

# An Improved Construction Method for MIHCs on Cycle Composition Networks

Hsun Su, Yuan-Kang Shih, and Shin-Shin Kao

**Abstract**—Many well-known interconnection networks, such as  $k$ -ary  $n$ -cubes, recursive circulant graphs, generalized recursive circulant graphs, circulant graphs and so on, are shown to belong to the family of cycle composition networks. Recently, various studies about mutually independent hamiltonian cycles, abbreviated as MIHC's, on interconnection networks are published. In this paper, using an improved construction method, we obtain MIHC's on cycle composition networks with a much weaker condition than the known result. In fact, we established the existence of MIHC's in the cycle composition networks and the result is optimal in the sense that the number of MIHC's we constructed is maximal.

**Keywords**—Hamiltonian cycle,  $k$ -ary  $n$ -cube, cycle composition networks, mutually independent.

## I. INTRODUCTION AND PRELIMINARIES

The architecture of an interconnection network is usually represented by a graph, in which vertices and edges correspond to processors and communication links, respectively. Thus, we use the terms graph and network interchangeably.

For the graph definitions and notations, we follow [1]. A graph  $G$  consists of a nonempty set  $V(G)$  and a subset  $E(G)$  of  $\{(u, v) \mid (u, v) \text{ is an unordered pair of } V(G)\}$ . The set  $V(G)$  is called the *vertex set* of  $G$  and  $E(G)$  is called the *edge set*. Two vertices  $u$  and  $v$  are *adjacent* if  $(u, v) \in E(G)$ . For a vertex  $u$  of  $G$ , we denote the *degree* of  $u$  by  $\deg(u) = |\{v \mid (u, v) \in E(G)\}|$ . A graph  $G$  is  *$r$ -regular* if for every vertex  $u \in G$ ,  $\deg(u) = r$ .

A *matching* of size  $n$  in a graph  $G$  is a set of  $n$  edges with no shared endpoints. The vertices belonging to the edges of a matching are *saturated* by the matching; the others are *unsaturated*. A *perfect matching* is a matching that saturates every vertex of  $G$ .

A *path* is represented by a finite sequence of vertices  $\langle v_0, v_1, v_2, \dots, v_n \rangle$ , where every two consecutive vertices are adjacent. The *length* of a path  $P$  is the number of edges in  $P$ . We write the path  $\langle v_0, v_1, v_2, \dots, v_n \rangle$  as  $\langle v_0, v_1, \dots, v_s, P_1, v_i, \dots, v_j, P_2, v_t, \dots, v_n \rangle$ , where  $P_1 = \langle v_s, v_{s+1}, \dots, v_i \rangle$  and  $P_2 = \langle v_j, v_{j+1}, \dots, v_t \rangle$ . A *hamiltonian path* between  $u$  and  $v$ , where  $u$  and  $v$  are two distinct vertices of  $G$ , is a path joining  $u$  to  $v$  that visits every vertex of  $G$  exactly once. Two paths  $P_1 = \langle u_0, u_1, \dots, u_m \rangle$  and  $P_2 = \langle v_0, v_1, \dots, v_m \rangle$  from  $a$  to  $b$  are *independent* if  $u_0 = v_0 = a$ ,  $u_m = v_m = b$ , and  $u_i \neq v_i$  for

$1 \leq i \leq m-1$  [9]. In [6], two paths  $P'_1 = \langle u_0, u_1, \dots, u_m \rangle$  and  $P'_2 = \langle v_0, v_1, \dots, v_m \rangle$  are *full-independent* if  $u_i \neq v_i$  for all  $0 \leq i \leq m$ . Paths with the same number of vertices are *mutually independent* (resp. *mutually full-independent*) if every two different paths are independent (resp. full-independent). A graph  $G$  is *hamiltonian connected* if there is a hamiltonian path joining any two distinct vertices of  $G$ . A graph  $G$  is called *1-vertex-fault-tolerant hamiltonian connected* if it remains hamiltonian connected after removing any vertex in  $G$ .

A *cycle* is a path of at least three vertices such that the first vertex is the same as the last vertex. A *hamiltonian cycle* of  $G$  is a cycle that traverses every vertex of  $G$  exactly once. A *hamiltonian graph* is a graph with a hamiltonian cycle. The *length* of a cycle  $C$  is the number of edges/vertices in  $C$ . Two cycles  $C_1 = \langle u_0, u_1, \dots, u_k, u_0 \rangle$  and  $C_2 = \langle v_0, v_1, \dots, v_k, v_0 \rangle$  beginning at  $s$  are *independent* if  $u_0 = v_0 = s$  and  $u_i \neq v_i$  for  $1 \leq i \leq k$  [11]. Cycles beginning at  $s$  with the same length are *mutually independent* if every two different cycles are independent. A graph  $G$  is said to *contain  $n$  MIHCs* if there exist  $n$  hamiltonian cycles in  $G$  beginning at any vertex  $s$  such that the  $n$  cycles are mutually independent. There are numerous studies in MIHCs. Readers can refer to [7]–[15].

