An iterative method for the least-squares symmetric solution of $AXB+CYD=F$ and its application

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Abstract—Based on the classical algorithm LSQR for solving (unconstrained) LS problem, an iterative method is proposed for the least-squares like-minimum-norm symmetric solution of $AXB+CYD=E$. As the application of this algorithm, an iterative method for the least-squares like-minimum-norm biymmetric solution of $AXB=E$ is also obtained. Numerical results are reported that show the efficiency of the proposed methods.

Keywords—Matrix equation, bisymmetric matrix, Least squares problem, like-minimum norm, Iterative algorithm.

I. INTRODUCTION

DENOTED by $\mathcal{R}^{m \times n}$ and $\mathcal{SR}^{n \times n}$ the set of $m \times n$ real matrices and the set of $n \times n$ real symmetric matrices, respectively. For any $A \in \mathcal{R}^{m \times n}$, $\mathcal{R}(A), A^T, A^\dagger, \|A\|_2$ and $\|A\|_F$ present the range, transpose, Moore-Penrose generalized inverse, Euclid norm and Frobenius norm, respectively. $A \otimes B$ represents the Kronecker product of matrices A and B. For $A = (a_{ij}) \in \mathcal{R}^{n \times n}$ and $a = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathcal{R}^n$, $\text{diag}(A)$ and $\text{diag}(a)$ represent the diagonal matrix with diagonal elements $a_{11}, a_{22}, \cdots, a_{nn}$ and $a_1, a_2, \cdots, a_n$, respectively. The sub-vector consisting of from αth component to βth component of x_i is denoted by $x_{\alpha:\beta,i}$.

Let m, n, m_1, m_2 be four positive integers, and let $E \in \mathcal{R}^{m \times n}, A \in \mathcal{R}^{m \times m_1}, B \in \mathcal{R}^{m_1 \times n}, C \in \mathcal{R}^{m \times m_2}$, and $D \in \mathcal{R}^{m_2 \times n}$. We consider the least squares problem

$$\min_{X,Y} \|AXB + CYD - E\|_F \qquad (1)$$

for $X \in \mathcal{SR}^{m_1 \times m_1}$ and $Y \in \mathcal{SR}^{m_2 \times m_2}$. Its corresponding linear matrix equation is

$$AXB + CYD = E. \qquad (2)$$

A matrix pair (X, Y) is referred to as a minimum norm solution if it minimizes

$$\|X\|_F^2 + \|Y\|_F^2,$$

and as a *like-minimum norm solution* if it minimizes

$$\|\text{tril}(X)\|_F^2 + \|\text{tril}(Y)\|_F^2,$$

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where $\text{tril}(X)$ represents extract lower triangular part of X, that is, if $X = (x_{ij}) \in \mathcal{R}^{n \times n}$, then

$$\text{tril}(X) = \begin{pmatrix} x_{11} & 0 & \cdots & 0 \\ x_{21} & x_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}.$$

Many results have been obtained about the matrix equation (2). For example, Chu[1] gave the consistency conditions and the minimum-norm solution by making use of the generalized singular value decomposition (GSVD). Huang[2] and Özgüler[3], respectively, gave the solvability conditions over a simple Artinian ring and a principal ideal domain by using the generalized inverse. However, we should point out that if the matrices A, B, C, D, and E are experimentally occurring in practice, they may not satisfy these solvability conditions because of the inconsistency of the matrix equation (2). Hence, we need to further study the least squares problem. For unconstrained problem (1), Xu, Wei, and Zheng[4] gave its solution by making use of the canonical correlation decomposition (CCD). In addition, Shim and Chen[5] presented its least-squares solution with the minimum norm by using the singular value decomposition (SVD) and the GSVD. The above methods, which can be called direct methods, have some difficulties when the scale of (1) is very great. Therefore iterative methods must be considered.

To our best knowledge, the method for the solution of (1) is not still discussed. In this paper, we develop an efficient iterative method for the least-squares solution of (1) with the like-minimum norm. When (2) has symmetric solutions, this method can get its like-minimum-norm symmetric solution. In section 2, we will review the LSQR algorithm for $\min_x \|Mx - f\|$, which is numerically very reliable even if M is ill-conditioned. Our algorithm for (1) will be proposed in section 3. As its application, we will study the method for the least-squares like-minimum-norm bisymmetric solution of $AXB = E$ in section 4. In section 5, two examples are reported that show the efficiency of the proposed algorithms.

