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Definable Subsets in Covering Approximation Spaces

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Abstract—Covering approximation spaces is a class of important generalization of approximation spaces. For a subset X of a covering approximation space $(U,\mathcal{C})$, is X definable or rough? The answer of this question is uncertain, which depends on covering approximation operators endowed on $(U,\mathcal{C})$. Note that there are many various covering approximation operators, which can be endowed on covering approximation spaces. This paper investigates covering approximation spaces endowed ten covering approximation operators respectively, and establishes some relations among definable subsets, inner definable subsets and outer definable subsets in covering approximation spaces, which deepens some results on definable subsets in approximation spaces.

Keywords—Covering approximation space, covering approximation operator, definable subset, inner definable subset, outer definable subset.

I. Introduction

In order to extract useful information hidden in voluminous data, many methods in addition to classical logic have been proposed. Pawlak rough-set theory, which was proposed by Z. Pawlak in [10], plays an important role in applications of these methods. Here, usefulness of approximation spaces has been demonstrated by many successful applications in pattern recognition and artificial intelligence (see [2], [4], [5], [6], [8], [9], [10], [11], [12], [20], for example).

Definition 1.1 ([12]): Let U, the universe of discourse, be a finite set and $\mathcal{C}$ be a partition on U. For $X \subset U$, Put

$$\underline{\mathcal{C}}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \subset X\}.$$

$$\overline{\mathcal{C}}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \bigcap X \neq \emptyset\};$$

(1) $(U;\mathcal{C})$ is called a approximation space.

(2) $\underline{C} : 2^U \longrightarrow 2^U$ is called lower approximation operator.

(3) $\overline{C} : 2^U \longrightarrow 2^U$ is called upper approximation operator.

(4) X is called a definable subset of $(U;\mathcal{C})$ if $\overline{\mathcal{C}}(X) = \underline{\mathcal{C}}(X)$.

(5) X is called a rough subset of $(U;\mathcal{C})$ if $\overline{\mathcal{C}}(X) \neq \underline{\mathcal{C}}(X)$.

Recently, D. Pei generalized definable subsets of approximation spaces to inner definable subsets and outer definable subsets.

Definition 1.2 ([13]): Let $(U;\mathcal{C})$ be a approximation space with approximation operators $\underline{C}$ and $\overline{C}$. A subset X of U is called an inner (resp. outer) definable subset of $(U;\mathcal{C})$ if $\underline{\mathcal{C}}(X) = X$ (resp. $\overline{\mathcal{C}}(X) = X$).

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For definable subsets, inner definable subsets and outer definable subsets of approximation spaces, D. Pei gave the following result.

Proposition 1.3 ([13]): Let $(U;\mathcal{C})$ be a approximation space with approximation operators $\underline{C}$ and $\overline{C}$, and $X \subset U$. Then the following are equivalent.

(1) X is a definable subset of $(U,\mathcal{C})$.

(2) X is an inner definable subset of $(U,\mathcal{C})$.

(3) X is an outer definable subset of $(U,\mathcal{C})$.

Note that, in the past years, approximation spaces and approximation operators have been extended to covering approximation spaces and covering approximation operators respectively (see [1], [3], [7], [14], [15], [16], [17], [18], [19], [20], [21], for example). It is natural to raise the following question.

Question 1.4: Can "approximation space" and "approximation operators" in Proposition 1.3 be replaced by "covering approximation space" and "covering approximation operators" respectively?

For a covering approximation space $(U;\mathcal{C})$, because there are many various covering approximation operators, which can be endowed on $(U;\mathcal{C})$, the answers of Question 1.4 are uncertain, which depend on covering approximation operators endowed on $(U,\mathcal{C})$. In this paper, we investigate covering approximation spaces endowed ten covering approximation operators respectively, and establishes some relations among definable subsets, inner definable subsets and outer definable subsets, which give some answers of Question 1.4 and deepens some results on definable subsets in approximation spaces.

II. Preliminaries

Definition 2.1 ([21]): Let U, the universe of discourse, be a finite set and $\mathcal{C}$ be a family of nonempty subsets of U.

(1) $\mathcal{C}$ is called a cover of U if $\bigcup\{K : K \in \mathcal{C}\} = U$.

(2) The pair $(U;\mathcal{C})$ is called a covering approximation space if $\mathcal{C}$ is a cover of U.

