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On the Wreath Product of Group by Some Other Groups

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***Abstract*—**In this paper, we will generate the wreath product $M_{11} wr M_{12}$ using only two permutations. Also, we will show the structure of some groups containing the wreath product $M_{11} wr M_{12}$. The structure of the groups founded is determined in terms of wreath product $(M_{11} wr M_{12})\, wr\, C_k$. Some related cases are also included. Also, we will show that S_{132K+1} and A_{132K+1} can be generated using the wreath product $(M_{11} wr M_{12})\, wr\, C_k$ and a transposition in S_{132K+1} and an element of order 3 in A_{132K+1}. We will also show that S_{132K+1} and A_{132K+1} can be generated using the wreath product $M_{11} wr M_{12}$ and an element of order $k+1$.

***Keywords*—**Group presentation, group generated by n-cycle, Wreath product, Mathieu group.

I. INTRODUCTION

HAMMAS and Al-Amri [1], have shown that A_{2n+1} of degree $2n+1$ can be generated using a copy of S_n and an element of order 3 in A_{2n+1}. They also gave the symmetric generating set of Groups A_{kn+1} and S_{kn+1} using S_n [5].

Shafee [2] showed that the groups A_{kn+1} and S_{kn+1} can be generated using the wreath product A_m wr S_a and an element of order *k*+1. Also she showed how to generate S_{kn+1} and A_{kn+1} symmetrically using *n* elements each of order *k*+1.

In [3], Shafee and Al-Amri have shown that the groups A_{110k+1} and S_{110k+1} can be generated using the wreath product $M_{11} wr M_{12}$ and an element of order *k*+1.

The Mathieu group M_{11} and M_{12} are two groups of the well known simple groups. In [6], they are fully described. In a matter of fact, they can be faintly presented in different ways. They have presentations in [6] as follows :

$M_{11} =< X, Y, Z \mid X^{11} = Y^5 = (XZ)^3 = 1, X^Y = X^4 = Y^Z = Y^2 >$

$M_{12} =< X, Y, Z \mid X^{11} = Y^2 = Z^2 = (XY)^3 = (XZ)^3 = (YZ) =^{10} 1,\ M_{11}$
$X^2(YZ)^2X = (YZ)^2 >$.

can be generated using two permutations, the first is of order 13 and an involution as follows : $M_{11} =< (1,2,...,11)(1,2,3,7,6)(4,8,5,9,10) >$. M_{12} can be generated using two permutations, the first is of order 17 and an involution as follow:

$M_{12} =< (1,2,...,11)(1,2,3,7,6)(4,8,5,9,10)(1,12)(2,11)(3,6)$
$(4,8)(5,9)(7,10) >$.

In this paper, we will generate the wreath product $M_{11} wr M_{12}$ using only two permutations. Also, we show the structure of some groups containing the wreath product $M_{11} wr M_{12}$. The structure of the groups founded is determined in terms of wreath product $(M_{11} wr M_{12})\, wr\, C_k$. Some related cases are also included. Also, we will show that S_{132K+1} and A_{132K+1} can be generated using the wreath product $(M_{11} wr M_{12})\, wr\, C_k$ and a transposition in S_{132K+1} and an element of order 3 in A_{132K+1}. We will also show that S_{132K+1} and A_{132K+1} can be generated using the wreath product $M_{11} wr M_{12}$ and an element of order $k+1$.

II. PRELIMINARY RESULTS

DEFINITION 2.1 Let *A* and *B* be groups of permutations on non empty sets Ω_1 and Ω_2 respectively. The wreath product of *A* and *B* is denote by *A* wr *B* and defined as *A* wr *B*= $A^{\Omega_2} \times_\theta B$, i.e., the direct product of $|\Omega_2|$ copies of *A* and a mapping θ

THEOREM 2.2 [4] Let G be the group generated by the *n*-cycle (1, 2, …, *n*) and the 2-cycle (*n*, *a*). If $1 < a < n$ is an integer with $n = am$, then $G \cong S_m$ wr C_a.

THEOREM 2.3 [4] Let $1 \le a \ne b < n$ be any integers. Let *n* be an odd integer and let G be the group generated by the *n*-cycle (1,2,…,*n*) and the 3-cycle (n, a, b). If the $hcf_{(n,a,b)}$=1, then $G = A_n$. While if *n* can be an even then $G = S_n$.

THEOREM 2.4 [4] Let $1 \le a < n$ be any integer. Let $G = \langle (1,2,...,n), (n,a) \rangle$. If $h.c.f.(n,a) = 1$, then $G = S_n$.

THEOREM 2.5 [4] Let $1 \le a \ne b < n$ be any integers. Let n be an even integer and let G be the group generated by the (*n*-1)-cycle $(1,2,...,n-1)$ and 3-cycle (*n*,*a*,*b*). Then $G = A_n$.

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III. The Results

THEOREM 3.1 The wreath product $M_{11}wrM_{12}$ can be generated using two permutations, the first is of order 132 and the second is of order 4.

