

MEGSOR Iterative Scheme for the Solution of 2D Elliptic PDE's

J. Sulaiman, M. Othman, and M. K. Hasan

Abstract—Recently, the findings on the MEG iterative scheme has demonstrated to accelerate the convergence rate in solving any system of linear equations generated by using approximation equations of boundary value problems. Based on the same scheme, the aim of this paper is to investigate the capability of a family of four-point block iterative methods with a weighted parameter, ω such as the 4 Point-EGSOR, 4 Point-EDGSOR, and 4 Point-MEGSOR in solving two-dimensional elliptic partial differential equations by using the second-order finite difference approximation. In fact, the formulation and implementation of three four-point block iterative methods are also presented. Finally, the experimental results show that the Four Point MEGSOR iterative scheme is superior as compared with the existing four point block schemes.

Keywords—MEG iteration, Second-order finite difference, Weighted parameter.

I. INTRODUCTION

IN solving science and engineering problems via numerical methods, there are many discretization techniques can be taken into account such as finite element, finite difference, finite volume, and boundary element methods, which can be used to discretize and to construct approximation equations for approximating the proposed problems. Then these approximation equations will be used to generate the corresponding systems of linear equations. Due to the large scale of linear systems, many studies on various iterative methods have been proposed to speed up the convergence rate in solving any system of linear equations. Therefore, Young [9, 10, 11, 12], Hackbusch [28] and Saad [26] have already elaborated and discussed the concept of various iterative methods. In addition, Evans [5] has also initiated 4 point block iterative methods via the Explicit Group (EG) iterative method for solving large linear system. Based on this method, further investigations have been extensively conducted by Ibrahim and Abdullah [3], Evans and Yousif [6, 7], Yousif and Evans [27] for demonstrating the capability of block iterative methods. Apart from this concept, Hadjidimos [4] has

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proposed the point iterative method together with two accelerated parameters, which is known as the Accelerated OverRelaxation (AOR) method. Later Martins *et al.* [21] have extended this work by the combination between the EG iterative method together with the AOR method and called as the Explicit Group AOR (EGAOR) method for solving elliptic partial differential equations. They stated that the 4 point-EGAOR method is superior compared to the existing point AOR method.

Besides those methods, the discovery on the concept of the half-sweep iterative method has been inspired by Abdullah [1] via Explicit Decoupled Group (EDG) iterative method in solving two-dimensional Poisson equations. The main advantage of this concept is that the half-sweep iterative method includes the reduction technique in order to reduce the computational complexity of linear systems generated from corresponding approximation equations. Following to this concept, Othman and Abdullah [16] have extended the basic idea of the concept to introduce the Modified Explicit Group (MEG) iterative method based on the quarter-sweep approach. They pointed out that this method is one of most efficient block iterative methods in solving any system of linear equations as compared with EG and EDG iterative methods. Due to the advantages of MEG iterative method, the main purpose of this paper is to investigate the capability of a family of four-point block iterative methods with a weighted parameter, ω such as the 4 Point-EGSOR, 4 Point-EDGSOR, and 4 Point-MEGSOR for solving the two-dimensional elliptic partial differential equations based on the second-order finite difference approximation equations. For comparison, these three four-point block iterative methods will be tested to solve three example problems for the two-dimensional.

To indicate the capability of three four-point block iterative methods, let us consider the two-dimensional elliptic partial differential equations as follows:

$$\begin{aligned} \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + r(x, y) \frac{\partial U}{\partial x} \\ + s(x, y) \frac{\partial U}{\partial y} + t(x, y) U = f(x, y), \end{aligned} \quad (x, y) \in D = [a, b] \times [a, b] \quad (1)$$

subject to the Dirichlet boundary conditions:

$$\begin{aligned} U(x,a) &= g_1(x), & a \leq x \leq b \\ U(x,b) &= g_2(x), & a \leq x \leq b \\ U(a,y) &= g_3(y), & a \leq y \leq b \\ U(b,y) &= g_4(y), & a \leq y \leq b \end{aligned}$$

where $f(x, y)$ is a known function. The coefficients $r(x, y)$, $s(x, y)$, and $t(x, y)$ are functions which only involve variables x and y . To approximate problem (1) via finite difference schemes, there are various ways to discretize Eq. (1) in order to construct approximation equations. Before imposing various discretization techniques for full-, half-, and quarter-sweep cases, let us consider uniformly node points as shown in Fig. 1. Then the solution domain, D in Fig. 1 needs to be partitioned uniformly in both x and y directions with a fixed mesh size, h which is defined in Eq. (2) as follows

$$\Delta x = \Delta y = \left. \frac{(b-a)}{M} = h, \quad M = n+1 \right\}. \quad (2)$$