Definition 2.2: Let $(U;\mathcal{C})$ be a covering approximation space. For $x \in U$, put $Md(x) = \{K : (x \in K \in \mathcal{C}) \bigwedge (x \in S \in \mathcal{C} \bigwedge S \subset K \Longrightarrow S = K)\}$ and $N(x) = \bigcap\{K : x \in K \in \mathcal{C}\}$. For each $n \in \{1,2,3,4,5,6,7,8,9,10\}$, $\underline{C_n}$ and $\overline{C_n}$ are defined as follows and are called n-th lower covering approximation operator and n-th upper covering approximation operator on $(U;\mathcal{C})$ respectively.

(1) $\underline{\mathcal{C}_1}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \subset X\}$;
$\overline{\mathcal{C}_1}(X) = \underline{\mathcal{C}_1}(X) \bigcup (\bigcup\{\bigcup Md(x) : x \in X - \underline{\mathcal{C}_1}(X)\})$.

(2) $\underline{\mathcal{C}_2}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \subset X\}$;
$\overline{\mathcal{C}_2}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \bigcap X \neq \emptyset\}$.

(3) $\underline{\mathcal{C}_3}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \subset X\}$;

$\overline{\mathcal{C}_3}(X) = \bigcup\{\bigcup Md(x) : x \in X\}$.

(4) $\underline{\mathcal{C}_4}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \subset X\}$;
$\overline{\mathcal{C}_4}(X) = \underline{\mathcal{C}_3}(X) \bigcup(\bigcup\{K : K \in \mathcal{C} \bigwedge K \bigcap(X - \underline{\mathcal{C}_4}(X)) \neq \emptyset\})$.

(5) $\underline{\mathcal{C}_5}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \subset X\}$;
$\overline{\mathcal{C}_5}(X) = \underline{\mathcal{C}_5}(X) \bigcup(\bigcup\{N(x) : x \in X - \underline{\mathcal{C}_5}(X)\})$.

(6) $\underline{\mathcal{C}_6}(X) = \{x \in U : N(x) \subset X\}$;
$\overline{\mathcal{C}_6}(X) = \{x \in U : N(x) \bigcap X \neq \emptyset\}$.

(7) $\underline{\mathcal{C}_7}(X) = \{x \in U : \forall K \in \mathcal{C}(x \in K \Longrightarrow K \subset X)\}$;
$\overline{\mathcal{C}_7}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \bigcap X \neq \emptyset\}$.

(8) $\underline{\mathcal{C}_8}(X) = \bigcup\{K : K \in \mathcal{C} \bigwedge K \subset X\}$;
$\overline{\mathcal{C}_8}(X) = U - \underline{\mathcal{C}_8}(U - X)$.

(9) $\underline{\mathcal{C}_9}(X) = \{x \in U : \forall u(x \in N(u) \Longrightarrow N(u) \subset X)\}$;
$\overline{\mathcal{C}_9}(X) = \bigcup\{N(x) : x \in U \bigwedge N(x) \bigcap X \neq \emptyset\}$.

(10) $\underline{\mathcal{C}_{10}}(X) = \{x \in U : \forall u(x \in N(u) \Longrightarrow u \in X)\}$;
$\overline{\mathcal{C}_{10}}(X) = \bigcup\{N(x) : x \in X\}$.

Remark 2.3: $\underline{C_n}$ and $\overline{C_n}$ (n=1,2,3) come from [21]; $\underline{C_4}$ and $\overline{C_4}$ come from [20]; $\underline{C_5}$ and $\overline{C_5}$ come from [19]; $\underline{C_6}$ and $\overline{C_6}$ come from [15], [20]; $\underline{C_7}$ and $\overline{C_7}$ come from [14]; $\underline{C_n}$ and $\overline{C_n}$ (n=8,9,10) come from [15].

Remark 2.4: Throughout this paper, $(U; \mathcal{C}_n)$ always denotes a covering approximation space $(U; \mathcal{C})$ with covering approximation operators $\underline{C_n}$ and $\overline{C_n}$, where $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Definition 2.5: Let $(U; \mathcal{C}_n)$ be a covering approximation space, where $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let $X \subset U$.

(1) X is called a definable subset of $(U; \mathcal{C}_n)$ if $\underline{\mathcal{C}_n}(X) = \overline{\mathcal{C}_n}(X)$.

(2) X is called an inner definable subset of $(U; \mathcal{C}_n)$ if $\underline{\mathcal{C}_n}(X) = X$.

