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On Cross-Ratio in some Moufang-Klingenberg Planes

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***Abstract*—In this paper we are interested in Moufang-Klingenberg planes $\mathbf{M}(\mathcal{A})$ defined over a local alternative ring $\mathcal{A}$ of dual numbers. We show that a collineation of $\mathbf{M}(\mathcal{A})$ preserve cross-ratio. Also, we obtain some results about harmonic points.

***Keywords*—Moufang-Klingenberg planes, local alternative ring, projective collineation, cross-ratio, harmonic points.

I. Introduction

In the Euclidean plane, Desargues established the fundemental fact that cross-ratio (a concept originally introduced by Pappus of Alexandria c.300 B.C) is invariant under projection [4, p. 133]. For this reason, cross-ratio is one of the most important concepts of projective geometry.

In this paper we deal with the class (which we will denote by $\mathbf{M}(\mathcal{A})$) of Moufang-Klingenberg (MK) planes coordinatized by a local alternative ring

$$\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$$

(an alternative field $\mathbf{A}$, $\varepsilon \notin \mathbf{A}$ and $\varepsilon^2 = 0$) introduced by Blunck in [8]. We will show that a collineation of $\mathbf{M}(\mathcal{A})$ given in [2] preserves cross-ratio. Moreover, we will obtain some results related to harmonic points. For more information about some well-known properties of cross-ratio in the case of Moufang planes or MK-planes $\mathbf{M}(\mathcal{A})$, respectively, it can be seen the papers of [10], [5], [9] or [8], [1].

The paper is organized as follows: Section 2 includes some basic definitions and results from the literature. In Section 3 we will give a collineation of $\mathbf{M}(\mathcal{A})$ from [2] and we show that this collineation preserves cross-ratio. Finally, we obtain some results on harmonic points.

II. Preliminaries

Let $\mathbf{M} = (\mathbf{P}, \mathbf{L}, \in, \sim)$ consist of an incidence structure $(\mathbf{P}, \mathbf{L}, \in)$ (points, lines, incidence) and an equivalence relation '$\sim$' (neighbour relation) on $\mathbf{P}$ and on $\mathbf{L}$, respectively. Then $\mathbf{M}$ is called a *projective Klingenberg plane* (PK-plane), if it satisfies the following axioms:

(PK1) If P, Q are non-neighbour points, then there is a unique line PQ through P and Q.

(PK2) If g, h are non-neighbour lines, then there is a unique point $g \cap h$ on both g and h.

(PK3) There is a projective plane $\mathbf{M}^* = (\mathbf{P}^*, \mathbf{L}^*, \in)$ and an incidence structure epimorphism $\Psi : \mathbf{M} \to \mathbf{M}^*$, such that the conditions

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim Q,\ \Psi(g) = \Psi(h) \Leftrightarrow g \sim h$$

hold for all $P, Q \in \mathbf{P}$, $g, h \in \mathbf{L}$.

A point $P \in \mathbf{P}$ is called *near* a line $g \in \mathbf{L}$ iff there exists a line $h \sim g$ such that $P \in h$.

Let $h, k \in \mathbf{L}$, $C \in \mathbf{P}$, C is not symmetric to h and k. Then the well-defined bijection

$$\sigma := \sigma_C(k, h) : \begin{cases} h \to k \\ X \to XC \cap k \end{cases}$$

mapping h to k is called a *perspectivity* from h to k with center C. A product of a finite number of perspectivities is called a *projectivity*.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a *collineation* of $\mathbf{M}$.

A *Moufang-Klingenberg plane* (MK-plane) is a PK-plane $\mathbf{M}$ that generalizes a Moufang plane, and for which $\mathbf{M}^*$ is a Moufang plane (for the exact definition see [3]).

An *alternative ring (field)* $\mathbf{R}$ is a not necessarily associative ring (field) that satisfies the alternative laws

$$a(ab) = a^2 b, (ba)a = ba^2, \forall a, b \in \mathbf{R}.$$

An alternative ring $\mathbf{R}$ with identity element 1 is called *local* if the set $\mathbf{I}$ of its non-unit elements is an ideal.