II. ALGORITHM LSQR

In the section, we briefly review the algorithm LSQR prosed by Paige and Sauders[6] for solving the following lease squares problem:

$$\min_{x \in \mathcal{R}^n} \|Mx - f\|_2 \qquad (3)$$

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with given $M \in \mathcal{R}^{m \times n}$ and $f \in \mathcal{R}^m$, whose normal equation is

$$M^T M x = M^T f. \tag{4}$$

Theoretically, LSQR converges within at most n iterations if exact arithmetic could be performed, where n is the length of x. In practice the iteration number of LSQR may be larger than n because of the computational errors. It was shown in [6] that LSQR is numerically more reliable even if M is ill-conditioned.

We summarize the LSQR algorithm as follows.

Algorithm LSQR

(1)Initialization.
$\beta_1 u_1 = f, \alpha_1 v_1 = M^T u_1, h_1 = v_1, x_0 = 0, \bar{\zeta}_1 = \beta_1, \bar{\rho}_1 = \alpha_1$.
(2)Iteration. For $i = 1, 2, \cdots$
  (i) bidiagonalization
    (a)$\beta_{i+1} u_{i+1} = M v_i - \alpha_i u_i$
    (b)$\alpha_{i+1} v_{i+1} = M^T u_{i+1} - \beta_{i+1} v_i$
  (ii)construct and use Givens rotation
    $\rho_i = \sqrt{\bar{\rho}_i^2 + \beta_{i+1}^2}$
    $c_i = \bar{\rho}_i / \rho_i, s_i = \beta_{i+1} / \rho_i, \theta_{i+1} = s_i \alpha_{i+1}$
    $\bar{\rho}_{i+1} = -c_i \alpha_{i+1}, \zeta_i = c_i \bar{\zeta}_i, \bar{\zeta}_{i+1} = s_i \bar{\zeta}_i$
  (iii) update x and h
    $x_i = x_{i-1} + (\zeta_i / \rho_i) h_i$
    $h_{i+1} = v_{i+1} - (\theta_{i+1} / \rho_i) h_i$
  (iv) check convergence.

It is well known that if the consistent system of linear equations $Mx = f$ has a solution $x^* \in \mathcal{R}(M^T)$, then x^* is the unique minimal norm solution of $Mx = f$. So, if Eq.(4) has a solution $x^* \in \mathcal{R}(M^T M) = \mathcal{R}(M^T)$, then x^* is the minimum norm solution of (3). It is obvious that x_k generated by Algorithm LSQR belongs to $\mathcal{R}(M^T)$ and this leads the following result.

Theorem 2.1. *The solution generated by Algorithm LSQR is the minimum norm solution of Eq.(3).*

Remark 2.1. Theoretically, when $\beta_{k+1} = 0$ or $\alpha_{k+1} = 0$ for some $k < \min\{m, n\}$, then recursions will stop. In both cases, x_k is the minimum norm least squares solution to Eq.(3). Also notice that $\|M^T(f - Mx_k)\|_2 = |\alpha_{k+1} \bar{\zeta}_{k+1} c_k| = 0$ is monotonically decreasing when k is increasing. Also notice that, at each step of the LSQR iteration, the main costs of computations are two matrix-vector products.

Remark 2.2. During the iterative processing, because of the round-off error, computed solution $\widehat{x}_k$ may make $\|M^T(f - M\widehat{x}_k)\|_2 \neq 0$ even $\|M^T(f - Mx_k)\|_2 = |\alpha_{k+1} \bar{\zeta}_{k+1} c_k| = 0$. Therefore we need to setup the stopping criteria to check the correct k. Paige and Sauders [6] discuss several choices of the stopping criteria. Sometimes we need to use restart strategy to improve the accuracy. In the numerical experiments provided in §5, we use $|\alpha_{k+1} \bar{\zeta}_{k+1} c_k| < \tau = 10^{-11}$ as the stopping criterion. We observe that this stopping criterion works well.