(3) X is called an outer definable subset of $(U; \mathcal{C}_n)$ if $\overline{\mathcal{C}_n}(X) = X$.

The following lemma comes from [14], [15], [19], [20], [21].

Lemma 2.6: Let $(U; \mathcal{C}_n)$ be a covering approximation space and $X \subset U$, where $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Then $\underline{\mathcal{C}_n}(X) \subset X \subset \overline{\mathcal{C}_n}(X)$.

By Definition 2.5 and Lemma 2.6, we have the following proposition.

Proposition 2.7: Let $(U; \mathcal{C}_n)$ be a covering approximation space and $X \subset U$, where $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Then X is a definable subset of $(U; \mathcal{C}_n)$ if and only if X is a both inner definable and outer definable subset of $(U; \mathcal{C}_n)$.

III. The Main Results

Theorem 3.1: Let $(U; \mathcal{C}_1)$ be a covering approximation space and $X \subset U$. Then the following are equivalent.

(1) X is a definable subset of $(U, \mathcal{C}_1)$.

(2) X is an inner definable subset of $(U, \mathcal{C}_1)$.

(3) X is an outer definable subset of $(U, \mathcal{C}_1)$.

Proof. (1) $\Longrightarrow$ (2): Let X be a definable subset of $(U, \mathcal{C}_1)$. By Proposition 2.7, X is an inner definable subset of $(U, \mathcal{C}_1)$.

(2) $\Longrightarrow$ (3): Let X be an inner definable subset of $(U, \mathcal{C}_1)$. Then $\underline{\mathcal{C}_1}(X) = X$, and so $X - \underline{\mathcal{C}_1}(X) = \emptyset$. It follows that $\overline{\mathcal{C}_1}(X) = \underline{\mathcal{C}_1}(X) \bigcup(\bigcup\{\bigcup Md(x) : x \in X - \underline{\mathcal{C}_1}(X)\}) = \underline{\mathcal{C}_1}(X) = X$. So X is an outer definable subset of $(U, \mathcal{C}_1)$.

(3) $\Longrightarrow$ (1): Let X be an outer definable subset of $(U, \mathcal{C}_1)$. Then $\overline{\mathcal{C}_1}(X) = X$. By Proposition 2.7, it suffices to prove that $\underline{\mathcal{C}_1}(X) = X$. Since $\underline{\mathcal{C}_1}(X) \subset X$ by Lemma 2.6, we only need to prove that $X \subset \underline{\mathcal{C}_1}(X)$. Let $y \in X = \overline{\mathcal{C}_1}(X) = \underline{\mathcal{C}_1}(X) \bigcup(\bigcup\{\bigcup Md(x) : x \in X - \underline{\mathcal{C}_1}(X)\})$. If $y \notin \underline{\mathcal{C}_1}(X)$, then $y \in \bigcup\{\bigcup Md(x) : x \in X - \underline{\mathcal{C}_1}(X)\}$, and so there is $z \in X - \underline{\mathcal{C}_1}(X)\}$ and $K \in Md(z) \subset \mathcal{C}$ such that $y \in K$. Since $K \subset \bigcup Md(z) \subset \overline{\mathcal{C}_1}(X) = X$ and $K \in \mathcal{C}$, $y \in K \subset \underline{\mathcal{C}_1}(X)$. This proves that $X \subset \underline{\mathcal{C}_1}(X)$.

Theorem 3.2: Let $(U; \mathcal{C}_2)$ be a covering approximation space and $X \subset U$. Consider the following conditions.

(1) X is a definable subset of $(U, \mathcal{C}_2)$.

(2) X is an inner definable subset of $(U, \mathcal{C}_2)$.

(3) X is an outer definable subset of $(U, \mathcal{C}_2)$.

Then (1) $\Longleftrightarrow$ (3) $\Longrightarrow$ (2) $\not\Longrightarrow$ (3).

Proof. (1) $\Longrightarrow$ (3): Let X be a definable subset of $(U, \mathcal{C}_2)$. By Proposition 2.7, X is an outer definable subset of $(U, \mathcal{C}_2)$.