Again, let us indicate these notations $x_i = ih$, and $y_j = jh$ ($i, j = 0, 1, 2, \dots, M$) where $U_{i,j} = U(x_i, y_j)$, $r_{i,j} = r(x_i, y_j)$, $s_{i,j} = s(x_i, y_j)$, $t_{i,j} = t(x_i, y_j)$, and $f_{i,j} = f(x_i, y_j)$.

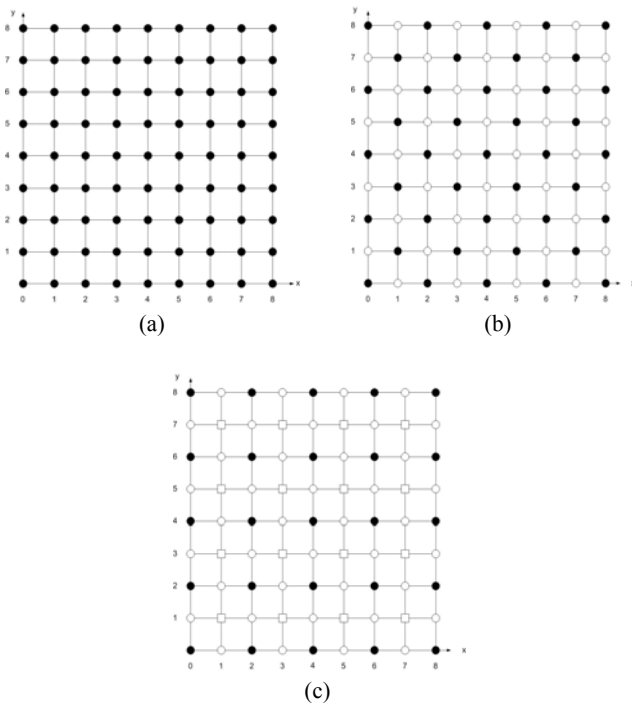


Fig. 1 (a), (b) and (c) The distribution of uniform node points for the full-, half-, and quarter-sweep cases respectively at $m = 8$

II. SECOND-ORDER QUARTER-SWEEP FINITE DIFFERENCE APPROXIMATION

Since four-point block iterative methods such as the 4 Point-EGSOR, 4 Point-EDGSOR, and 4 Point-MEGSOR are categorised as linear solvers, problem (1) firstly needs to be approximated by approximation equations. For simplicity, the next discussion will be restricted to construct second-order

finite difference approximation equations for full-, half- or quarter-sweep cases. Then the formulation of 4 Point-EGSOR, 4 Point-EDGSOR, and 4 Point-MEGSOR iterative methods will be derived to be used in solving corresponding linear systems generated from second-order finite difference approximation equations. For implementation of these block iterative methods, the finite grid networks as shown in Fig. 1 are used as a guideline on how to implement point block iterations by using the natural ordering strategy.

Eventually, other than second-order finite difference schemes being considered to approximate problem (1) at the point (x_i, y_j) , some researchers have suggested high order (Rosser [13]; Gupta *et al.* [20]) schemes to derive high-order finite difference approximation equations. In fact, investigation of nine-point finite difference stencils has been also conducted in forming various nine-point schemes such as fourth-order standard nine-point stencil [22], fourth-order rotated nine-point stencil [24], and fourth-order spark nine-point stencil [25]. Both low and high order schemes will result in their linear system with different properties of their coefficient matrix.