In 2008, Kueng et al. introduced the cycle composition networks [4], abbreviated as CCN's. Let  $k \geq 4$ ,  $n \geq 6$ ,  $r \geq 2$  be integers, and  $G_i$  be a  $r$ -regular graph with  $n$  vertices for  $0 \leq i \leq k-1$ . From now on, all additions and subtractions are considered modulo  $k$ . Let  $M_{i,j}$  be an arbitrary perfect matching between the vertices of  $G_i$  and those of  $G_j$  and  $\mathcal{M} = \bigcup_{i=0}^{k-1} M_{i,i+1}$ . The *cycle composition network*  $\tilde{G} = \text{CCN}(G_0, G_1, \dots, G_{k-1}; \mathcal{M})$  is defined to be the graph with the vertex set  $V(\tilde{G}) = \bigcup_{i=0}^{k-1} V(G_i)$  and the edge set  $E(\tilde{G}) = \bigcup_{i=0}^{k-1} (E(G_i)) \cup \mathcal{M}$ . We abbreviate  $\text{CCN}(G_0, G_1, \dots, G_{k-1}; \mathcal{M})$  as  $\text{CCN}_k$ . See Figure 1 for an illustration. Many well-known interconnection networks, such as  $k$ -ary  $n$ -cubes, recursive circulant graphs, generalized recursive circulant graphs, circulant graphs and so on, are shown to belong to the family of CCN's. Hence, CCN's have attracted many studies and research interests [2]–[4].

Suppose that each  $G_i$  contains  $r$  MIHCs. More precisely, for any vertex  $s_i$  of  $G_i$ , where  $0 \leq i \leq k-1$ , there exist  $r$  hamiltonian cycles in  $G_i$  beginning at the vertex  $s_i$  such that the  $r$  cycles are mutually independent in  $G_i$ . Is it true that  $\tilde{G} = \text{CCN}(G_0, G_1, \dots, G_{k-1}; \mathcal{M})$  contains  $(r+2)$  MIHCs? In [5], M.-F. Hsieh et. al. derived the following result.

**Theorem 1:** For  $k \geq 6$ , let  $\{G_i\}_{i=0}^{k-1}$  be  $k$   $r$ -regular hamil-

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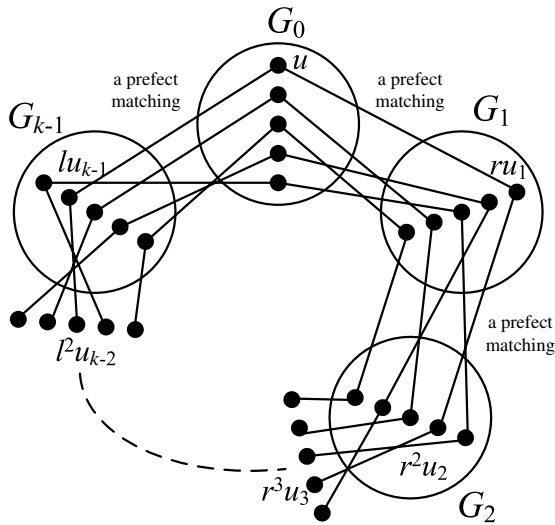


Fig. 1. An illustration for  $CCN_k = CCN(G_0, G_1, \dots, G_{k-1}; \mathcal{M})$ .

tonian graphs with  $n$  vertices. Suppose that each  $G_i$  contains  $r$  MIHCs and  $r$  mutually full-independent hamiltonian paths between any  $r$  pairs of distinct vertices of  $G_i$ , and is 1-vertex-fault-tolerant hamiltonian connected. Then there exist  $r + 2$  MIHCs in  $CCN_k$ .

Obviously, each vertex of  $\tilde{G}$  has exactly  $r + 2$  neighbors. However, the requirement that each  $G_i$  contains  $r$  mutually full-independent hamiltonian paths between any  $r$  pairs of vertices of  $G_i$  seems to be unnecessarily strict. Besides, to check whether this requirement is satisfied on each  $G_i$  is a difficult task. Consequently, Theorem 1 is of little practical use. In this paper, using a different construction scheme, we are able to achieve the same result with a much weaker condition on  $G_i$ . (See Theorem 4).