(3) $\Longrightarrow$ (2): Let X be an outer definable subset of $(U; \mathcal{C}_2)$. Then $\overline{\mathcal{C}_2}(X) = X$. Let $x \in X = \overline{\mathcal{C}_2}(X)$, then there is $K \in \mathcal{C}$ such that $x \in K$ and $K \bigcap X \neq \emptyset$, and so $K \subset \overline{\mathcal{C}_2}(X) = X$. It follows that $x \in K \subset \underline{\mathcal{C}_2}(X)$. This proves that $X \subset \underline{\mathcal{C}_2}(X)$. Since $\underline{\mathcal{C}_2}(X) \subset X$ by Lemma 2.6, $\underline{\mathcal{C}_2}(X) = X$. So X is an inner definable subset of $(U, \mathcal{C}_2)$.

(3) $\Longrightarrow$ (1): Let X be an outer definable subset of $(U; \mathcal{C}_2)$. By (3) $\Longrightarrow$ (2), X is an inner definable subset of $(U, \mathcal{C}_2)$. By Proposition 2.7, X is a definable subset of $(U, \mathcal{C}_2)$.

(2) $\not\Longrightarrow$ (3): Consider a covering approximation space $(U; \mathcal{C}_2)$ and $X \subset U$, where $U = \{a, b, c\}$, $\mathcal{C} = \{\{a, b\}, \{b, c\}\}$, $X = \{a, b\}$. It is not difficult to check that $\underline{\mathcal{C}_2}(X) = X$ and $\overline{\mathcal{C}_3}(X) = U \neq X$. So X is an inner definable subset of $(U; \mathcal{C}_2)$ and is not an outer definable subset of $(U; \mathcal{C}_2)$.

Theorem 3.3: Let $(U; \mathcal{C}_3)$ be a covering approximation space and $X \subset U$. Consider the following conditions.

(1) X is a definable subset of $(U, \mathcal{C}_3)$.

(2) X is an inner definable subset of $(U, \mathcal{C}_3)$.

(3) X is an outer definable subset of $(U, \mathcal{C}_3)$.

Then (1) $\Longleftrightarrow$ (3) $\Longrightarrow$ (2) $\not\Longrightarrow$ (3).

Proof. (1) $\Longrightarrow$ (3): Let X be a definable subset of $(U, \mathcal{C}_3)$. By Proposition 2.7, X is an outer definable subset of $(U, \mathcal{C}_3)$.

(3) $\Longrightarrow$ (2): Let X is an outer definable subset of $(U; \mathcal{C}_3)$. Then $\overline{\mathcal{C}_3}(X) = X$. Let $x \in X$, then $x \in \bigcup Md(x) \subset \overline{\mathcal{C}_3}(X) = X$. Thus there is $K \in Md(x) \subset \mathcal{C}$ such that $x \in K \subset X$. It follows that $x \in K \subset \underline{\mathcal{C}_3}(X)$. This proves that $X \subset \underline{\mathcal{C}_3}(X)$. Since $\underline{\mathcal{C}_3}(X) \subset X$ by Lemma 2.6, $\underline{\mathcal{C}_3}(X) = X$. So X is an inner definable subset of $(U, \mathcal{C}_3)$.

(3) $\Longrightarrow$ (1): Let X be an outer definable subset of $(U; \mathcal{C}_3)$. By (3) $\Longrightarrow$ (2), X is an inner definable subset of $(U, \mathcal{C}_3)$. By Proposition 2.7, X is a definable subset of $(U, \mathcal{C}_3)$.

(2) $\not\Longrightarrow$ (3): Consider a covering approximation space $(U; \mathcal{C}_3)$ and $X \subset U$, where $U = \{a, b, c\}$, $\mathcal{C} = \{\{a, b\}, \{b, c\}\}$, $X = \{a, b\}$. It is not difficult to check that $\underline{\mathcal{C}_3}(X) = X$ and $\overline{\mathcal{C}_3}(X) = U \neq X$. So X is an inner definable subset of $(U; \mathcal{C}_3)$ and is not an outer definable subset of $(U; \mathcal{C}_3)$.

Theorem 3.4: Let $(U; \mathcal{C}_4)$ be a covering approximation space and $X \subset U$. Then the following are equivalent.

(1) X is a definable subset of $(U, \mathcal{C}_4)$.

(2) X is an inner definable subset of $(U, \mathcal{C}_4)$.

(3) X is an outer definable subset of $(U, \mathcal{C}_4)$.

Proof. (1) $\Longrightarrow$ (2): Let X be a definable subset of $(U, \mathcal{C}_4)$. By Proposition 2.7, X is an inner definable subset of $(U, \mathcal{C}_4)$.

