptions of Theorem 4 for a uniform partition σ_n of $[a, b]$, the reminder satisfies the estimation

$$|R(f, \sigma_n)| \leq \frac{17}{128} \cdot \frac{L}{n}(b-a)^2.$$

Remark 4: If we want to approximate the integral $\int_a^b f(t)\mathrm{d}t$ by the composite spline rule (3) with an accuracy less than $\epsilon > 0$, we need at least $n_\epsilon \in \mathbb{N}$ points for the division σ_n, where

$$n_\epsilon = \left[\frac{17}{128} \cdot \frac{L}{\epsilon}(b-a)^2\right] + 1,$$

and $[r]$ denotes the integer part of $r \in \mathbb{R}$.

V. Spline Inequality for Functions of Bounded Variation

In this section, we present an inequality of the spline type for functions of bounded variation and show that the error estimate in the spline rule is less than the one in Simpson's rule for functions of bounded variation.

Dragomir [2] proved the following inequality of Simpson's type for functions of bounded variation:

Theorem 5: Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation. Then

$$\left|\int_a^b f(x)\mathrm{d}x - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f(\frac{a+b}{2})\right]\right| \leq \frac{1}{3}(b-a)\bigvee_a^b(f), \tag{12}$$

where $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a, b]$ and the constant $\frac{1}{3}$ is the best constant.

We have the following lemma from [2]:

Lemma 2: Let $p, v : [a, b] \to \mathbb{R}$ be continuous and of bounded variation, respectively, on $[a, b]$. Then

$$\left|\int_a^b p(x)\mathrm{d}v(x)\right| \leq \max_{x\in[a,b]} |p(x)| \bigvee_a^b(v). \tag{13}$$

Let $s(x)$ be a function defined as

$$s(x) = \begin{cases} x - \frac{13a+3b}{16} & x \in [a, \frac{a+b}{2}) \\ x - \frac{3a+13b}{16} & x \in [\frac{a+b}{2}, b]. \end{cases}$$

With the same procedure which passed in the previous section, we have

$$\int_a^b s(x)\mathrm{d}f(x) = \frac{b-a}{16}\left[3f(a) + 10f(\frac{a+b}{2}) + 3f(b)\right] - \int_a^b f(x)\mathrm{d}x,$$

Now, let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Using lemma 2, one obtains

$$\left|\int_a^b s(x)\mathrm{d}f(x)\right| \leq \max_{x\in[a,b]} |s(x)| \bigvee_a^b(f). \tag{14}$$

Since s is a non-decreasing function on $[a, \frac{a+b}{2})$ and $[\frac{a+b}{2}, b]$ and

$$s(a) = \frac{3(a-b)}{16}, \qquad s((\frac{a+b}{2})^-) = \frac{5(b-a)}{16},$$

$$s((\frac{a+b}{2})^+) = \frac{5(a-b)}{16}, \qquad s(b) = \frac{3(b-a)}{16},$$

we deduce that

$$\max_{x\in[a,b]} |s(x)| = \frac{5(b-a)}{16}.$$

Thus, we have proved the following theorem:

Theorem 6: Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation. Then the inequality

$$\left|\int_a^b f(x)\mathrm{d}x - \frac{b-a}{16}\left[3f(a) + 10f(\frac{a+b}{2}) + 3f(b)\right]\right| \leq \frac{5}{16}(b-a)\bigvee_a^b(f), \tag{15}$$

holds, where $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a, b]$.

Remark 5: We state that the constant $\frac{5}{16}$ which occurred in the inequality is the best constant. In order to show this, let us assume that for a constant $C > 0$, the following inequality holds:

$$\left|\int_a^b f(x)\mathrm{d}x - \frac{b-a}{16}\left[3f(a) + 10f(\frac{a+b}{2}) + 3f(b)\right]\right| \leq C(b-a)\bigvee_a^b(f). \tag{16}$$

Let $f:[a,b]\to\mathbb{R}$ be a function defined as

$$f(x)=\begin{cases} 1 & x\in[a,\frac{a+b}{2})\cup(\frac{a+b}{2},b] \\ -1 & x=\frac{a+b}{2}. \end{cases}$$

Then we have

$$\left|\int_a^b f(x)\mathrm{d}x-\frac{b-a}{16}\left[3f(a)+10f(\frac{a+b}{2})+3f(b)\right]\right| \leq \frac{5}{4}(b-a) \tag{17}$$

and

$$(b-a)\bigvee_a^b(f)=4(b-a). \tag{18}$$

Now, using the above inequality, we get $4C(b-a)\geq\frac{5}{4}(b-a)$ which implies that $C\geq\frac{5}{16}$ and then $\frac{5}{16}$ is the best possible constant in (15).