As mentioned in the first paragraph, second-order finite difference schemes will be considered to derive the full-, half-, and quarter-sweep five-point approximation equations for problem (1). Using second-order central difference schemes, the full-, and quarter-sweep five-point approximation equations for problem (1) can be shown respectively as:

$$\begin{aligned} (2 - hr_{i,j})U_{i-1,j} + (2 + hr_{i,j})U_{i+1,j} + (2 - hs_{i,j})U_{i,j-1} \\ + (2 + hs_{i,j})U_{i,j+1} - (8 - 2h^2t_{i,j})U_{i,j} = 2h^2f_{i,j} \end{aligned} \quad (3)$$

and

$$\begin{aligned} (1 - hr_{i,j})U_{i-2,j} + (1 + hr_{i,j})U_{i+2,j} + (1 - hs_{i,j})U_{i,j-2} \\ + (1 + hs_{i,j})U_{i,j+2} - (4 - 4h^2t_{i,j})U_{i,j} = 4h^2f_{i,j} \end{aligned} \quad (4)$$

For formulating and implementing the 4 Point-EDGSOR iterative method, the five-point rotated finite difference approximation equation needs to be constructed by rotating the i -plane axis and the j -plane axis clockwise by 45° . Then the five-point rotated finite difference approximation equation can be easily shown as [22]

$$\begin{aligned} (2 - hr_{i,j} - hs_{i,j})U_{i-1,j-1} + (2 - hr_{i,j} + hs_{i,j})U_{i-1,j+1} \\ + (2 + hr_{i,j} - hs_{i,j})U_{i+1,j-1} + (2 + hr_{i,j} + hs_{i,j})U_{i+1,j+1} \\ - (8 - 4h^2t_{i,j})U_{i,j} = 4h^2f_{i,j} \end{aligned} \quad (5)$$

Again, illustration of the computational molecules for Eqs. (3), (4), and (5) based on five-point finite difference schemes can be seen in Fig. 2.

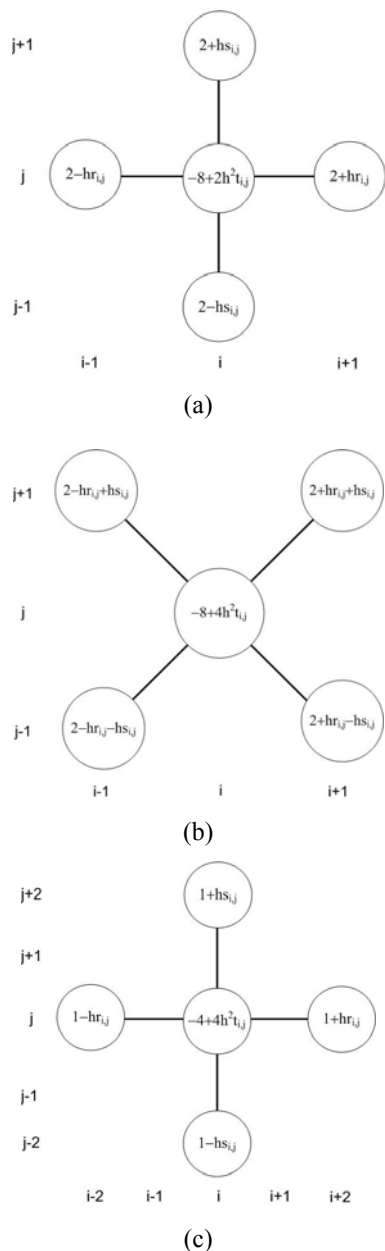


Fig. 2 Computational molecules in case of (a) the full-sweep, (b) the half-sweep, and (c) the quarter-sweep cases

By considering interior node points (i, j) of type ● as shown in Fig. 1, formulation and implementation of 4 Point-EGSOR, 4 Point-EDGSOR and 4 Point-MEGSOR iterative methods will be applied onto these interior node points until the criterion of convergence test is satisfied. Afterwards calculations of approximate solution for the remaining points of other type will be obtained directly by using direct methods (Abdullah [1]; Abdullah & Ali [2]; Ibrahim & Abdullah [3]; Yousif & Evans [27]; Othman & Abdullah [14, 15, 16]; Sulaiman *et al.* [18, 19]). Basically, the execution times for the half- and quarter-sweep iterative schemes may be reduced faster than the full-sweep case. Actually, this is due to both iterations have about half and quarter of the entire node points respectively.