(2) $\Longrightarrow$ (3): Let X be an inner definable subset of $(U, \mathcal{C}_4)$. Then $\underline{\mathcal{C}_4}(X) = X$, and so $X - \underline{\mathcal{C}_4}(X) = \emptyset$. It follows that $\overline{\mathcal{C}_4}(X) = \underline{\mathcal{C}_4}(X) \bigcup (\bigcup\{K : K \in \mathcal{C} \bigwedge K \bigcap (X - \underline{\mathcal{C}_4}(X)) \neq \emptyset\}) = \underline{\mathcal{C}_4}(X) = X$. So X is an outer definable subset of $(U, \mathcal{C}_4)$.

(3) $\Longrightarrow$ (1): Let X be an outer definable subset of $(U, \mathcal{C}_4)$. Then $\overline{\mathcal{C}_4}(X) = X$. By Proposition 2.7, it suffices to prove that $\underline{\mathcal{C}_4}(X) = X$. Since $\underline{\mathcal{C}_4}(X) \subset X$ by Lemma 2.6, we only need to prove that $X \subset \underline{\mathcal{C}_4}(X)$. Assume that $X \not\subset \underline{\mathcal{C}_4}(X)$. Then there is $x \in X - \underline{\mathcal{C}_4}(X))$, i.e., $x \in X = \overline{\mathcal{C}_4}(X) = \underline{\mathcal{C}_4}(X) \bigcup (\bigcup\{K : K \in \mathcal{C} \bigwedge K \bigcap (X - \underline{\mathcal{C}_4}(X)) \neq \emptyset\})$ and $x \notin \underline{\mathcal{C}_4}(X)$. So there is $K \in \mathcal{C}$ such that $x \in K$ and $K \bigcap (X - \underline{\mathcal{C}_4}(X)) \neq \emptyset$. So $K \subset \overline{\mathcal{C}_4}(X) = X$. It follows that $x \in K \subset \underline{\mathcal{C}_4}(X)$. This contradicts that $x \notin \underline{\mathcal{C}_4}(X)$.

Theorem 3.5: Let $(U; \mathcal{C}_5)$ be a covering approximation space and $X \subset U$. Consider the following conditions.

(1) X is a definable subset of $(U, \mathcal{C}_5)$.

(2) X is an inner definable subset of $(U, \mathcal{C}_5)$.

(3) X is an outer definable subset of $(U, \mathcal{C}_5)$.

Then (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) $\not\Longrightarrow$ (2).

Proof. (1) $\Longrightarrow$ (2): Let X be a definable subset of $(U, \mathcal{C}_5)$. By Proposition 2.7, X is an inner definable subset of $(U, \mathcal{C}_5)$.

(2) $\Longrightarrow$ (3): Let X be an inner definable subset of $(U; \mathcal{C}_5)$ Then $\underline{\mathcal{C}_5}(X) = X$, and so $X - \underline{\mathcal{C}_5}(X) = \emptyset$. It follows that $\overline{\mathcal{C}_5}(X) = \underline{\mathcal{C}_5}(X) \bigcup (\bigcup\{N(x) : x \in X - \underline{\mathcal{C}_5}(X)\}) = \underline{\mathcal{C}_5}(X) = X$. So X is an outer definable subset of $(U, \mathcal{C}_5)$.

(2) $\Longrightarrow$ (1): Let X be an inner definable subset of $(U; \mathcal{C}_5)$. By (2) $\Longrightarrow$ (3), X is an outer definable subset of $(U, \mathcal{C}_5)$. By Proposition 2.7, X is a definable subset of $(U, \mathcal{C}_5)$.

(3) $\not\Longrightarrow$ (2): Consider a covering approximation space $(U; \mathcal{C}_5)$ and $X \subset U$, where $U = \{a, b, c\}$, $\mathcal{C} = \{\{a, b\}, \{b, c\}, \{c, a\}\}$, $X = \{a\}$. It is not difficult to check that $\overline{\mathcal{C}_3}(X) = X$ and $\underline{\mathcal{C}_5}(X) = \emptyset \neq X$. So X is an outer definable subset of $(U; \mathcal{C}_5)$ and is not an inner definable subset of $(U; \mathcal{C}_5)$.

Theorem 3.6: Let $(U; \mathcal{C}_6)$ be a covering approximation space and $X \subset U$. Consider the following conditions.

(1) X is a definable subset of $(U, \mathcal{C}_6)$.

(2) X is an inner definable subset of $(U, \mathcal{C}_6)$.