Corollary 5: Suppose that $f:[a,b]\to\mathbb{R}$ is a differentiable function whose derivative is continuous on (a,b) and

$$\|f'\|_1:=\int_a^b|f'(x)|\mathrm{d}x<\infty,$$

then

$$\left|\int_a^b f(x)\mathrm{d}x-\frac{b-a}{16}\left[3f(a)+10f(\frac{a+b}{2})+3f(b)\right]\right| \leq \frac{5}{16}\|f'\|_1(b-a)^2. \tag{19}$$

Remark 6: By comparing (12) and (15), we conclude that the error estimate in (15) is less than the one in (12).

Via Theorem 6, we obtain the following estimation for the remainder $R(f,\sigma_n)$.

Corollary 6: Under the assumptions of Theorem 6, the remainder term $R(f,\sigma_n)$ in the composite spline quadrature formula satisfies

$$|R(f,\sigma_n)|\leq\frac{5}{16}\cdot\gamma(h)\bigvee_a^b(f), \tag{20}$$

where $\gamma(h):=\max\{h_i|\ i=0,1,...,n-1\}$.

The case of uniform partition of $[a,b]$ are presented in the following corollary:

Corollary 7: Under the assumptions of Theorem 6 for a uniform partition σ_n of $[a,b]$, the reminder satisfies the estimation

$$|R(f,\sigma_n)|\leq\frac{5}{16}\cdot\frac{b-a}{n}\bigvee_a^b(f).$$

Remark 7: If we want to approximate the integral $\int_a^b f(t)\mathrm{d}t$ by the composite spline rule (3) with an accuracy less than $\epsilon>0$, we need at least $n_\epsilon\in\mathbb{N}$ points for the division σ_n, where

$$n_\epsilon=\left[\frac{5}{16}\cdot\frac{b-a}{\epsilon}\bigvee_a^b(f)\right]+1,$$

and $[r]$ denotes the integer part of $r\in\mathbb{R}$.

VI. Spline Inequality in Terms of the p-Norm

In this section, we present an inequality of the spline type for functions whose derivatives belong to $L^p([a,b])$ and show that the error estimate in the spline rule is less than the one in Simpson's rule for functions whose derivatives belong to $L^p([a,b])$.

Dragomir [3] proved the following inequality of Simpson's type for functions whose derivatives belong to $L^p([a,b])$:

Theorem 7: Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ whose derivative belongs to $L^p([a,b])$. Then

$$\left|\int_a^b f(x)\mathrm{d}x-\frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f(\frac{a+b}{2})\right]\right| \leq\frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{1/q}(b-a)^{1+1/q}\|f'\|_p, \tag{21}$$

where $1/p+1/q=1, p>1$.

Let $s(x)$ be a function defined as

$$s(x)=\begin{cases} x-\frac{13a+3b}{16} & x\in[a,\frac{a+b}{2}) \\ x-\frac{3a+13b}{16} & x\in[\frac{a+b}{2},b]. \end{cases}$$

With the same procedure which passed in the third section, we have

$$\int_a^b s(x)\mathrm{d}f(x) = \frac{b-a}{16}\left[3f(a)+10f(\frac{a+b}{2})+3f(b)\right]-\int_a^b f(x)\mathrm{d}x,$$

Applying Hölder's inequality, we obtain

$$\int_a^b s(x)f'(x)\mathrm{d}x=\left(\int_a^b|s(x)|^q\mathrm{d}x\right)^{1/q}\|f'\|_p.$$

On the other hand,

$$\begin{aligned}\int_a^b|s(x)|^q\mathrm{d}x &= \int_a^{\frac{a+b}{2}}\left|x-\frac{13a+3b}{16}\right|^q\mathrm{d}x + \int_{\frac{a+b}{2}}^b\left|x-\frac{3a+13b}{16}\right|^q\mathrm{d}x \\ &= \int_a^{\frac{13a+3b}{16}}\left(\frac{13a+3b}{16}-x\right)^q\mathrm{d}x + \int_{\frac{13a+3b}{16}}^{\frac{a+b}{2}}\left(x-\frac{13a+3b}{16}\right)^q\mathrm{d}x \\ &+ \int_{\frac{a+b}{2}}^{\frac{3a+13b}{16}}\left(\frac{3a+13b}{16}-x\right)^q\mathrm{d}x + \int_{\frac{3a+13b}{16}}^b\left(x-\frac{3a+13b}{16}\right)^q\mathrm{d}x \\ &= \frac{(3^{q+1}+5^{q+1})(b-a)^{q+1}}{8(q+1)16^q}.\end{aligned}$$