We are now ready to give consecutively two important lemmas related to alternative rings.

Lemma 2.1: The subring generated by any two elements of an alternative ring is associative (cf. [12, Theorem 3.1]).

Lemma 2.2: The identities

$$x(y(xz)) = (xyx)z$$
$$((yx)z)x = y(xzx)$$
$$(xy)(zx) = x(yz)x$$

which are known as Moufang identities are satisfied in every alternative ring (cf. [11, p. 160]).

We summarize some basic concepts about the coordinatization of MK-planes from [3].

Let $\mathbf{R}$ be a local alternative ring. Then $\mathbf{M}(\mathbf{R}) = (\mathbf{P}, \mathbf{L}, \in, \sim)$ is the incidence structure with neighbour relation defined

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as follows:

$$
\begin{aligned}
\mathbf{P} &= \{(x,y,1):\ x,y\in\mathbf{R}\}\\
&\quad\cup\{(1,y,z): y\in\mathbf{R},\ z\in\mathbf{I}\}\\
&\quad\cup\{(w,1,z)|: w,z\in\mathbf{I}\},\\
\mathbf{L} &= \{[m,1,p]:\ m,p\in\mathbf{R}\}\\
&\quad\cup\{[1,n,p]: p\in\mathbf{R},\ n\in\mathbf{I}\}\\
&\quad\cup\{[q,n,1]: q,n\in\mathbf{I}\}\\
[m,1,p] &= \{(x,xm+p,1): x\in\mathbf{R}\}\\
&\quad\cup\{(1,zp+m,z): z\in\mathbf{I}\}\\
[1,n,p] &= \{(yn+p,y,1): y\in\mathbf{R}\}\\
&\quad\cup\{(zp+n,1,z): z\in\mathbf{I}\}\\
[q,n,1] &= \{(1,y,yn+q): y\in\mathbf{R}\}\\
&\quad\cup\{(w,1,wq+n): w\in\mathbf{I}\}
\end{aligned}
$$

and

$$
\begin{aligned}
&P=(x_1,x_2,x_3)\sim(y_1,y_2,y_3)=Q\Leftrightarrow\\
&x_i-y_i\in\mathbf{I}\ (i=1,2,3)),\forall P,Q\in\mathbf{P}\\
&g=[x_1,x_2,x_3]\sim[y_1,y_2,y_3]=h\Leftrightarrow\\
&x_i-y_i\in\mathbf{I}\ (i=1,2,3)),\forall g,h\in\mathbf{L}.
\end{aligned}
$$

Now it is time to give the following theorem from [3].

Theorem 2.1: $\mathbf{M}(\mathbf{R})$ is an MK-plane, and each MK-plane is isomorphic to some $\mathbf{M}(\mathbf{R})$.

Let $\mathbf{A}$ be an alternative field and $\varepsilon\notin\mathbf{A}$. Consider $\mathcal{A}:=\mathbf{A}(\varepsilon)=\mathbf{A}+\mathbf{A}\varepsilon$ with componentwise addition and multiplication as follows:

$$(a_1+a_2\varepsilon)(b_1+b_2\varepsilon)=a_1b_1+(a_1b_2+a_2b_1)\varepsilon,$$

where $a_i,b_i\in\mathbf{A}$ for $i=1,2$. Then $\mathcal{A}$ is a local alternative ring with ideal $\mathbf{I}=\mathbf{A}\varepsilon$ of non-units. The set of formal inverses of the non-units of $\mathcal{A}$ is denoted as $\mathbf{I}^{-1}$. Calculations with the elements of $\mathbf{I}^{-1}$ are defined as follows [7]:

$$
\begin{aligned}
(a\varepsilon)^{-1}+t &:= (a\varepsilon)^{-1}:=t+(a\varepsilon)^{-1}\\
q(a\varepsilon)^{-1} &:= \left(aq^{-1}\varepsilon\right)^{-1}\\
(a\varepsilon)^{-1}q &:= \left(q^{-1}a\varepsilon\right)^{-1}\\
\left((a\varepsilon)^{-1}\right)^{-1} &:= a\varepsilon,
\end{aligned}
$$

where $(a\varepsilon)^{-1}\in\mathbf{I}^{-1},\ t\in\mathcal{A},\ q\in\mathcal{A}\setminus\mathbf{I}$. (Other terms are not defined.). For more information about $\mathcal{A}$ and its relation to MK-planes, the reader is referred to the papers of Blunck [7], [8]. In [8], the centre $\mathbf{Z}(\mathcal{A})$ is defined to be the (commutative, associative) subring of $\mathcal{A}$ which is commuting and associating with all elements of $\mathcal{A}$. It is $\mathbf{Z}(\mathcal{A}):=\mathbf{Z}(\varepsilon)=\mathbf{Z}+\mathbf{Z}\varepsilon$, where $\mathbf{Z}=\{z\in\mathbf{A}: za=az,\ \forall a\in\mathbf{A}\}$ is the centre of $\mathbf{A}$. If $\mathbf{A}$ is not associative, then $\mathbf{A}$ is a Cayley division algebra over its centre $\mathbf{Z}$.

Throughout this paper we assume $char\mathbf{A}\neq\mathbf{2}$ and we restrict ourselves to the MK-planes $\mathbf{M}(\mathcal{A})$

Blunck [8] gives the following algebraic definition of the cross-ratio for the points on the line $g:=[1,0,0]$ in $\mathbf{M}(\mathcal{A})$.

$$
\begin{aligned}
&(A,B;C,D):=(a,b;c,d)\\
&=<\left((a-d)^{-1}(b-d)\right)\left((b-c)^{-1}(a-c)\right)>\\
&(Z,B;C,D):=\left(z^{-1},b;c,d\right)\\
&=<\left((1-dz)^{-1}(b-d)\right)\left((b-c)^{-1}(1-cz)\right)>\\
&(A,Z;C,D):=\left(a,z^{-1};c,d\right)\\
&=<\left((a-d)^{-1}(1-dz)\right)\left((1-cz)^{-1}(a-c)\right)>\\
&(A,B;Z,D):=\left(a,b;z^{-1},d\right)\\
&=<\left((a-d)^{-1}(b-d)\right)\left((1-zb)^{-1}(1-za)\right)>\\
&(A,B;C,Z):=\left(a,b;c,z^{-1}\right)\\
&=<\left((1-za)^{-1}(1-zb)\right)\left((b-c)^{-1}(a-c)\right)>,
\end{aligned}
$$

where $A=(0,a,1),B=(0,b,1),C=(0,c,1),D=(0,d,1),Z=(0,1,z)$ are pairwise non-neighbour points of g and $<x>=\{y^{-1}xy:\ y\in\mathcal{A}\}$.

In [7, Theorem 2], it is shown that the transformations

$$
\begin{aligned}
t_u(x) &= x+u; u\in\mathcal{A}\\
r_u(x) &= xu; u\in\mathcal{A}\setminus\mathbf{I}\\
i(x) &= x^{-1}\\
l_u(x) &= ux=\left(ir_u^{-1}i\right)(x);\ u\in\mathcal{A}\setminus\mathbf{I}
\end{aligned}
$$

which are defined on the line g preserve cross-ratios. In [6, Corollary (iii)], it is also shown that the group generated by these transformations, which is denoted by Λ, equals to the group of projectivities of a line in $\mathbf{M}(\mathcal{A})$. The elements preserving cross-ratio of the group Λ defined on g will act a very important role in the proof of Theorem 3.1.

We give the following result from [1, Theorem 8]. This result states a simple way for calculation of the cross-ratio of the points on any line in $\mathbf{M}(\mathcal{A})$.