(3) X is an outer definable subset of $(U, \mathcal{C}_6)$.

Then (1) $\Longleftrightarrow$ both (2) and (3); (2) $\not\Longrightarrow$ (3); (3) $\not\Longrightarrow$ (2).

Proof. (1) $\Longleftrightarrow$ both (2) and (3): It holds By Proposition 2.7.

(2) $\not\Longrightarrow$ (3): Consider a covering approximation space $(U; \mathcal{C}_6)$ and $X \subset U$, where $U = \{a, b\}$, $\mathcal{C} = \{\{a\}, \{a, b\}\}$, $X = \{a\}$. It is not difficult to check that $\underline{\mathcal{C}_6}(X) = X$ and $\overline{\mathcal{C}_6}(X) = \{a, b\} \neq X$. So X is an inner definable subset of $(U; \mathcal{C}_6)$ and is not an outer definable subset of $(U; \mathcal{C}_6)$.

(3) $\not\Longrightarrow$ (2): Consider a covering approximation space $(U; \mathcal{C}_6)$ and $X \subset U$, where $U = \{a, b\}$, $\mathcal{C} = \{\{a\}, \{a, b\}\}$, $X = \{b\}$. It is not difficult to check that $\overline{\mathcal{C}_6}(X) = X$ and $\underline{\mathcal{C}_6}(X) = \emptyset \neq X$. So X is an outer definable subset of $(U; \mathcal{C}_6)$ and is not an inner definable subset of $(U; \mathcal{C}_6)$.

Theorem 3.7: Let $(U; \mathcal{C}_7)$ be a covering approximation space and $X \subset U$. Then the following are equivalent.

(1) X is a definable subset of $(U, \mathcal{C}_7)$.

(2) X is an inner definable subset of $(U, \mathcal{C}_7)$.

(3) X is an outer definable subset of $(U, \mathcal{C}_7)$.

Proof. (1) $\Longrightarrow$ (2): Let X be a definable subset of $(U, \mathcal{C}_7)$. By Proposition 2.7, X is an inner definable subset of $(U, \mathcal{C}_7)$.

(2) $\Longrightarrow$ (3): Let X be an inner definable subset of $(U, \mathcal{C}_7)$. Then $\underline{\mathcal{C}_7}(X) = X$. Since $X \subset \overline{\mathcal{C}_7}(X)$ by Lemma 2.6, It suffices to prove that $\overline{\mathcal{C}_7}(X) \subset X$. Let $y \in \overline{\mathcal{C}_7}(X)$. Then there is $K \in \mathcal{C}$ such that $x \in K$ and $K \bigcap X \neq \emptyset$. Pick $z \in K \bigcap X$, then $z \in X = \underline{\mathcal{C}_7}(X) = \{x \in U : \forall K \in \mathcal{C}(x \in K \Longrightarrow K \subset X)\}$. Since $z \in K$, $K \subset X$, and hence $y \in K \subset X$. This proves that $\overline{\mathcal{C}_7}(X) \subset X$.

(3) $\Longrightarrow$ (1): Let X be an outer definable subset of $(U, \mathcal{C}_7)$. Then $\overline{\mathcal{C}_7}(X) = X$. By Proposition 2.7, it suffices to prove that $\underline{\mathcal{C}_7}(X) = X$. Since $\underline{\mathcal{C}_7}(X) \subset X$ by Lemma 2.6, we only need to prove that $X \subset \underline{\mathcal{C}_7}(X)$. Let $x \in X$. For each $K \in \mathcal{C}$, if $x \in K$, then $x \in K \bigcap X \neq \emptyset$, and so $K \subset \overline{\mathcal{C}_7}(X) = X$. This proves that $x \in \underline{\mathcal{C}_7}(X)$. Consequently, $X \subset \underline{\mathcal{C}_7}(X)$.

Theorem 3.8: Let $(U; \mathcal{C}_8)$ be a covering approximation space and $X \subset U$. Consider the following conditions.

(1) X is a definable subset of $(U, \mathcal{C}_8)$.

(2) X is an inner definable subset of $(U, \mathcal{C}_8)$.

(3) X is an outer definable subset of $(U, \mathcal{C}_8)$.

Then (1) $\Longleftrightarrow$ both (2) and (3); (2) $\not\Longrightarrow$ (3); (3) $\not\Longrightarrow$ (2).

Proof. (1) $\Longleftrightarrow$ both (2) and (3): It holds By Proposition 2.7.