III. FORMULATION AND IMPLEMENTATION OF THE 4 POINT-MEG ITERATIVE METHOD

As being explained in this first section, the advantage of family of four-point block iterative methods with a weighted parameter, ω mainly on 4 Point-EGSOR, 4 Point-EDGSOR, and 4 Point-MEGSOR will be examined in solving for problem (1). From the previous studies (Young 1954, 1971, 1972, 1976; Sulaiman *et al.* [15,16]), this parameter, ω has been known to speed up the convergence rate of iterative methods in solving any linear system.

In order to develop formulation of 4 point-MEGSOR iterative method, the basic concept of this method needs to be explored. According to Othman and Abdullah (2000), the 4 Point-MEG iterative method has been developed based on second-order quarter-sweep finite difference approximation equation for solving two-dimensional Poisson equation. Later Othman *et al.* [17] and Sulaiman *et al.* [19] have extensively extended and applied the concept of quarter-sweep iteration to various problems. Based on the same concept together with the weighted parameter, ω , formulation of 4 point-MEGSOR iterative method will be presented. For simplicity, let any four point group be considered to form a (4×4) linear system given as (Othman & Abdullah [16]; Othman *et al.* [17])

$$\begin{bmatrix} a_1 & c_1 & e_1 & 0 \\ b_2 & a_2 & 0 & e_2 \\ d_3 & 0 & a_3 & c_3 \\ 0 & d_4 & b_4 & a_4 \end{bmatrix} \begin{bmatrix} U_{i,j} \\ U_{i+2,j} \\ U_{i,j+2} \\ U_{i+2,j+2} \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix} \quad (6)$$

where,

$$\begin{aligned} a_1 &= -4 + 4h^2t_{i,j}, & a_2 &= -4 + 4h^2t_{i+2,j}, \\ b_1 &= 1 - hr_{i,j}, & b_2 &= 1 - hr_{i+2,j}, \\ c_1 &= 1 + hr_{i,j}, & c_2 &= 1 + hr_{i+2,j}, \\ d_1 &= 1 - hs_{i,j}, & d_2 &= 1 - hs_{i+2,j}, \\ e_1 &= 1 + hs_{i,j}, & e_2 &= 1 + hs_{i+2,j}, \\ a_3 &= -4 + 4h^2t_{i,j+2}, & a_4 &= -4 + 4h^2t_{i+2,j+2}, \\ b_3 &= 1 - hr_{i,j+2}, & b_4 &= 1 - hr_{i+2,j+2}, \\ c_3 &= 1 + hr_{i,j+2}, & c_4 &= 1 + hr_{i+2,j+2}, \\ d_3 &= 1 - hs_{i,j+2}, & d_4 &= 1 - hs_{i+2,j+2}, \\ e_3 &= 1 + hs_{i,j+2}, & e_4 &= 1 + hs_{i+2,j+2} \end{aligned}$$

$$\begin{aligned} S_1 &= 4h^2 f_{i,j} - b_1 U_{i-2,j} - d_1 U_{i,j-2}, \\ S_2 &= 4h^2 f_{i+2,j} - c_2 U_{i+4,j} - d_2 U_{i+2,j-2}, \\ S_3 &= 4h^2 f_{i,j+2} - b_3 U_{i-2,j+2} - e_3 U_{i,j+4}, \\ S_4 &= 4h^2 f_{i+2,j+2} - c_4 U_{i+4,j+2} - e_4 U_{i+2,j+4}. \end{aligned}$$

Then let the linear system (6) be rewritten as follows:

$$AU = S \quad (7)$$

where,

$$A = \begin{bmatrix} a_1 & c_1 & e_1 & 0 \\ b_2 & a_2 & 0 & e_2 \\ d_3 & 0 & a_3 & c_3 \\ 0 & d_4 & b_4 & a_4 \end{bmatrix},$$

$$\tilde{U} = [U_{i,j} \quad U_{i+2,j} \quad U_{i,j+2} \quad U_{i+2,j+2}]^T,$$

$$\tilde{S} = [S_1 \quad S_2 \quad S_3 \quad S_4]^T.$$

Now by considering the inverse matrix of system (7), we can show that the 4 Point-MEG iterative scheme can be generally expressed as:

$$\begin{bmatrix} U_{i,j} \\ U_{i+2,j} \\ U_{i,j+2} \\ U_{i+2,j+2} \end{bmatrix}^{(k+1)} = \begin{bmatrix} a_1 & c_1 & e_1 & 0 \\ b_2 & a_2 & 0 & e_2 \\ d_3 & 0 & a_3 & c_3 \\ 0 & d_4 & b_4 & a_4 \end{bmatrix}^{-1} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix}. \quad (8)$$