Theorem 2.2: Let $\{O,U,V,E\}$ be the basis of $\mathbf{M}(\mathcal{A})$ where $O=(0,0,1),U=(1,0,0),V=(0,1,0),E=(1,1,1)$ (see [3, Section 4]). Then, according to types of lines, the cross-ratio of the points on the line l can be calculated as follows:

If $A,B,C,D\ Z$ are the pairwise non-neighbour points

(a) of the line $l=[m,1,k]$, where $A=(a,am+k,1),B=(b,bm+k,1),C=(c,cm+k,1),D=(d,dm+k,1)$ are not near to the line $UV=[0,0,1]$ and $Z=(1,m+zp,z)$ is near to UV;
(b) of the line $l=[1,n,p]$, where $A=(an+p,a,1),B=(bn+p,b,1),C=(cn+p,c,1),D=(dn+p,d,1)$ are not neighbour to V and $Z=(n+zp,1,z)\sim V$;
(c) of the line $l=[q,n,1]$, where $A=(1,a,q+an),B=(1,b,q+bn),C=(1,c,q+cn),D=(1,d,q+dn)$ are not neighbour to V and $Z=(z,1,zq+n)\sim V$;

then

$$\begin{aligned}(A,B;C,D) &= (a,b;c,d)\\ (Z,B;C,D) &= \left(z^{-1},b;c,d\right)\\ (A,Z;C,D) &= \left(a,z^{-1};c,d\right)\\ (A,B;Z,D) &= \left(a,b;z^{-1},d\right)\\ (A,B;C,Z) &= \left(a,b;c,z^{-1}\right).\end{aligned}$$

We can give an important theorem, from [1, Theorem 9], about cross-ratio.

Theorem 2.3: In $\mathbf{M}(\mathcal{A})$, perspectivities preserve cross-ratios.

Now we give a definition in $\mathbf{M}(\mathcal{A})$, well known from the case of Moufang planes [10]. In $\mathbf{M}(\mathcal{A})$, any pairwise non-neighbour four points $A,B,C,D \in l$ are called as *harmonic* if $(A,B;C,D) =< -1 >$ and we let $h(A,B,C,D)$ represent the statement: A,B,C,D are harmonic.

III. On Cross-Ratio in $\mathbf{M}(\mathcal{A})$.

In this section we will give a collineation of $\mathbf{M}(\mathcal{A})$, from [2]. Next, we show that the collineation preserve cross-ratios. Now we start with giving the collineation of $\mathbf{M}(\mathcal{A})$, where $w,z,q,n \in \mathbf{A}$:For any $s \notin \mathbf{I}$, the map J_s transforms points and lines as follows:

$$\begin{aligned}(x,y,1) &\rightarrow \left(ys^{-1},xs,1\right)\\ (1,y,z\varepsilon) &\rightarrow \left(1,sy^{-1}s,s(y^{-1}z)\right)\ \ if\ y\notin \mathbf{I}\\ (1,y,z\varepsilon) &\rightarrow \left(s^{-1}ys^{-1},1,s^{-1}z\right)\ \ if\ y\in \mathbf{I}\\ (w\varepsilon,1,z\varepsilon) &\rightarrow (1,sws,sz)\end{aligned}$$

and

$$\begin{aligned}[m,1,k] &\rightarrow \left[sm^{-1}s,1,-\left(km^{-1}\right)s\right]\ \ if\ m\notin \mathbf{I}\\ [m,1,k] &\rightarrow \left[1,s^{-1}ms^{-1},ks^{-1}\right]\ \ if\ m\in \mathbf{I}\\ [1,n\varepsilon,p] &\rightarrow [sns,1,ps]\\ [q\varepsilon,n\varepsilon,1] &\rightarrow \left[sn,s^{-1}q,1\right].\end{aligned}$$

Now we are ready to give the following

Theorem 3.1: The collineation J_s preserve cross-ratio.