(2) $\not\Longrightarrow$ (3): Consider a covering approximation space $(U; \mathcal{C}_8)$ and $X \subset U$, where $U = \{a, b, c\}$, $\mathcal{C} = \{\{a, b\}, \{b\}, \{b, c\}\}$, $X = \{b\}$. It is not difficult to check that $\underline{\mathcal{C}_8}(X) = X$ and $\overline{\mathcal{C}_8}(X) = U \neq X$. So X is an inner definable subset of $(U; \mathcal{C}_8)$ and is not an outer definable subset of $(U; \mathcal{C}_8)$.

(3) $\not\Longrightarrow$ (2): Consider a covering approximation space $(U; \mathcal{C}_8)$ and $X \subset U$, where $U = \{a, b, c\}$, $\mathcal{C} = \{\{a, b\}, \{b\}, \{b, c\}\}$, $X = \{a, c\}$. It is not difficult to check that $\overline{\mathcal{C}_8}(X) = X$ and $\underline{\mathcal{C}_8}(X) = \emptyset \neq X$. So X is an outer definable subset of $(U; \mathcal{C}_8)$ and is not an inner definable subset of $(U; \mathcal{C}_8)$.

Theorem 3.9: Let $(U; \mathcal{C}_9)$ be a covering approximation space and $X \subset U$. Then the following are equivalent.

(1) X is a definable subset of $(U, \mathcal{C}_9)$.

(2) X is an inner definable subset of $(U, \mathcal{C}_9)$.

(3) X is an outer definable subset of $(U, \mathcal{C}_9)$.

Proof. (1) $\Longrightarrow$ (2): Let X be a definable subset of $(U, \mathcal{C}_9)$. By Proposition 2.7, X is an inner definable subset of $(U, \mathcal{C}_9)$.

(2) $\Longrightarrow$ (3): Let X be an inner definable subset of $(U, \mathcal{C}_9)$. Then $\underline{\mathcal{C}_9}(X) = X$. Let $y \in \overline{\mathcal{C}_9}(X) = \bigcup\{N(x) : x \in U \bigwedge N(x) \bigcap X \neq \emptyset\}$. Then there is $z \in U$ such that $y \in N(z)$ and $N(z) \bigcap X \neq \emptyset$. Pick $v \in N(z) \bigcap X$, then $v \in X = \underline{\mathcal{C}_9}(X) = \{x \in U : \forall u(x \in N(u) \Longrightarrow N(u) \subset X)\}$. Since $v \in N(z)$, $N(z) \subset X$. So $y \in N(z) \subset X$. This proves that $\overline{\mathcal{C}_9}(X) \subset X$. By Lemma 2.6, $X \subset \overline{\mathcal{C}_9}(X)$. It follows that $\overline{\mathcal{C}_9}(X) = X$. So X is an outer definable subset of $(U, \mathcal{C}_9)$.

(3) $\Longrightarrow$ (1): Let X be an outer definable subset of $(U, \mathcal{C}_9)$. Then $\overline{\mathcal{C}_9}(X) = X$. By Proposition 2.7, it suffices to prove that $\underline{\mathcal{C}_9}(X) = X$. Since $\underline{\mathcal{C}_9}(X) \subset X$ by Lemma 2.6, we only need to prove that $X \subset \underline{\mathcal{C}_9}(X)$. Assume that $X \not\subset \underline{\mathcal{C}_9}(X)$. Then there is $y \in X$ and $y \notin \underline{\mathcal{C}_9}(X) = \{x \in U : \forall u(x \in N(u) \Longrightarrow N(u) \subset X)\}$. So there is $v \in U$ such that $y \in N(v) \not\subset X$. Pick $z \in N(v)$ such that $z \notin X$. Note that $y \in N(v) \bigcap X$, so

$N(v) \bigcap X \neq \emptyset$. Thus $z \in \bigcup\{N(x) : x \in U \bigwedge N(x) \bigcap X \neq \emptyset\} = \overline{\mathcal{C}_9}(X) = X$. This contradicts that $z \notin X$.

Theorem 3.10: Let $(U;\mathcal{C}_{10})$ be a covering approximation space and $X \subset U$. Consider the following conditions.

(1) X is a definable subset of $(U, \mathcal{C}_{10})$.

(2) X is an inner definable subset of $(U, \mathcal{C}_{10})$.

(3) X is an outer definable subset of $(U, \mathcal{C}_{10})$.