The following notations are defined for the rest of the paper. Let  $u_i$  be a vertex of  $G_i$  for some  $i$ . We use  $l^j u_{i-j}$  to denote the vertex of  $G_{i-j}$  such that there exists a path in  $\mathcal{M}$  of the form  $\langle u_i, l u_{i-1}, l^2 u_{i-2}, \dots, l^j u_{i-j} \rangle$ . Similarly, we use  $r^j u_{i+j}$  to denote the vertex of  $G_{i+j}$  such that there exists a path in  $\mathcal{M}$  of the form  $\langle u_i, r u_{i+1}, r^2 u_{i+2}, \dots, r^j u_{i+j} \rangle$ . W.L.O.G., let  $u \in G_0$  and  $u = u_0$ . See Figure 1 for an illustration. It is possible that there exists a cycle beginning at  $u$  with the length  $k$  of the form  $\langle u = u_0, r u_1, r^2 u_2, \dots, r^{k-2} u_{k-2}, r^{k-1} u_{k-1}, u \rangle$ . More specifically,  $r^i u_i = l^{k-i} u_{i-k}$  for any  $1 \leq i \leq k - 1$ .

## II. MAIN RESULTS

Let  $k \geq 4$  and  $n \geq 6$ . Throughout this section, we use the symbol  $CCN_k$  for  $CCN(G_0, G_1, \dots, G_{k-1}; \mathcal{M})$ , which is a cycle composition network composed of  $k$  graphs  $\{G_i \mid G_i \text{ is a } r\text{-regular graph with } |G_i| = n \text{ for } 0 \leq i \leq k - 1\}$  and  $k$  perfect matchings  $\mathcal{M} = \bigcup_{i=0}^{k-1} M_{i,i+1}$ , for simplicity.

**Lemma 1:** Consider any  $CCN_k$ . Suppose that  $G_0$  contains  $r$  MIHCs beginning at any given vertex, denoted by  $\{\bar{C}_0^i \mid 0 \leq i \leq r - 1\}$ , and there exists some edge  $(a_0, b_0)$  such that  $(a_0, b_0) \in \bar{C}_0^i$  for all  $0 \leq i \leq r - 1$ . Let  $a_1 \in V(G_1)$  and

$b_{r-1} \in V(G_{r-1})$  be arbitrary. If there is a path  $P$  between  $a_1$  and  $b_{r-1}$  such that  $P$  visits each vertex of  $\bigcup_{i=1}^{k-1} G_i$  in  $CCN_k$  exactly once, then  $CCN_k$  contains  $r$  MIHCs starting with any vertex in  $G_0$  and passing through a common edge.

**Proof:** W.L.O.G., let  $s_0 \in V(G_0)$  be the beginning vertex. Obviously, for  $1 \leq i \leq r - 1$ ,  $\bar{C}_0^i$  is of the form  $\langle s_0, A^i, a_0, b_0, B^i, s_0 \rangle$ , where  $A^i$  and  $B^i$  are two disjoint paths in  $G_0$  such that  $A^i$  is between  $s_0$  and  $a_0$ ,  $B^i$  is between  $b_0$  and  $s_0$ , and  $A^i \cup B^i = V(G_0)$ . Since  $\{\bar{C}_0^i \mid 0 \leq i \leq r - 1\}$  are MIHCs in  $G_0$ , it must be  $|A^i| \neq |A^j|$  and  $|B^i| \neq |B^j|$  for  $i \neq j$ . Otherwise,  $a_0$  or  $b_0$  might appear at the same timestep on different  $\bar{C}_0^i$ 's.

Note that  $ra_1 \in V(G_1)$  and  $lb_{r-1} \in V(G_{r-1})$ . It is known that there is a path  $P$  between  $ra_1$  and  $lb_{r-1}$  such that  $P$  visits every vertex of  $\bigcup_{i=1}^{k-1} G_i$  in  $CCN_k$  exactly once. Let  $C_i = \langle s_0, A^i, a_0, ra_1, P, lb_{r-1}, b_0, B^i, s_0 \rangle$ . It is easy to see that  $\{C_i \mid 1 \leq i \leq r - 1\}$  forms a set of  $r$  MIHCs of  $CCN_k$ , and each  $C_i$  contains the edge  $(a_0, ra_1)$ , which is the common edge. ■

**Theorem 2:** Consider  $CCN_4$ . For  $0 \leq i \leq 3$ , suppose that  $G_i$  satisfies the following two requirements – (1)  $G_i$  is 1-vertex-fault-tolerant hamiltonian connected. (2) Starting from any vertex of  $G_i$ , there exist  $r$  MIHCs passing through a common edge of  $G_i$ . Then  $CCN_4$  contains  $r + 2$  MIHCs passing through a common edge.