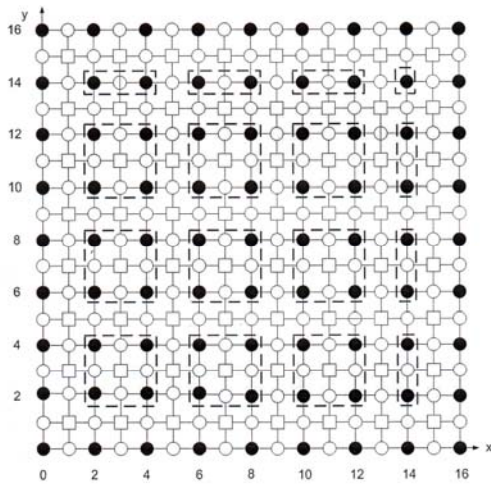


Fig. 3 Implementation of the 4 Point-MEGSOR at the solution domain at $m=16$

Then by adding the weighted parameter, ω into the scheme (8), the formulation of the 4 Point-MEGSOR can be easily defined as:

$$\begin{bmatrix} U_{i,j} \\ U_{i+2,j} \\ U_{i,j+2} \\ U_{i+2,j+2} \end{bmatrix}^{(k+1)} = (1-\omega) \begin{bmatrix} U_{i,j} \\ U_{i+2,j} \\ U_{i,j+2} \\ U_{i+2,j+2} \end{bmatrix}^{(k)} + \omega \begin{bmatrix} a_1 & c_1 & e_1 & 0 \\ b_2 & a_2 & 0 & e_2 \\ d_3 & 0 & a_3 & c_3 \\ 0 & d_4 & b_4 & a_4 \end{bmatrix}^{-1} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix} \quad (9)$$

where the value of ω is in the range of $[1, 2)$. If $\omega = 1$, this method will be the special case of the existing 4 Point-MEG method. Based on Fig. 3, it can be seen that several completed group of four points based on interior node points of type \bullet were presented. For the case of remaining same

node points near the right and top boundaries, however, there exist several ungroup points scheme. According to Abdullah and Evans (1989), these ungroup points can be treated as a group of two points and the single point schemes shown in Fig. 3. For comparison, other two four-point block iterative methods such as the 4 Point-EGSOR and the 4 Point-EDGSOR methods will be also formulated and then implemented.

IV. NUMERICAL RESULTS

In this section, there are three problems, which are categorized as two-dimensional elliptic partial differential equations in order to demonstrate the effectiveness of the 4 Point-MEGSOR scheme using the quarter-sweep five-point finite difference approximation equation in Eq. (4) for problem (1). For comparison, three measurement parameters such as the number of iterations, execution time and maximum absolute error will be considered. In all three problems, the iterations of all methods were stopped, when the absolute error tolerance, $\varepsilon = 10^{-10}$ was achieved. The following three two-dimensional were being used to conduct some numerical experiments of 4 Point-EGSOR, 4 Point-EDGSOR, and 4 Point-MEGSOR methods as follows:

Example 1 (Abdullah [1])

The following 2D Poisson equation was being used to conduct some numerical experiments of 4 Point-EGSOR, 4 Point-EDGSOR, and 4 Point-MEGSOR methods:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = (x^2 + y^2)e^{xy}, \quad 0 \leq x, y \leq 1. \quad (10)$$

Then boundary conditions and the exact solution of the problem (10) are defined by:

$$U(x, y) = e^{xy}, \quad 0 \leq x, y \leq 1. \quad (11)$$

All results of numerical experiments for example 1, obtained from implementation of four-point block iterative methods with or without the weighted parameter, ω are tabulated in Tables I and II at different values of mesh sizes, M .