Proof: Let A,B,C,D and Z be the points with the property given in the statement of Theorem 2.2. Then, it is obvious that

$$\begin{aligned}(A,B;C,D) &= (a,b;c,d)\\ (Z,B;C,D) &= \left(z^{-1},b;c,d\right)\\ (A,Z;C,D) &= \left(a,z^{-1};c,d\right)\\ (A,B;Z,D) &= \left(a,b;z^{-1},d\right)\\ (A,B;C,Z) &= \left(a,b;c,z^{-1}\right),\end{aligned}\tag{1}$$

where $z\in\mathbf{I}$. In this case we must find the effect of φ to the points of any line where φ is the collineations J_s.

Let $\varphi = \mathrm{J}_s$. If $l=[m,1,k]$, then

$$\begin{aligned}\varphi(X) &= \varphi(x,xm+k,1)\\ &= ((xm+k)s^{-1},xs,1)\\ \varphi(Z) &= \varphi(1,m+zk,z)\\ &= (1,s(m+zk)^{-1}s,s((m+zk)^{-1}z))\\ &\quad for\ m+zk\notin\mathbf{I}\\ \varphi(Z) &= \varphi(1,m+zk,z)\\ &= (s^{-1}(m+zk)s^{-1},1,s^{-1}z),\\ &\quad for\ m+zk\in\mathbf{I}\\ \varphi(l) &= \left[sm^{-1}s,1,-\left(km^{-1}\right)s\right]\ \ for\ m\notin\mathbf{I}\\ \varphi(l) &= \left[1,s^{-1}ms^{-1},ks^{-1}\right]\ \ for\ m\in\mathbf{I}.\end{aligned}$$

In this case, from (a) of Theorem 2.2, the cross-ratio of the points of $[sm^{-1}s,1,-\left(km^{-1}\right)s]$ is as follows:

$$\begin{aligned}&(\varphi(A),\varphi(B);\varphi(C),\varphi(D))\\ &=((am+k)s^{-1},(bm+k)s^{-1};\\ &(cm+k)s^{-1},(dm+k)s^{-1})\\ &=^{\sigma}(a,b;c,d)\\ &(\varphi(Z),\varphi(B);\varphi(C),\varphi(D))\\ =\ &(\left(s\left((m+zk)^{-1}z\right)\right)^{-1},(bm+k)s^{-1};\\ &(cm+k)s^{-1},(dm+k)s^{-1})\\ =\ &^{\sigma}\left(z^{-1},b;c,d\right),\end{aligned}$$

where $\sigma = r_{m^{-1}}\circ t_{-k}\circ r_s\in\Lambda$. From (b) of Theorem 2.2, the cross-ratio of the points of $\left[1,s^{-1}ms^{-1},ks^{-1}\right]$ is as follows:

$$\begin{aligned}&(\varphi(A),\varphi(B);\varphi(C),\varphi(D))\\ &=(as,bs;cs,ds)=^{\sigma}(a,b;c,d)\\ &(\varphi(Z),\varphi(B);\varphi(C),\varphi(D))\\ &=(z^{-1}s,bs;cs,ds)=^{\sigma}\left(z^{-1},b;c,d\right),\end{aligned}$$

where $\sigma = r_{s^{-1}}\in\Lambda$.

If $l=[1,n,p]$, then

$$\begin{aligned}\varphi(X)&=\varphi(xn+p,x,1)=(xs^{-1},(xn+p)s,1)\\ \varphi(Z)&=\varphi(n+zp,1,z)=(1,s(n+zp)s,sz)\end{aligned}$$

and

$$\varphi(l)=[sns,1,ps].$$

In this case, from (a) of Theorem 2.2, the cross-ratio of the points of $[sns,1,ps]$ is as follows:

$$\begin{aligned}&(\varphi(A),\varphi(B);\varphi(C),\varphi(D))\\ &=(as^{-1},bs^{-1};cs^{-1},ds^{-1})=^{\sigma}(a,b;c,d)\\ &(\varphi(Z),\varphi(B);\varphi(C),\varphi(D))\\ &=(z^{-1}s^{-1},bs^{-1};cs^{-1},ds^{-1})=^{\sigma}\left(z^{-1},b;c,d\right),\end{aligned}$$