III. THE MATRIX-FORM LSQR ALGORITHM FOR (1)

A symmetric matrix is uniquely determined by part of its elements, namely some independent elements. For a matrix $X = (x_1, x_2, \cdots, x_n) \in \mathcal{R}^{n \times n}$, we define

$$\text{vec}(X) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{R}^{n^2}, \text{vec}_i(X) = \begin{pmatrix} x_{1:n,1} \\ x_{2:n,2} \\ \vdots \\ x_{nn} \end{pmatrix} \in \mathcal{R}^N,$$

where $N \equiv n(n+1)/2$. Obviously, there is an one to one linear mapping from the long-vector space

$$\text{vec}(\mathcal{SR}^{n \times n}) = \{\text{vec}(X) | X \in \mathcal{SR}^{n \times n}\}$$

to the independent parameter space

$$\text{vec}_i(\mathcal{SR}^{n \times n}) = \{\text{vec}_i(X) | X \in \mathcal{SR}^{n \times n}\}.$$

Let us denote by $\mathcal{F}(n)$ the matrix that defines linear mapping form $\text{vec}_i(\mathcal{SR}^{n \times n})$ to $\text{vec}(\mathcal{SR}^{n \times n})$,

$$X \in \mathcal{SR}^{n \times n}, \quad \text{vec}(X) = \mathcal{F}(n)\text{vec}_i(X).$$

We call $\mathcal{F}(n) \in \mathcal{R}^{n^2 \times N}$ a symmetry constraint matrix of degree n, which will be simply denoted by $\mathcal{F}$ if n can be ignored without misunderstanding.

Now, we discuss the constrained problem (1), which is equivalent to

$$\min_{\varphi} \|M\varphi - f\|_2 \tag{5}$$

with

$$M = ((B^T \otimes A)\mathcal{F}_1, (D^T \otimes C)\mathcal{F}_2) \in \mathcal{R}^{mn \times \frac{(m_1+1)m_1 + (n_1+1)n_1}{2}},$$

$$f = \text{vec}(E) \in \mathcal{R}^{mn}, \varphi = \begin{pmatrix} \text{vec}_i(X) \\ \text{vec}_i(Y) \end{pmatrix} \in \mathcal{R}^{\frac{(m_1+1)m_1 + (n_1+1)n_1}{2}}, \tag{6}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are the symmetry constraint matrices of degree m_1 and m_2, respectively. So the normal equation of (1) and (5) is

$$M^T M \varphi = M^T \text{vec}(E). \tag{7}$$

The following result will help us to deduce a matrix-form LSQR method for (1).

Lemma 3.1.[10] *Let $U \in \mathcal{R}^{m \times n}$ and $Z = A^T U B^T$. Then for the symmetry constraint $\mathcal{F}_1$,*

$$((B^T \otimes A)\mathcal{F}_1)^T vec(U) = \text{vec}_i(Z + Z^T - \text{diag}(Z)).$$

Next we will apply LSQR on the LS problems (5). The vector iterations of LSQR will be rewritten into matrix form so that the Kronecker product can be released. To this end, it is required to transform the vector v and u in Algorithm LSQR back to the matrix forms. So we must transform $v = M^T u$ and $u = Mv$ back to the matrix forms. Notice that we do not want to construct the matrix M explicitly.

Let $u = \text{vec}(U) \in \mathcal{R}^{mn}$ with $U \in \mathcal{R}^{m \times n}, v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathcal{R}^{\frac{(m_1+1)m_1 + (n_1+1)n_1}{2}}$, where $v_1 = \text{vec}_i(V_1)$ and $v_2 = \text{vec}_i(V_2)$

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with $V_1 \in \mathcal{SR}^{m_1 \times m_1}$ and $V_2 \in \mathcal{R}^{n_1 \times n_1}$. Then

$$
\begin{aligned}
U &= \operatorname{matr}(u) = \operatorname{matr}(Mv) \\
&= \operatorname{matr}\left(((B^T \otimes A)\mathcal{F}_1, (D^T \otimes C)\mathcal{F}_2) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) \\
&= \operatorname{matr}((B^T \otimes A)\mathcal{F}_1 \operatorname{vec}_i(V_1) + (D^T \otimes C)\mathcal{F}_2 \operatorname{vec}_i(V_2)) \\
&= \operatorname{matr}((B^T \otimes A)\operatorname{vec}(V_1) + (D^T \otimes C)\operatorname{vec}(V_2)) \\
&= \operatorname{matr}(\operatorname{vec}(AV_1 B) + \operatorname{vec}(CV_2 D)) \\
&= AV_1 B + CV_2 D,
\end{aligned}
$$