Through the observation in Tables I and II, it clearly shows that the four point block iterative methods with the weighted parameter, ω are superior as compared to the corresponding block iterative methods without ω . However in terms of using ω , it can be seen that numbers of iterations decreased approximately 41.50-49.16% and 16.41-17.51% respectively correspond to 4 Point-MEGSOR and 4 Point-EDGSOR methods compared to 4 Point-EGSOR method. In fact, the execution time against the mesh size for both 4 Point-MEGSOR and 4 Point-EDGSOR methods are much faster about 60.00-71.22% and 16.24-25.71% respectively than 4 Point-EGSOR method.

TABLE I

COMPARISON OF NUMBER OF ITERATIONS K, EXECUTION TIME (IN SECONDS) AND MAXIMUM ABSOLUTE ERRORS OF THE 4 POINT-EG, 4 POINT-EDG, AND 4 POINT-MEG ITERATIVE METHODS FOR EXAMPLE 1

No. of iterations					
Method	Mesh grid				
	100	200	300	400	500
4 EG	8894	32753	69990	119756	181461
4 EDG	6779	25000	53473	91562	138823
4 MEG	2401	8894	19082	32753	49761
Execution Time (Seconds)					
Method (r opt)	Mesh grid				
	100	200	300	400	500
4 EG	7.80	94.41	462.76	1315.25	3175.65
4 EDG	4.04	53.67	255.64	770.36	1836.09
4 MEG	1.24	13.53	63.47	185.08	447.42
Maximum Absolute Errors					
Method (r opt)	Mesh grid				
	100	200	300	400	500
4 EG	2.88e-7	1.52e-7	4.32e-7	7.97e-7	1.26e-6
4 EDG	1.66e-5	4.02e-6	1.55e-6	5.88e-7	3.58e-7
4 MEG	1.27e-6	2.89e-7	9.20e-8	1.52e-7	2.83e-7

TABLE II

COMPARISON OF NUMBER OF ITERATIONS K, EXECUTION TIME (IN SECONDS) AND MAXIMUM ABSOLUTE ERRORS OF THE 4 POINT-EGSOR, 4 POINT-EDGSOR, AND 4 POINT-MEGSOR ITERATIVE METHODS FOR EXAMPLE 1

No. of iterations					
Method (r opt)	Mesh grid				
	100	200	300	400	500
4 EGSOR	297	582	865	1143	1423
4 EDGSOR	245	482	723	952	1186
4 MEGSOR	151	297	506	585	728
Execution Time (Seconds)					
Method (r opt)	Mesh grid				
	100	200	300	400	500
4 EGSOR	0.35	1.76	5.69	12.60	24.32
4 EDGSOR	0.26	1.37	4.50	9.90	20.37
4 MEGSOR	0.14	0.60	2.00	3.70	7.00
Maximum Absolute Errors					
Method (r opt)	Mesh grid				
	100	200	300	400	500
4 EGSOR	3.15e-7	7.86e-8	3.49e-8	1.93e-8	1.24e-8
4 EDGSOR	1.66e-5	4.15e-6	1.85e-6	1.04e-6	6.63e-7
4 MEGSOR	1.28e-6	3.16e-7	1.40e-7	7.89e-8	5.02e-8

Example 2 (Evans [5])

Some numerical experiments were conducted in solving the following 2D Helmholtz equation:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \rho U = 6 - \rho(2x^2 + y^2), \quad 0 \leq x, y \leq 1. \quad (12)$$

Then boundary conditions and the exact solution of the problem (12) are defined by:

$$U(x, y) = 2x^2 + y^2, \quad 0 \leq x, y \leq 1. \quad (13)$$

All results of numerical experiments for example 2, obtained from implementation of four-point block iterative methods with or without the weighted parameter, ω are tabulated in Tables III and IV at different values of mesh sizes, m for a fixed $\rho = 25.0$.

Similarly from Tables III and IV, the findings for all three four-point block iterative methods with the weighted parameter, ω show that numbers of iterations for the 4 Point-MEGSOR and 4 Point-EDGSOR methods have declined by 47.72-49.53% and 17.04-20.28% respectively compared to 4 Point-EGSOR method. Again, the execution times for 4 Point-MEGSOR and 4 Point-EDGSOR methods are much faster

approximately 64.29-74.34-72.35% and 27.97-36.02% respectively than 4 Point-EGSOR method.