where $\sigma = r_s\in\Lambda$. If $l=[q,n,1]$, then

$$\begin{aligned}\varphi(X) &= \varphi(1,x,q+xn)=(1,sx^{-1}s,s(x^{-1}(q+xn)))\\ &\quad for\ x\notin\mathbf{I}\\ \varphi(X) &= \varphi(1,x,q+xn)=(s^{-1}xs^{-1},1,s^{-1}(q+xn))\\ &\quad for\ x\in\mathbf{I}\\ \varphi(Z) &= \varphi(z,1,zq+n)=(1,szs,s(zq+n))\end{aligned}$$

and

$$\varphi(l) = \left[sn, s^{-1}q, 1\right].$$

In this case, from (c) of Theorem 2.2, the cross-ratio of the points of $\left[sn, s^{-1}q, 1\right]$ is as follows:

$$\begin{aligned}
&(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) \\
&= \left(sa^{-1}s, sb^{-1}s; sc^{-1}s, sd^{-1}s\right) \\
&=^{\sigma} (a, b; c, d), \\
&(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) \\
&= \left(szs, sb^{-1}s; sc^{-1}s, sd^{-1}s\right) \\
&=^{\sigma} \left(z^{-1}, b; c, d\right),
\end{aligned}$$

where $\sigma = i \circ l_{s^{-1}} \circ r_{s^{-1}} \in \Lambda$. Consequently, by considering other all cases we get

$$\begin{aligned}
(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (a, b; c, d) \\
(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left(z^{-1}, b; c, d\right) \\
(\varphi(A), \varphi(Z); \varphi(C), \varphi(D)) &= \left(a, z^{-1}; c, d\right) \\
(\varphi(A), \varphi(B); \varphi(Z), \varphi(D)) &= \left(a, b; z^{-1}, d\right) \\
(\varphi(A), \varphi(B); \varphi(C), \varphi(Z)) &= \left(a, b; c, z^{-1}\right)
\end{aligned}$$

for collineation φ. Combining the last result and the result of (1), the proof is completed. ■

Now we are ready to give the other results of the paper. On $\mathcal{A}$ we give the following theorem, an alternate definition of harmonicty and given for an alternative ring $\mathbf{A}$ with $char\mathbf{A} \neq \mathbf{2}$.

Theorem 3.2: Let $a, b, c, d \in \mathcal{A}$. Then $h(a, b, c, d)$ if and only if

1) if $a, b, c, d \in \mathcal{A}$, $2(a-b)^{-1} = (a-c)^{-1} + (a-d)^{-1}$.
2) if $a = z^{-1}$, $2(d-c)^{-1} + (c-b)^{-1} = z \in \mathbf{I}$.
3) if $b = z^{-1}$, $2(c-d)^{-1} + (d-a)^{-1} = z \in \mathbf{I}$.
4) if $c = z^{-1}$, $2(b-a)^{-1} + (d-b)^{-1} = z \in \mathbf{I}$.
5) if $d = z^{-1}$, $2(a-b)^{-1} + (c-a)^{-1} = z \in \mathbf{I}$.

Proof: 1. From the definition of cross-ratio,

$$h(a, b, c, d) = \left((a-d)^{-1}(b-d)\right)\left((b-c)^{-1}(a-c)\right) = -1.$$

By direct computation (with Lemma 2.1),

$$\begin{aligned}
&(a-d)^{-1}(b-d) = -(a-c)^{-1}(b-c) \\
&(a-d)^{-1}(b-a+a-d) = -(a-c)^{-1}(b-a+a-c) \\
&(a-d)^{-1}(b-a) + 1 = -(a-c)^{-1}(b-a) - 1 \\
&2 = -(a-c)^{-1}(b-a) - (a-d)^{-1}(b-a) \\
&2(a-b)^{-1} = (a-c)^{-1} + (a-d)^{-1}.
\end{aligned}$$