**Proof:** W.L.O.G., let  $s_0$  be an arbitrary vertex of  $G_0$ . We want to construct  $r + 2$  MIHCs starting at  $s_0$  in  $CCN_4$ . It is known that  $G_0$  contains  $r$  MIHCs beginning at  $s_0$  and passing through a common edge of  $G_0$ . Let  $u_1$  and  $v_2$  be any two vertices in  $G_1$  and  $G_2$ , respectively. Since  $G_i$  is hamiltonian connected, there exist three hamiltonian paths  $P_1$ ,  $P_2$  and  $P_3$ , such that  $P_1$  connects  $ra_1$  and  $u_1$  in  $G_1$ ,  $P_2$  connects  $ru_2$  and  $v_2$  in  $G_2$ , and  $P_3$  connects  $rv_3$  and  $lb_3$  in  $G_3$ . Then  $P = \langle ra_1, P_1, u_1, ru_2, P_2, v_2, rv_3, P_3, lb_3 \rangle$  is a path between  $ra_1$  and  $lb_3$  that visits each vertex of  $\{G_i \mid 1 \leq i \leq 3\}$  exactly once. By Lemma 1,  $CCN_4$  contains  $r$  MIHCs, denoted by  $\{C_i \mid 0 \leq i \leq r - 1\}$ , and each  $C_i$  contains the common edge  $(a_0, ra_1)$ .

Now, we construct the  $(r + 1)$ -th MIHC of  $CCN_4$  beginning at  $s_0$ . In  $G_3$ , choose a vertex  $x_3$  which is adjacent to  $ls_3$  and  $x_3 \neq lb_3$ . Since  $G_3$  is 1-vertex-fault-tolerant hamiltonian connected, there is a hamiltonian path  $T_3$  of  $G_3 - \{ls_3\}$  between  $la_3$  and  $x_3$ . We can write  $T_3$  as  $\langle la_3, Q_3, y_3, x_3 \rangle$ . Since  $G_i$  is 1-vertex-fault-tolerant hamiltonian connected,  $G_0 - \{s_0\}$  contains a hamiltonian path  $Q_0$  that connects  $ry_0$  and  $a_0$ ,  $G_1 - \{rs_1\}$  contains a hamiltonian path  $Q_1$  that connects  $ra_1$  and  $l^2x_1$ , and  $G_2 - \{lx_2\}$  contains a hamiltonian path  $Q_2$  that connects  $r^2s_2$  and  $l^2a_2$ . Let  $C_r = \langle s_0, rs_1, r^2s_2, Q_2, l^2a_2, la_3, Q_3, y_3, ry_0, Q_0, a_0, ra_1, Q_1, l^2x_1, lx_2, x_3, ls_3, s_0 \rangle$ . It is easy to see that  $C_r$  is mutually independent of the  $r$  MIHCs  $\{C_i \mid 0 \leq i \leq r - 1\}$  constructed by Lemma 1, and  $C_r$  passes through the edge  $(a_0, ra_1)$ , which is underlined in  $C_r$ .

Finally, we construct the  $(r + 2)$ -th MIHC of  $CCN_4$  beginning at  $s_0$ . In  $G_2$ , choose a vertex  $w_2$  such that  $w_2$  is adjacent to  $r^2s_2$  and  $w_2 \neq lx_2$ . Choose another vertex  $z_2$  in  $G_2$  such that  $z_2 \neq r^2a_2$  and  $z_2 \neq r^2s_2$ . Since  $G_3$  is hamiltonian connected, there is a hamiltonian path  $T_3$  of  $G_3 - \{ls_3\}$  between  $la_3$  and  $x_3$ . We can write  $T_3$  as  $\langle la_3, Q_3, y_3, x_3 \rangle$ . Since  $G_i$  is 1-vertex-fault-tolerant hamiltonian connected,  $G_0 - \{s_0\}$  contains a hamiltonian path  $Q_0$  that connects  $ry_0$  and  $a_0$ ,  $G_1 - \{rs_1\}$  contains a hamiltonian path  $Q_1$  that connects  $ra_1$  and  $l^2x_1$ , and  $G_2 - \{lx_2\}$  contains a hamiltonian path  $Q_2$  that connects  $r^2s_2$  and  $l^2a_2$ . Let  $C_{r+1} = \langle s_0, rs_1, r^2s_2, Q_2, l^2a_2, la_3, Q_3, y_3, ry_0, Q_0, a_0, ra_1, Q_1, l^2x_1, lx_2, x_3, ls_3, s_0 \rangle$ . It is easy to see that  $C_{r+1}$  is mutually independent of the  $r + 1$  MIHCs  $\{C_i \mid 0 \leq i \leq r\}$  constructed by Lemma 1 and Theorem 2, and  $C_{r+1}$  passes through the edge  $(a_0, ra_1)$ , which is underlined in  $C_{r+1}$ .