TABLE III

COMPARISON OF NUMBER OF ITERATIONS K, EXECUTION TIME (IN SECONDS) AND MAXIMUM ABSOLUTE ERRORS OF THE 4 POINT-EG, 4 POINT-EDG, AND 4 POINT-MEG ITERATIVE METHODS FOR EXAMPLE 2 AT $\rho=25$

No. of iterations					
Method	Mesh grid				
	100	200	300	400	500
4 EG	4032	14860	31787	54439	82556
4 EDG	3070	11334	24269	41594	63113
4 MEG	1091	4032	8653	14860	22588
Execution Time (Seconds)					
Method (r opt)	Mesh grid				
	100	200	300	400	500
4 EG	3.20	39.30	197.90	577.62	1355.29
4 EDG	2.04	26.69	134.62	394.44	958.27
4 MEG	0.53	6.01	28.78	84.68	200.35
Maximum Absolute Errors					
Method (r opt)	Mesh grid				
	100	200	300	400	500
4 EG	2.22e-8	8.93e-8	2.01e-7	3.58e-7	5.59e-7
4 EDG	1.67e-8	6.70e-8	1.51e-7	2.68e-7	4.19e-7
4 MEG	5.46e-9	2.22e-8	5.02e-8	8.93e-8	1.40e-7

TABLE IV

COMPARISON OF NUMBER OF ITERATIONS K, EXECUTION TIME (IN SECONDS) AND MAXIMUM ABSOLUTE ERRORS OF THE 4 POINT-EGSOR, 4 POINT-EDGSOR, AND 4 POINT-MEGSOR ITERATIVE METHODS FOR EXAMPLE 2 AT $\rho=25$

No. of iterations					
Method (r opt)	Mesh grid				
	100	200	300	400	500
4 EGSOR	212	417	624	822	1027
4 EDGSOR	169	336	501	665	852
4 MEGSOR	107	218	316	417	519
Execution Time (Seconds)					
Method (r opt)	Mesh grid				
	100	200	300	400	500
4 EGSOR	0.28	1.43	4.72	10.41	19.95
4 EDGSOR	0.19	1.03	3.02	6.74	13.46
4 MEGSOR	0.10	0.45	1.34	2.72	5.04
Maximum Absolute Errors					
Method (r opt)	Mesh grid				
	100	200	300	400	500
4 EGSOR	5.77e-10	7.60e-10	1.39e-9	2.02e-9	2.48e-9
4 EDGSOR	3.52e-10	9.85e-10	9.01e-10	1.84e-9	8.75e-11
4 MEGSOR	1.56e-10	1.05e-10	8.70e-10	7.60e-10	2.91e-10

Example 3 (Ali & Ling [23])

The following two-dimensional steady convection-diffusion equation was being used to conduct some numerical experiments of 4 Point-EGSOR, 4 Point-EDGSOR, and 4 Point-MEGSOR methods as follows

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + r(x, y) \frac{\partial U}{\partial x} + s(x, y) \frac{\partial U}{\partial y} = \left((1 + Re)x^2 + (1 - Re)y^2 \right) e^{xy}, \quad 0 \leq x, y \leq 1. \quad (14)$$

Then boundary conditions and the exact solution of the problem (14) are defined by

$$U(x, y) = e^{xy}, \quad 0 \leq x, y \leq 1. \quad (15)$$

with $r(x, y) = -Re y$ and $s(x, y) = Re x$. The Reynolds number, Re is a scalar.

All results of numerical experiments for example 3, obtained from implementation of four point block iterative

methods with or without the weighted parameter, ω are tabulated in Tables V and VI at different values of mesh sizes, m for a fixed $Re = 10.0$.

Inline with the results for examples 1 and 2, the observation in Table VI has shown that numbers of iterations for the 4 Point-MEGSOR and 4 Point-EDGSOR methods have declined by 48.28-50.00% and 18.07-20.89% respectively compared to 4 Point-EGSOR method. Again, the execution times for 4 Point-MEGSOR and 4 Point-EDGSOR methods are much faster approximately 63.64-72.35% and 48.49% respectively than 4 Point-EGSOR method.