2. From the definition of cross-ratio,

$$\begin{aligned}
&h\left(z^{-1}, b, c, d\right) \\
&= \left((1-dz)^{-1}(b-d)\right)\left((b-c)^{-1}(1-cz)\right) = -1.
\end{aligned}$$

By direct computation (Lemma 2.1),

$$\begin{aligned}
&(b-c)^{-1}(1-cz) = -(b-d)^{-1}(1-dz) \\
&(b-c)^{-1}(1-cz) = -(b-d)^{-1}(1-cz+cz-dz) \\
&(b-c)^{-1}(1-cz) = -(b-d)^{-1}(1-cz) \\
&-(b-d)^{-1}((c-d)z) \\
&\left((b-c)^{-1} + (b-d)^{-1}\right)(1-cz) = -(b-d)^{-1}((c-d)z) \\
&(b-c)^{-1} + (b-d)^{-1} = -\left((b-d)^{-1}((c-d)z)\right)(1+cz) \\
&(b-c)^{-1} + (b-d)^{-1} = -(b-d)^{-1}((c-d)z) \\
&(b-d)(b-c)^{-1} + 1 = -(c-d)z \\
&(b-c+c-d)(b-c)^{-1} + 1 = -(c-d)z \\
&2 + (c-d)(b-c)^{-1} = -(c-d)z \\
&2(c-d)^{-1} + (b-c)^{-1} = -z \\
&2(d-c)^{-1} + (c-b)^{-1} = z \in \mathbf{I},
\end{aligned}$$

where $zz = 0$ since $z \in \mathbf{I}$.

3. The proof is same the proof of 2.

4. From the definition of cross-ratio,

$$\begin{aligned}
&h\left(a, b, z^{-1}, d\right) \\
&= \left((a-d)^{-1}(b-d)\right)\left((1-zb)^{-1}(1-za)\right) = -1.
\end{aligned}$$

By direct computation (Lemma 2.1),

$$\begin{aligned}
&(1-zb)^{-1}(1-za) = -(b-d)^{-1}(a-d) \\
&(1+zb)(1-za) = -(b-d)^{-1}(a-b+b-d) \\
&1 + zb - za = -(b-d)^{-1}(a-b) - 1 \\
&2 + z(b-a) = -(b-d)^{-1}(a-b) \\
&2(b-a)^{-1} + z = (b-d)^{-1} \\
&2(b-a)^{-1} + (d-b)^{-1} = z \in \mathbf{I},
\end{aligned}$$

where $(1-zb)^{-1} = 1 + zb$ and $zz = 0$.

5. The proof is same the proof of 4. ■

Now, we give the following theorem, given as without proof in [10] for $\mathbf{A}$.

Theorem 3.3: On $\mathcal{A}$, the followings is valid:

1) $h\left(0, a, 0^{-1}, \frac{a}{2}\right)$
2) $h\left(a, b, 0^{-1}, \frac{a+b}{2}\right)$
3) $h\left(a, -a, 0^{-1}, 0\right)$
4) $h\left(1, -1, a, a^{-1}\right)$
5) $h\left(a^2, 1, a, -a\right)$

Proof: 1. By the definition of cross-ratio, since

$$\left(0, a, 0^{-1}, \frac{a}{2}\right) = \left(0 - \frac{a}{2}\right)^{-1}\left(a - \frac{a}{2}\right) = \frac{-2}{a}\frac{a}{2} = -1,$$

then $h\left(0, a, 0^{-1}, \frac{a}{2}\right)$.

2. By the definition of cross-ratio, since

$$\begin{aligned}
\left(a, b, 0^{-1}, \frac{a+b}{2}\right) &= \left(a - \frac{a+b}{2}\right)^{-1}\left(b - \frac{a+b}{2}\right) \\
&= \left(\frac{a-b}{2}\right)^{-1}\left(\frac{b-a}{2}\right) = -1,
\end{aligned}$$

then $h\left(a, b, 0^{-1}, \frac{a+b}{2}\right)$.