$$
\begin{aligned}
v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= M^T u = \begin{pmatrix} ((B^T \otimes A)\mathcal{F}_1)^T \operatorname{vec}(U) \\ ((D^T \otimes C)\mathcal{F}_2)^T \operatorname{vec}(U) \end{pmatrix} \\
&= \begin{pmatrix} \operatorname{vec}_i(Z_1 + Z_1^T - \operatorname{diag}(Z_1)) \\ \operatorname{vec}_i(Z_2 + Z_2^T - \operatorname{diag}(Z_2)) \end{pmatrix}
\end{aligned}
$$

where

$$
Z_1 = A^T U B^T, \quad Z_2 = C^T U D^T.
$$

So v_1 and v_2 can be formally transformed to symmetric matrices:

$$
V_1 = Z_1 + Z_1^T - \operatorname{diag}(Z_1), \quad V_2 = Z_2 + Z_2^T - \operatorname{diag}(Z_2).
$$

Notice that α in Algorithm LSQR should be

$$
\begin{aligned}
\|v\|_2 &= \left\| \begin{pmatrix} \operatorname{vec}_i(Z_1 + Z_1^T - \operatorname{diag}(Z_1)) \\ \operatorname{vec}_i(Z_2 + Z_2^T - \operatorname{diag}(Z_2)) \end{pmatrix} \right\|_2 \\
&= \sqrt{\|\operatorname{vec}_i(V_1)\|_F^2 + \|\operatorname{vec}_i(V_2)\|_F^2}.
\end{aligned}
$$

Next, we give the algorithm for the like-minimum-norm solution of (1).

Algorithm $[X, Y] = LSQR_M(A, B, C, D, E)$

(1)Initialization.

$X_0 = 0 (\in \mathcal{SR}^{m_1 \times n_1}), Y_0 = 0 (\in \mathcal{SR}^{m_2 \times n_2})$,
$\beta_1 = \|E\|_F, U_1 = E/\beta_1$,
$Z_1 = A^T U_1 B^T, Z_2 = C^T U D^T$,
$\bar{V}_1^{(1)} = Z_1 + Z_1^T - \operatorname{diag}(Z_1)$,
$\bar{V}_1^{(2)} = Z_2 + Z_2^T - \operatorname{diag}(Z_2)$,
$\alpha_1 = (\|\operatorname{vec}_i(\bar{V}_1^{(1)})\|_F^2 + \|\operatorname{vec}_i(\bar{V}_1^{(2)})\|_F^2)^{1/2}$,
$V_1^{(1)} = \bar{V}_1^{(1)}/\alpha_1, V_1^{(2)} = \bar{V}_1^{(2)}/\alpha_1$,
$H_1^{(1)} = V_1^{(1)}, H_1^{(2)} = V_1^{(2)}, \bar{\xi}_1 = \beta_1, \bar{\rho}_1 = \alpha_1$.

(2)Iteration. For $i = 1, 2, \cdots$

$\bar{U}_{i+1} = A V_i^{(1)} B + C V_i^{(2)} D - \alpha_i U_i$,
$\beta_{i+1} = \|\bar{U}_{i+1}\|_F, U_{i+1} = \bar{U}_{i+1}/\beta_{i+1}$,
$Z_1 = A^T U_{i+1} B^T, Z_2 = C^T U_{i+1} D^T$,
$\bar{V}_{i+1}^{(1)} = Z_1 + Z_1^T - \operatorname{diag}(Z_1) - \beta_{i+1} V_i^{(1)}$,
$\bar{V}_{i+1}^{(2)} = Z_2 + Z_2^T - \operatorname{diag}(Z_2) - \beta_{i+1} V_i^{(2)}$,
$\alpha_1 = (\|\operatorname{vec}_i(\bar{V}_{i+1}^{(1)})\|_F^2 + \|\operatorname{vec}_i(\bar{V}_{i+1}^{(2)})\|_F^2)^{1/2}$,
$V_{i+1}^{(1)} = \bar{V}_{i+1}^{(1)}/\alpha_{i+1}, V_{i+1}^{(2)} = \bar{V}_{i+1}^{(2)}/\alpha_{i+1}$,
$\rho_i = \sqrt{\bar{\rho}_i^2 + \beta_{i+1}^2}, c_i = \bar{\rho}_i/\rho_i$,
$s_i = \beta_{i+1}/\rho_i, \theta_{i+1} = s_i \alpha_{i+1}$,
$\bar{\rho}_{i+1} = -c_i \alpha_{i+1}, \zeta_i = c_i \bar{\zeta}_i, \bar{\zeta}_{i+1} = s_i \bar{\zeta}_i$,
$X_i = X_{i-1} + (\xi_i/\rho_i) H_i^{(1)}, Y_i = Y_{i-1} + (\xi_i/\rho_i) H_i^{(2)}$,
$H_{i+1}^{(1)} = V_{i+1}^{(1)} - (\theta_i/\rho_i) H_i^{(1)}, H_{i+1}^{(2)} = V_{i+1}^{(2)} - (\theta_i/\rho_i) H_i^{(2)}$.