Therefore, we have proved the following theorem:

Theorem 8: Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ whose derivative belongs to $L^p([a,b])$. Then the inequality

$$\left|\int_a^b f(x)\mathrm{d}x - \frac{b-a}{16}\left[3f(a)+10f(\frac{a+b}{2})+3f(b)\right]\right| \leq \frac{1}{16}\left[\frac{3^{q+1}+5^{q+1}}{8(q+1)}\right]^{1/q}(b-a)^{1+1/q}\|f'\|_p, \qquad (22)$$

holds, where $1/p + 1/q = 1, p > 1$.

Remark 8: By comparing (21) and (22), we conclude that the error estimate in (22) is less than the one in (21).

Via Theorem 8, we obtain the following estimation for the remainder $R(f, \sigma_n)$.

Corollary 8: Under the assumptions of Theorem 8, the remainder term $R(f, \sigma_n)$ in the composite spline quadrature formula satisfies

$$|R(f,\sigma_n)| \leq \frac{1}{16}\left[\frac{3^{q+1}+5^{q+1}}{8(q+1)}\right]^{1/q}\left(\sum_{j=0}^{n-1} h_j^{q+1}\right)^{1/q}\|f'\|_p \ . \ (23)$$

The case of uniform partition of $[a, b]$ are presented in the following corollary:

Corollary 9: Under the assumptions of Theorem 8 for a uniform partition σ_n of $[a, b]$, the reminder satisfies the estimation

$$|R(f,\sigma_n)| \leq \frac{1}{16n}\left[\frac{3^{q+1}+5^{q+1}}{8(q+1)}\right]^{1/q}(b-a)^{1+1/q}\|f'\|_p \ .$$

Remark 9: If we want to approximate the integral $\int_a^b f(t)\mathrm{d}t$ by the composite spline rule (3) with an accuracy less than $\epsilon > 0$, we need at least $n_\epsilon \in \mathbb{N}$ points for the division σ_n, where

$$n_\epsilon = \left[\frac{1}{16\epsilon}\left(\frac{3^{q+1}+5^{q+1}}{8(q+1)}\right)^{1/q}(b-a)^{1+1/q}\|f'\|_p\right] + 1,$$

and $[r]$ denotes the integer part of $r \in \mathbb{R}$.

VII. Numerical Results

According to section 3, the error of the spline rule like the error of the trapezoidal rule decreases with the minimum rate of h^2. Also, it was shown that the error of the spline rule is less than the error of the trapezoidal rule . According to these facts, we compare the spline rule with the trapezoidal rule to show that, practically, the spline rule is better than the trapezoidal rule. In Table I, some examples are presented to compare the spline rule with the trapezoidal rule on the interval $[0, 1]$ which is divided into ten partitions. All the programs have been written in MATLAB environment.

VIII. Conclusions

In this paper, a new quadrature rule which is derived from cubic spline functions along with the error analysis was presented. Moreover, some estimates for the reminder, when the integrand was a Lipschitzian function, a function of bounded variation or a function whose derivative belongs to L^p, were given and it was observed that these error estimates were less than the similar ones for Simpson's rule. Also, we presented some examples to show that, mainly, the spline rule is better than the trapezoidal rule.

TABLE I
Numerical Results

$f(x)$	Exact	Spline rule	Trapezoidal rule
$\cos(x) - x$	0.341470	0.341296	0.340769
$\frac{1}{x^4+4x^2+3}$	0.241549	0.241510	0.241392
$x\log(\sqrt{x+1})$	0.698959	0.699018	0.699197
$\log(x^2+1)$	1.169229	1.169188	1.169062
$e^{2x}\cos(e^x)$	-1.176887	-1.181349	-1.195234
x^3+x-1	-0.25	-0.249375	-0.2475
$e^{t^2+7t-30}-1$	-1	0.999999	0.999999

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