tonian connected, there exists a hamiltonian path  $R_3$  of  $G_3$  that connects  $ls_3$  and  $rz_3$ . Since  $G_i$  is 1-vertex-fault-tolerant hamiltonian connected,  $G_0 - \{s_0\}$  contains a hamiltonian path  $R_0$  between  $r^2z_0$  and  $a_0$ ,  $G_1 - \{rs_1\}$  contains a hamiltonian path  $R_1$  between  $ra_1$  and  $lz_1$ , and  $G_2 - \{r^2s_2\}$  contains a hamiltonian path  $R_2$  between  $z_2$  and  $w_2$ . Let  $C_{r+1} = \langle s_0, ls_3, R_3, rz_3, r^2z_0, R_0, a_0, ra_1, R_1, lz_1, z_2, R_2, w_2, r^2s_2, rs_1, s_0 \rangle$ . Consequently,  $C_{r+1}$  is mutually independent of  $\{C_i \mid 0 \leq i \leq r\}$  constructed above and passes through the common edge  $(a_0, ra_1)$ , which is underlined. See Figure 2 for an illustration. ■

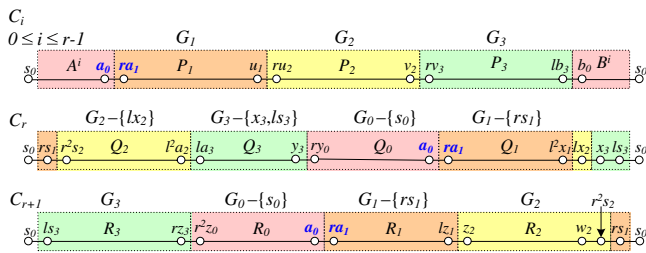


Fig. 2. An illustration of Theorem 2.

**Theorem 3:** Suppose that  $CCN_5$  is constructed by five  $r$ -regular graphs  $G_i$  with  $n$  vertices for  $0 \leq i \leq 4$ , and each  $G_i$  is 1-vertex-fault-tolerant hamiltonian connected and contains  $r$  MIHCs passing through a fixed edge from any vertex of  $G_i$ . Then  $CCN_5$  contains  $r + 2$  MIHCs passing through a fixed edge.

**Proof:** W.L.O.G., we let  $s_0$  be the beginning vertex and  $\overline{C}_0^0, \overline{C}_0^1, \dots, \overline{C}_0^{r-1}$  be the  $r$  MIHCs beginning at  $s_0$  and passing through the fixed edge  $(a_0, b_0)$  in  $G_0$ . Hence we can write  $\overline{C}_0^i$  as  $\langle s_0, A^i, a_0, b_0, B^i, s_0 \rangle$  for  $0 \leq i \leq r-1$ . We will construct the  $r + 2$  MIHCs beginning at  $s_0$  and passing through a fixed edge in  $CCN_5$ .

Consider the first  $r$  MIHCs of  $CCN_5$  beginning at  $s_0$ . Let  $u_1, v_2$  and  $u_3$  be any three vertices in  $G_1, G_2$  and  $G_3$ , respectively. There exist four hamiltonian paths  $P_1, P_2, P_3$  and  $P_4$  joining from  $ra_1$  to  $u_1, ru_2$  to  $v_2, rv_3$  to  $u_3$  and  $ru_4$  to  $lb_4$  in  $G_1, G_2, G_3$  and  $G_4$ , respectively. Set  $C_i = \langle s_0, A^i, a_0, ra_1, P_1, u_1, ru_2, P_2, v_2, rv_3, P_3, u_3, ru_4, P_4, lb_4, b_0, B^i, s_0 \rangle$  for  $0 \leq i \leq r-1$ . Then, the  $r$  MIHCs are  $C_0, C_1, \dots, C_{r-1}$ , which pass through the fixed edge  $(a_0, ra_1)$ .

Now, we consider the  $(r + 1)$ -th MIHC of  $CCN_5$  beginning at  $s_0$ . In  $G_4$ , choose a vertex  $x_4$  which is adjacent to  $ls_4$  and  $x_4 \neq lb_4$ . Since  $G_4$  is 1-vertex-fault-tolerant hamiltonian connected, there is a hamiltonian path  $T_4$  of  $G_4 - \{ls_4\}$  between  $la_4$  and  $x_4$ . W.L.O.G.,  $T_4$  can be written as  $\langle la_4, Q_4, y_4, x_4 \rangle$ . In  $G_3$ , choose a vertex  $lx_3 \neq r^3s_3$ . If  $lx_3 = r^3s_3$ , we have to choose another vertex  $x_4$ , which is adjacent to  $ls_4$ , for  $lx_3 \neq r^3s_3$ . Since  $G_3$  is 1-vertex-fault-tolerant hamiltonian connected, there is a hamiltonian path  $Q_3$  of  $G_3 - \{lx_3\}$  between  $rd_3$  and  $l^2a_3$ , it can be written as  $\langle rd_3, Q_3, l^2a_3 \rangle$ . Let  $d_2$  be any vertex in  $G_2$  not adjacent to  $l^2a_3$ . Using the 1-vertex-fault-tolerant hamiltonian connected property of  $G_0, G_1$  and  $G_2$ , there exist three hamiltonian paths  $Q_0, Q_1$  and