(3)check convergence.

IV. THE BISYMMETRIC SOLUTION OF $\min_X \|AXB - E\|_F$

As the application of the method proposed in section 3, it is easy to find the iterative method for the bisymmetric solution of $\min_X \|AXB - E\|_F$. First, we give the definition of bisymmetric matrices.

Definition 4.1. *Let* $A = (a_{ij}) \in \mathcal{R}^{n \times n}$. *If* A *satisfies*

$$
a_{ij} = a_{ji} = a_{n+1-i, n+1-j}, i, j = 1, 2, \cdots, n,
$$

then A *is called an* $n \times n$ *centrosymmetric matrix; the set of all centrosymmetric matrices is denoted by* $\mathcal{BSR}^{n \times n}$.

The bisymmetric and anti-bisymmetric matrices play an important role in many areas. [8] and [9] studied respectively the solution and the least squares bisymmetric solution of $A^T X A = D$ using GSVD and CCD. In this section we will consider the following problem:

$$
\min_{X \in \mathcal{BSR}^{n \times n}} \|AXB - E\|_F, \tag{8}
$$

with $A \in \mathcal{R}^{m \times n}, B \in \mathcal{R}^{n \times p}$ and $E \in \mathcal{R}^{m \times p}$.

We first characterize the set of all $n \times n$ bisymmetric matrices. Let

$$
K = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & I_k \\ J_k & -J_k \end{pmatrix}, & n = 2k, \\[2ex] \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ J_k & 0 & -J_k \end{pmatrix}, & n = 2k+1, \end{cases}
$$

where I_k is the $k \times k$ identity matrix and J_n is the $n \times n$ anti-identity natrix, i.e. J_n has 1 on the anti-diagonal and zeros elsewhere. Clearly K is orthogonal for all n. The matrix K plays an important role in analyzing the properties of bisymmetric matrices. In particular, we have the following splitting of bisymmetric matrices into smaller submatrices using K.

Lemma 4.1.[8] *Let* $\mathcal{BSR}^{n \times n}$ *and* k *be the set of all bisymmetric matrices in* $\mathcal{R}^{n \times n}$ *and the largest integer less than or equal to* $n/2$, *respectively. Then*

$$
\mathcal{BSR}^{n \times n} = \left\{ K \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix} K^T \middle| Q \in \mathcal{SR}^{(n-k) \times (n-k)}, R \in \mathcal{SR}^{k \times k} \right\}
$$

It follows form Lemma 4.1 that the problem (8) is equivalent to the problem

$$
\min_{\substack{G_1 \in \mathcal{SR}^{(n-k) \times (n-k)} \\ G_2 \in \mathcal{SR}^{k \times k}}} \|A_1 G_1 B_1 + A_2 G_2 B_2 - E\|_F \tag{9}
$$

with corresponding partition

$$
A K_n = [A_1, A_2] \quad \text{and} \quad K_n^T B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}. \tag{10}
$$

Now we can give the algorithm for the like-minimum-norm solution of (8) as follows:

Algorithm $X = LSQR_B(A, B, E)$

(1).Input A, B, E and obtain A_1, B_1, A_2, B_2 from (10);

(2).$[G_1, G_2] = LSQR_M(A_1, B_1, A_2, B_2, E)$;

(3).Output $X = K_n \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} K_n^T$.

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V. Numerical examples

In this section, we present some numerical examples to illustrate the efficiency of our algorithms.