TABLE V

COMPARISON OF NUMBER OF ITERATIONS K, EXECUTION TIME (IN SECONDS) AND MAXIMUM ABSOLUTE ERRORS OF THE 4 POINT-EG, 4 POINT-EDG, AND 4 POINT-MEG ITERATIVE METHODS FOR EXAMPLE 3 AT $Re = 10$

Method	No. of iterations				
	Mesh grid				
	100	200	300	400	500
4 EG	6012	22251	47708	81846	124287
4 EDG	4576	16964	36404	62493	94949
4 MEG	1617	6012	12935	22251	33866

Method (r opt)	Execution Time (Seconds)				
	Mesh grid				
	100	200	300	400	500
4 EG	6.40	87.15	422.68	1270.32	2978.93
4 EDG	2.62	37.30	182.94	558.72	1320.89
4 MEG	1.10	9.91	46.09	137.12	322.50

Method (r opt)	Maximum Absolute Errors				
	Mesh grid				
	100	200	300	400	500
4 EG	1.91e-6	5.01e-7	3.01e-7	5.03e-7	7.94e-7
4 EDG	1.33e-5	3.26e-6	1.32e-6	5.93e-7	2.69e-7
4 MEG	7.61e-6	1.91e-6	8.61e-7	5.01e-7	3.47e-7

TABLE VI

COMPARISON OF NUMBER OF ITERATIONS K, EXECUTION TIME (IN SECONDS) AND MAXIMUM ABSOLUTE ERRORS OF THE 4 POINT-EGSOR, 4 POINT-EDGSOR, AND 4 POINT-MEGSOR ITERATIVE METHODS FOR EXAMPLE 3 $Re = 10$

Method (r opt)	No. of iterations				
	Mesh grid				
	100	200	300	400	500
4 EGSOR	286	553	819	1106	1353
4 EDGSOR	229	449	671	875	1085
4 MEGSOR	147	286	418	553	688

Method (r opt)	Execution Time (Seconds)				
	Mesh grid				
	100	200	300	400	500
4 EGSOR	0.33	1.92	6.13	14.19	26.62
4 EDGSOR	0.17	1.11	3.61	8.31	16.03
4 MEGSOR	0.12	0.61	1.82	3.94	7.36

Method (r opt)	Maximum Absolute Errors				
	Mesh grid				
	100	200	300	400	500
4 EGSOR	1.90e-6	4.77e-7	2.12e-7	1.19e-7	7.67e-8
4 EDGSOR	1.34e-5	3.34e-6	1.48e-6	8.34e-7	5.33e-7
4 MEGSOR	7.61e-6	1.91e-6	8.48e-7	4.77e-7	3.05e-7s

V. CONCLUSION

For the proposed problems, the formulation and implementation of the 4 Point-MEG iterative method with or without the weighted parameter, ω by using the second-order five-point finite difference schemes have been derived to form the corresponding approximation equations as shown in Eq. (4). From observation of all experimental results by imposing the SOR technique, it can be also observed that the number of

iterations and the execution time for all four point block iterative methods such as EGSOR, EDGSOR and MEGSOR have been declined tremendously as compared with the corresponding block iterative methods without weighted parameter, ω . Overall, the 4 Point-MEGSOR method is superior to the 4 Point-EGSOR and EDGSOR methods in terms of a number of iterations and the execution time. This is due to the reduction technique applied to the 4 Point-MEGSOR method which reduces approximately 75% of its computational complexity as compared to the 4 Point-EGSOR method. In fact, these conclusions are inline with the results of Othman and Abdullah [16]. The approximate solutions for all methods are in good agreement.

For future works, the capability of 4 Point-MEGSOR should be investigated for solving other multi-dimensional partial differential equations (Evans [5]; Evans & Sahimi [8]; Ibrahim & Abdullah [3]) and being used as a smoother in multigrid solvers (Hackbusch [28]; Othman & Abdullah [14, 15]). Also, further studies for various point block iterative methods can be also examined (Yousif & Evans [27]; Martins *et al.* [21]).

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