$Q_2$  of  $G_0 - \{s_0\}, G_1 - \{rs_1\}$  and  $G_2 - \{l^2x_2\}$  from  $ry_0$  to  $a_0, ra_1$  to  $l^3x_1$  and  $r^2s_2$  to  $d_2$ , respectively. Let  $C_r = \langle s_0, rs_1, r^2s_2, Q_2, d_2, rd_3, Q_3, l^2a_3, la_4, Q_4, y_4, ry_0, Q_0, a_0, ra_1, Q_1, l^3x_1, l^2x_2, lx_3, x_4, ls_4, s_0 \rangle$ . Therefore,  $C_r$  is mutually independent of the first  $r$  MIHCs  $C_0, C_1, \dots, C_{r-1}$  and passes the fixed edge  $(a_0, ra_1)$ .

Finally, we consider the last MIHC of  $CCN_5$  beginning at  $s_0$ . Let  $w_1, z_2$  and  $w_4$  be any three vertices in  $G_1, G_2$  and  $G_4$ , where  $w_4$  is not adjacent to  $a_0$  and  $z_2$  is not adjacent to  $r^3s_3$ . There exist two hamiltonian paths  $R_3$  and  $R_4$  joining from  $rz_3$  to  $r^3s_3$  and  $ls_4$  to  $w_4$  in  $G_3$  and  $G_4$ . And using the 1-vertex-fault-tolerant hamiltonian connected property of  $G_0, G_1$  and  $G_2$ , there exist three hamiltonian paths  $R_0, R_1$  and  $R_2$  of  $G_0 - \{s_0\}, G_1 - \{rs_1\}$  and  $G_2 - \{r^2s_2\}$  from  $rw_0$  to  $a_0, ra_1$  to  $w_1$  and  $rw_2$  to  $z_2$ , respectively. We let  $C_{r+1} = \langle s_0, ls_4, R_4, w_4, rw_0, R_0, a_0, ra_1, R_1, w_1, rw_2, R_2, z_2, rz_3, R_3, r^3s_3, r^2s_2, rs_1, s_0 \rangle$ . So,  $C_{r+1}$  is mutually independent of the first  $r + 1$  MIHCs  $C_0, C_1, \dots, C_r$  and passes the fixed edge  $(a_0, ra_1)$ . See Figure 3 for an illustration. ■

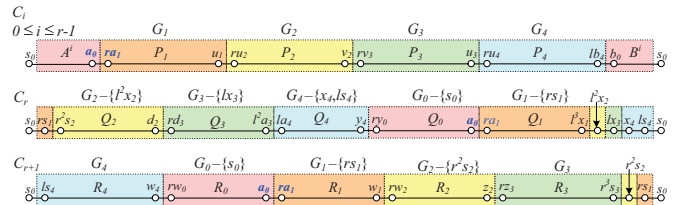


Fig. 3. An illustration of Theorem 3.

**Theorem 4:** Suppose that  $CCN_k$  is constructed by  $k$   $r$ -regular graphs  $G_i$  with  $n$  vertices for  $0 \leq i \leq k-1$ . If each  $G_i$  is 1-vertex-fault-tolerant hamiltonian connected and contains  $r$  MIHCs passing through a fixed edge from any vertex of  $G_i$ , then  $CCN_k$  contains  $r + 2$  MIHCs passing through a fixed edge.

**Proof:** W.L.O.G., we let  $s_0$  be the beginning vertex and  $\overline{C}_0^0, \overline{C}_0^1, \dots, \overline{C}_0^{r-1}$  be the  $r$  MIHCs beginning at  $s_0$  and passing through the fixed edge  $(a_0, b_0)$  in  $G_0$ . Hence we can write  $\overline{C}_0^i$  as  $\langle s_0, A^i, a_0, b_0, B^i, s_0 \rangle$  for  $0 \leq i \leq r-1$ . We will construct the  $r + 2$  MIHCs beginning at  $s_0$  passing through a fixed edge in  $CCN_k$ .