Example 1. Find the like-minimum-norm symmetric solution of (1). Let

$$A = \left(\begin{array}{cc} hilb(4) & zeros(4,3) \\ eye(4) & ones(4,3) \end{array} \right),$$

$$B = \left(\begin{array}{cc} ones(3,5) & zeros(3,4) \\ zeros(4,5) & pascal(4) \end{array} \right),$$

$$C = \left(\begin{array}{c} magic(5) \\ ones(3,4) \end{array} \right),$$

$$D = \left(\begin{array}{cc} hankel(1:4) & zeros(4,5) \\ zeros(1,4) & zeros(1,5) \end{array} \right),$$

$$E_1 = \left(\begin{array}{cc} toeplitz(1:8) & ones(8,1) \end{array} \right),$$

$$X = ones(7,7), Y = zeros(5,5), E_2 = AXB + CYD$$

such that $AXB + CYD = E_1$ has no solution and $AXB + CYD = E_2$ has symmetric solutions, where $hilb(n), pascal(n)$ and $magic(n)$ denote Hilbert matrix, Pascal matrix and Magic matrix of order n, respectively; and $toeplitz(1:n)$ and $hankel(1:n)$ denote Toeplitz matrix and Hankel matrix of order n, respectively, with their first rows being $(1, 2, \cdots, n)$.

For M and φ defined by (6), we define
$\delta_k \equiv \|M^T M \varphi_k - M^T f\|_2 = |\alpha_k \zeta_{k+1} c_k|,$
$\eta_k \equiv \|AX_k B + CY_k D - E_1\|_F = \|M\varphi_k - f\|_2 = |\bar{\zeta}_{k+1}|.$

Fig.1 plots the functions of the error δ_k and η_k for Algorithm $LSQR_M$ and show that Algorithm $LSQR_M$ is very efficient for this example.
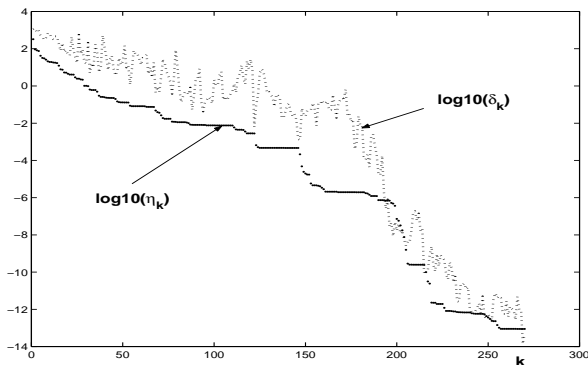

Fig. 1.   The error of the computed solutions by Algorithm $LSQR_M$

The computed solution (X_{246}, Y_{246}) of $AXB + CYD = E_2$ using 246 iterations has the error

$$\eta_{246} = 10^{-13.0527},$$

and satisfies $\|\mathrm{tril}(X_{246})\|_F^2 + \|\mathrm{tril}(Y_{246})\|_F^2 = 26.8000$, and $\|X_{246}\|_F^2 + \|Y_{246}\|_F^2 = 50.4400$.
$\|X\|_F^2 + \|Y\|_F^2 = 49$ and $\|\mathrm{tril}(X)\|_F^2 + \|\mathrm{tril}(Y)\|_F^2 = 28$ imply that (X_{246}, Y_{246}) is not the minimum-norm symmetric solution, but the like-minimum-norm symmetric solution.

Example 2. Find the minimum-norm of (8). Let

$$A = \left(\begin{array}{cc} hilb(4) & zeros(4,1) \\ eye(4) & ones(4,1) \end{array} \right),$$

$$B = \left(\begin{array}{cc} ones(1,5) & zeros(3,4) \\ zeros(1,5) & pascal(4) \end{array} \right),$$

$$E = \left(\begin{array}{cc} toeplitz(1:8) & ones(8,1) \end{array} \right).$$

Algorithm $LSQR_C$ is very efficient for this example, too. The computed minimum norm solution using 18 iterations is $X_{18} =$

$$\left(\begin{array}{ccccc} -0.3573 & 0.5120 & 0.5027 & -1.4904 & 0.8402 \\ 0.5120 & -0.0697 & -2.4868 & 4.2716 & -1.4904 \\ 0.5027 & -2.4868 & 5.1777 & -2.4868 & 0.5027 \\ -1.4904 & 4.2716 & -2.4868 & -0.0697 & 0.5120 \\ 0.8402 & -1.4904 & 0.5027 & 0.5120 & -0.3573 \end{array} \right)$$

with the error

$$\delta_k = \|M^T M \varphi_k - M^T f\|_2 = 10^{-16.0575},$$

where $\varphi = \left(\begin{array}{c} \mathrm{vec}_i(G_1) \\ \mathrm{vec}_i(G_2) \end{array} \right)$ and $M = ((B_1^T \otimes A_1)\mathcal{F}_1, (B_2^T \otimes A_2)\mathcal{F}_2)$ with A_1, A_2, B_1, B_2, defined by (10).

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