3. By the definition of cross-ratio, since

$$\left(a, -a, 0^{-1}, 0\right) = (a-0)^{-1}(-a-0) = -1,$$

then $h\left(a, -a, 0^{-1}, 0\right)$.

4. By the definition of cross-ratio, since

$$\begin{aligned}
\left(1, -1, a, a^{-1}\right) &= \left(\left(1-a^{-1}\right)^{-1}\left(-1-a^{-1}\right)\right) \\
&\quad \left((-1-a)^{-1}(1-a)\right) \\
&= \left(\left(a^{-1}-1\right)^{-1} - \left(1-a^{-1}\right)^{-1} a^{-1}\right) \\
&\quad \left((-1-a)^{-1} + (1+a)^{-1} a\right) \\
&= \left(\left(a^{-1}-1\right)^{-1} - \left(a\left(1-a^{-1}\right)\right)^{-1}\right) \\
&\quad \left((-1-a)^{-1} + \left(a^{-1}(1+a)\right)^{-1}\right) \\
&= \left(\left(a^{-1}-1\right)^{-1} - (a-1)^{-1}\right) \\
&\quad \left(-(1+a)^{-1} + \left(a^{-1}+1\right)^{-1}\right) \\
&= (a^{-1}-1)^{-1}(a^{-1}+1)^{-1} - (1+a)^{-1} \\
&\quad - (a-1)^{-1}\left(\left(a^{-1}+1\right)^{-1} - (1+a)^{-1}\right) \\
&= \left(a^{-1}-1\right)^{-1}\left(a^{-1}+1\right)^{-1} - \left(a^{-1}-1\right)^{-1} \\
&\quad (1+a)^{-1} - (a-1)^{-1}\left(a^{-1}+1\right)^{-1} \\
&\quad + (a-1)^{-1}(1+a)^{-1} \\
&= \left(\left(a^{-1}+1\right)\left(a^{-1}-1\right)\right)^{-1} \\
&\quad - \left((1+a)\left(a^{-1}-1\right)\right)^{-1} \\
&\quad - \left(\left(a^{-1}+1\right)(a-1)\right)^{-1} \\
&\quad + \left((1+a)(a-1)\right)^{-1} \\
&= \left(a^{-1}a^{-1} - a^{-1} + a^{-1} - 1\right)^{-1} \\
&\quad - \left(a^{-1} - 1 + 1 - a\right)^{-1} \\
&\quad - \left(1 - a^{-1} + a - 1\right)^{-1} + (a - 1 + aa - a)^{-1} \\
&= \left(a^{-1}a^{-1} - 1\right)^{-1} - \left(a^{-1} - a\right)^{-1} \\
&\quad - \left(-a^{-1} + a\right)^{-1} + (-1 + aa)^{-1} \\
&= \left(a^{-1}\left(a^{-1} - a\right)\right)^{-1} - \left(a^{-1} - a\right)^{-1} \\
&\quad + \left(a^{-1} - a\right)^{-1} - \left(a\left(a^{-1} - a\right)\right)^{-1} \\
&= \left(a^{-1} - a\right)^{-1} a - \left(a^{-1} - a\right)^{-1} a^{-1} \\
&= \left(a^{-1} - a\right)^{-1}\left(a - a^{-1}\right) \\
&= -1,
\end{aligned}$$

then $h\left(1, -1, a, a^{-1}\right)$.

5. By the definition of cross-ratio, since

$$\begin{aligned}
\left(a^2, 1, a, -a\right) &= \left(\left(a^2+a\right)^{-1}(1+a)\right)\left((1-a)^{-1}\left(a^2-a\right)\right) \\
&= \left(((a+1)a)^{-1}(1+a)\right)\left((1-a)^{-1}((a-1)a)\right) \\
&= \left(a^{-1}(a+1)^{-1}(1+a)\right)\left((1-a)^{-1}(a-1)a\right) \\
&= a^{-1}(-a) \\
&= -1,
\end{aligned}$$

then $h\left(a^2, 1, a, -a\right)$. ■

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