Consider the first  $r$  MIHCs of  $CCN_k$  beginning at  $s_0$ . We choose distinct vertices  $u_i, v_i \in G_i$  for  $2 \leq i \leq k-3$  such that  $(v_{i-1}, u_i) \in E(CC_N_k)$  for  $3 \leq i \leq k-3$ . Let  $U_1(u_2, v_{k-3}) = \langle u_2, P_2, v_2, u_3, P_3, v_3, \dots, u_{k-3}, P_{k-3}, v_{k-3} \rangle$ , where  $P_i$  is a hamiltonian path of  $C_i$  between  $u_i$  and  $v_i$  for  $2 \leq i \leq k-3$ . Let  $u_1$  and  $u_{k-2}$  be any two vertices in  $G_1$  and  $G_{k-2}$ , respectively. There exist three hamiltonian paths  $P_1, P_{k-2}$  and  $P_{k-1}$  joining from  $ra_1$  to  $u_1, rv_{k-2}$  to  $u_{k-2}$  and  $ru_{k-1}$  to  $lb_{k-1}$  in  $G_1, G_{k-2}$  and  $G_{k-1}$ , respectively. Set  $C_i = \langle s_0, A^i, a_0, ra_1, P_1, u_1, ru_2, U_1(ru_2, v_{k-3}), v_{k-3}, rv_{k-2}, P_{k-2}, u_{k-2}, ru_{k-1}, P_{k-1}, lb_{k-1}, b_0, B^i, s_0 \rangle$  for  $0 \leq i \leq r-1$ . Then, the  $r$  MIHCs are  $C_0, C_1, \dots, C_{r-1}$  which pass through the fixed edge  $(a_0, ra_1)$ .

Now, we consider the  $(r + 1)$ -th MIHC of  $CCN_k$  beginning at  $s_0$ . We choose distinct vertices  $c_i, d_i \in G_i$  for  $2 \leq i \leq k-3$  such that  $(c_{i-1}, d_i) \in E(CC_N_k)$  for

$3 \leq i \leq k-3$ . Since  $G_i$  is 1-vertex-fault-tolerant hamiltonian connected for  $2 \leq i \leq k-3$ , let  $U_x(c_2, d_{k-3}) = \langle c_2, Q_2, d_2, c_3, Q_3, d_3, \dots, c_{k-3}, Q_{k-3}, d_{k-3} \rangle$ , where  $Q_i$  is a hamiltonian path of  $C_i - \{l^{k-i-1}x_i\}$  between  $c_i$  and  $d_i$  for  $2 \leq i \leq k-3$ . In  $G_{k-1}$ , choose a vertex  $x_{k-1}$  which is adjacent to  $ls_{k-1}$  and  $x_{k-1} \neq lb_{k-1}$ . Since  $G_{k-1}$  is 1-vertex-fault-tolerant hamiltonian connected, there is a hamiltonian path  $T_{k-1}$  of  $G_{k-1} - \{ls_{k-1}\}$  between  $la_{k-1}$  and  $x_{k-1}$ . W.L.O.G.,  $T_{k-1}$  can be written as  $\langle la_{k-1}, Q_{k-1}, y_{k-1}, x_{k-1} \rangle$ . Let  $d_{k-2}$  be any vertex in  $G_{k-2} - \{lx_{k-2}, l^2a_{k-2}\}$ . Using the 1-vertex-fault-tolerant hamiltonian connected property of  $G_0, G_1$  and  $G_{k-2}$ , there exist three hamiltonian paths  $Q_0, Q_1$  and  $Q_{k-2}$  of  $G_0 - \{s_0\}, G_1 - \{rs_1\}$  and  $G_{k-2} - \{lx_2\}$  from  $ry_0$  to  $a_0, ra_1$  to  $l^{k-2}x_1$  and  $d_{k-2}$  to  $l^2a_{k-2}$ , respectively. And let  $X_Q = \langle l^{k-3}x_2, l^{k-4}x_3, \dots, lx_{k-2} \rangle$ . Let  $C_r = \langle s_0, rs_1, r^2s_2, U_x(r^2s_2, ld_{k-2}), ld_{k-2}, d_{k-2}, Q_{k-2}, l^2a_{k-2}, la_{k-1}, Q_{k-1}, y_{k-1}, ry_0, Q_0, a_0, ra_1, Q_1, l^{k-2}x_1, l^{k-3}x_2, X_Q, lx_{k-2}, x_{k-1}, ls_{k-1}, s_0 \rangle$ . Therefore,  $C_r$  is mutually independent of the first  $r$  MIHCs  $C_0, C_1, \dots, C_{r-1}$  and passes the fixed edge  $(a_0, ra_1)$ .