Then (1) $\Longleftrightarrow$ both (2) and (3); (2) $\not\Longrightarrow$ (3); (3) $\not\Longrightarrow$ (2).

Proof. (1) $\Longleftrightarrow$ both (2) and (3): It holds By Proposition 2.7.

(2) $\not\Longrightarrow$ (3): Consider a covering approximation space $(U;\mathcal{C}_{10})$ and $X \subset U$, where $U = \{a,b,c\}$, $\mathcal{C} = \{\{a,b\},\{b,c\}\}$, $X = \{a,c\}$. It is not difficult to check that $\underline{\mathcal{C}_{10}}(X) = X$ and $\overline{\mathcal{C}_{10}}(X) = U \neq X$. So X is an inner definable subset of $(U;\mathcal{C}_{10})$ and is not an outer definable subset of $(U;\mathcal{C}_{10})$.

(3) $\not\Longrightarrow$ (2): Consider a covering approximation space $(U;\mathcal{C}_{10})$ and $X \subset U$, where $U = \{a,b,c\}$, $\mathcal{C} = \{\{a,b\},\{b,c\}\}$, $X = \{a,b\}$. It is not difficult to check that $\overline{\mathcal{C}_{10}}(X) = X$ and $\underline{\mathcal{C}_{10}}(X) = \{a\} \neq X$. So X is an outer definable subset of $(U;\mathcal{C}_{10})$ and is not an inner definable subset of $(U;\mathcal{C}_{10})$.

IV. Postscript

In [15], the following covering approximation operators are given for covering approximation spaces.

Definition 4.1: Let $(U;\mathcal{C})$ be a covering approximation space and $X \subset U$. Put

$\underline{\mathcal{D}}(X) = \{x \in U : \exists u(u \in N(x) \bigwedge N(u) \subset X)\}$;

$\overline{\mathcal{D}}(X) = \{x \in U : \forall u(u \in N(x) \Longrightarrow N(u) \bigcap X \neq \emptyset)\}$.

$\underline{\mathcal{D}}$ and $\overline{\mathcal{D}}$ are called lower covering approximation operator and upper covering approximation operator on $(U;\mathcal{C})$ respectively.

We use $(U;\mathcal{D})$ to denote covering approximation space $(U;\mathcal{C})$ with covering approximation operator $\underline{\mathcal{D}}$ and $\overline{\mathcal{D}}$.

Definition 4.2: Let $(U;\mathcal{D})$ be a covering approximation space.

(1) X is called a definable subset of $(U;\mathcal{D})$ if $\underline{\mathcal{D}}(X) = \overline{\mathcal{D}}(X)$.

(2) X is called an inner definable subset of $(U;\mathcal{D})$ if $\underline{\mathcal{D}}(X) = X$.

(3) X is called an outer definable subset of $(U;\mathcal{D})$ if $\overline{\mathcal{D}}(X) = X$.

It is clear that the following proposition holds.

Proposition 4.3: If X is a both inner definable and outer definable subset of covering approximation space $(U;\mathcal{D})$, then X is a definable subset of $(U;\mathcal{D})$.

However, a definable subset of covering approximation space $(U;\mathcal{D})$ need not to be an inner definable or outer definable subset of $(U;\mathcal{D})$. In fact, we have the following example.

Example 4.4: Let $U = \{a,b\}$, $\mathcal{C} = \{\{a\},\{a,b\}\}$, $X = \{a\}$. It is not difficult to check that $\underline{\mathcal{D}}(X) = \overline{\mathcal{D}}(X) = U$. So $\underline{\mathcal{D}}(X) \neq X$ and $\overline{\mathcal{D}}(X) \neq X$. It follows that X is a definable subset of $(U;\mathcal{D})$, but X is a neither inner definable nor outer definable subset of $(U;\mathcal{D})$.

Just as above example, for a covering approximation space $(U;\mathcal{D})$ and $X \subset U$, the following implication does not hold in general.

$$X \subset U \Longrightarrow \underline{\mathcal{D}}(X) \subset X \subset \overline{\mathcal{D}}(X).$$

This make it interesting to investigate definable subsets of $(U;\mathcal{D})$. The following question is still open, which is worthy to be considered in subsequent research.

Question 4.5: Let $(U;\mathcal{D})$ be a covering approximation space. Whether there are some relations among definable subsets, inner definable subsets and outer definable subsets of $(U;\mathcal{D})$?

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