Finally, we construct the last MIHC of  $CCN_k$  beginning at  $s_0$ . We choose distinct vertices  $w_i, z_i \in G_i$  for  $2 \leq i \leq k-3$  such that  $(w_{i-1}, z_i) \in E(CCN_k)$  for  $3 \leq i \leq k-3$ . Since  $G_i$  is 1-vertex-fault-tolerant hamiltonian connected for  $2 \leq i \leq k-3$ , let  $U_x(w_2, z_{k-3}) = \langle w_2, R_2, z_2, w_3, R_3, z_3, \dots, w_{k-3}, R_{k-3}, z_{k-3} \rangle$ , where  $R_i$  is a hamiltonian path of  $C_i - \{r^i s_i\}$  between  $w_i$  and  $z_i$  for  $2 \leq i \leq k-3$ . Let  $w_1, w_{k-2}$  and  $z_{k-1}$  be any three vertices in  $G_1, G_{k-2}$  and  $G_{k-1}$ , where  $z_{k-1}$  is not adjacent to  $a_0$ . There exist a hamiltonian path  $R_{k-2}$  joining from  $l^2s_{k-2}$  to  $w_{k-2}$  in  $G_{k-2}$ . Using the 1-vertex-fault-tolerant hamiltonian connected property of  $G_0, G_1$  and  $G_{k-1}$ , there exist three hamiltonian paths  $R_0, R_1$  and  $R_{k-1}$  of  $G_0 - \{s_0\}, G_1 - \{rs_1\}$  and  $G_{k-1} - \{ls_{k-1}\}$  from  $rz_0$  to  $a_0, ra_1$  to  $w_1$  and  $rw_{k-1}$  to  $z_{k-1}$ , respectively. Let  $S_R = \langle r^{k-3}s_{k-3}, r^{k-2}s_{k-2}, \dots, rs_1 \rangle$ , and  $z_{k-3}$  be adjacent to  $r^{k-3}s_{k-3}$ . We let  $C_{r+1} = \langle s_0, ls_{k-1}, l^2s_{k-2}, R_{k-2}, w_{k-2}, rw_{k-1}, R_{k-1}, z_{k-1}, rz_0, R_0, a_0, ra_1, R_1, w_1, rw_2, U_s(rw_2, z_{k-3}), z_{k-3}, r^{k-3}s_{k-3}, S_R, rs_1, s_0 \rangle$ . To avoid the collision of  $U_1(rw_2, v_{k-3})$  and  $U_s(rw_2, z_{k-3})$ , which means  $|S_R| + |R_{k-3}| \leq |B^i| + |P_{k-1}| + |P_{k-1}|$ , we have  $(k-3) + (n-1) \leq 1 + n + n \Rightarrow k \leq n+5$ . So,  $C_{r+1}$  is mutually independent of the first  $r+1$  MIHCs  $C_0, C_1, \dots, C_r$  and passes the fixed edge  $(a_0, ra_1)$ . See Figure 4 for an illustration. ■

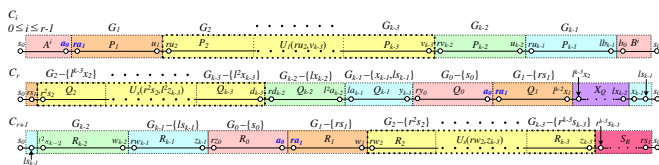


Fig. 4. An illustration of Theorem 4.

### III. CONCLUSION

Let  $k \geq 4, n \geq 6, r \geq 2$  be integers and  $G_i$  be a  $r$ -regular graph with  $n$  vertices for  $0 \leq i \leq k-1$ . In this paper, we prove that under a much weaker condition than [5],

given any vertex  $u$  of the cycle composition network  $CCN_k = CCN(G_0, G_1, \dots, G_{k-1}; \mathcal{M})$ , there exist  $(r+2)$  hamiltonian cycles in  $CCN_k$  beginning at  $u$  such that the  $(r+2)$  cycles are mutually independent. The result is optimal since each vertex of the cycle composition network has exactly  $(r+2)$  neighbors. It is known that many well-known interconnection networks, such as  $k$ -ary  $n$ -cubes, recursive circulant graphs  $G(cd^m, d)$ , generalized recursive circulant graphs  $G(h_k, h_{k-1}, \dots, h_1)$ , circulant graphs  $C(n : c_1, c_2, \dots, c_k)$  and so on, belong to the family of the cycle composition networks. To our knowledge, the above results of  $G(cd^m, d)$ ,  $G(h_k, h_{k-1}, \dots, h_1)$  and  $C(n : c_1, c_2, \dots, c_k)$  have not been published yet. Our study has established the existence of MIHCs in these three families as long as the conditions of Theorem 4 are verified.

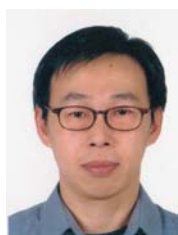
### ACKNOWLEDGEMENT

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