Proof : Let $G = \langle X,Y\rangle$, where: X=(1, 2, 3, 4, …, 132), which is a cycle of order 252, Y=(1, 9)(2, 6)(4, 5)(7, 8)(12, 20, 23, 31)(13, 17)(15, 16)(18, 19)(24, 28)(26, 27) (29, 30)(34, 42, 56, 64)(35, 39)(37, 38)(40, 41)(45, 53)(46, 50)(48, 49)(51, 52)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71)(73, 74), which is the product of two cycles each of order 4 and twenty four transpositions. Let $\alpha_1 = \left((XY)^6[X,Y]^5\right)^{18}$. Then

$$\alpha_1 = (11, 22, 33, 44, 55, 66, 132),$$

which is a cycle of order 7. Let $\alpha_2 = \alpha_1^{-1}X$. It is easy to show that

$$\alpha_2 = (1, 2, 3, …, 17)(18, 19, 20, …, 22) … (67, 68, 69, 132),$$

which is the product of seven cycles each of order 11. Let: $\beta_1 = (Y^2)^{(XY)^{18}}$=(9, 20)(12, 23)(31, 53)(34, 56), $\beta_2 = \beta_1 Y^{-1}$=(1, 9, 12, 20)(2, 6)(4, 5) (7, 8)(13, 17)(15, 16)(18, 19)(23, 31, 45, 53)(24, 28)(26, 27)(29, 30)(34, 42)(35, 39)(37, 38)(40, 41)(46, 50)(48, 49)(51, 52)(56, 64)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71) (73, 74), $\beta_3 = (Y^3\beta_2)^2$=(1, 45)(12, 23), $\beta_4 = \beta_3^{(\alpha_2^{-1}\alpha_1^3)}$=(11, 44)(55, 66) and $\beta_5 = \beta_4^{\beta_3^{\alpha_2^{-1}}}$ = (17, 221)(68, 85).

Let $\alpha_3 = \beta_5^{\beta_3^{(\alpha_2^{-1}\alpha_1)}}$. Hence

α_3=(12, 24)(48, 60).

Let $\alpha_4 = YX^{-1}\alpha_3^{-1}X$. We can conclude that

α_4 =(1,9)(2,6)(4,5)(7,8)(12,20)(13,17)(15,16)(18,19)(23,31)(24,28)(26,27)(29,30)(34,42)(35,39)(37,38)(40,41)(45,53)(46,50)(48,49)(51,52)(56,64)(57,61)(59,60)(62,63)(67,75)(68,72)(70,71)(73,74),

which is the product of twenty eight transpositions. Let $K = \langle\alpha_2, \alpha_4\rangle$. Let $\theta : K \to M_{12})$ be the mapping defined by

$\theta(12i+j) = j \quad \forall\ 1 \le i \le 10,\ \forall\ 1 \le j \le 12$

Since $\theta(\alpha_2) = (1, 2, …, 12)$ and $\theta(\alpha_4) = (1, 9)(2, 6)(4, 5)(7, 8)$, then $K \cong \theta(K) = M_{12}$. Let $H_0 = \langle\alpha_1, \alpha_3\rangle$. Then $H_0 \cong M_{11}$. Moreover, K conjugates H_0 into H_1, H_1 into H_2 and so it conjugates H_{16} into H_0 , where

$H_i =< (i,12+i,34+i,51+i,68+i,85+i,102+i,....,221+i)(i,12+i)(34+i,68+i) >$

$\forall\ 1 \le i \le 10$. Hence we get $M_{11}wrM_{12}) \subseteq G$. On the other hand, Since X= $\alpha_1\alpha_2$ and Y=$\alpha_4\alpha_3^X$, then $G \subseteq M_{11}wrM_{12}$. Hence $G = M_{11}wrM_{12}$ ◊

THEOREM 3.2 The wreath product $(M_{11}wrM_{12})\,wr\,C_k$ can be generated using two permutations, the first is of order 132*k* and an involution, for all integers $k \ge 1$.

Proof : Let σ = (1, 2, …, 132*k*) and τ =(*k*, 9*k*)(2*k*, 6*k*)(4*k*, 5*k*)(7*k*, 8*k*)(12*k*, 20*k*, 23*k*, 31*k*)(13*k*, 17*k*)(15*k*, 16*k*)(18*k*, 19*k*)(24*k*, 28*k*)(26*k*, 27*k*)(29*k*, 30*k*)(34*k*, 42*k*, 56*k*, 64*k*)(35*k*, 39*k*)(37*k*, 38*k*)(40*k*, 41*k*)(45*k*, 53*k*)(46*k*, 50*k*)(48*k*, 49*k*)(51*k*, 52*k*)(57*k*, 61*k*) (59*k*, 60*k*) (62*k*, 63*k*)(67*k*, 75*k*)(68*k*, 72*k*)(70*k*, 71*k*). If *k*=1, then we get the group $M_{11}wrM_{12}$ which can be considered as the trivial wreath product $(M_{11}wrM_{12})\,wr\,C_k$ wr<id>. Assume that $k > 1$. Let $\alpha = \prod_{i=0}^{12} \tau^{\sigma^{ik}}$, we get an element $\delta = \alpha^{45}$=(*k*, 2*k*, 3*k*, …, 132*k*). Let $G_i = \langle \delta^{\sigma^i}, \tau^{\sigma^i}\rangle$, be the groups acts on the sets Γ_i ={ i, *k*+i, 2*k*+i,…, 131*k*+*i* }, for all $1 \le i \le k$. Since $\bigcap_{i=1}^{k} \Gamma_i = \varphi$, then we get the direct product $G_1 \times G_2 \times … \times G_k$, where, by theorem 3.1 each $G_i \cong M_{11}wrM_{12}$. Let $\beta = \delta^{-1}\sigma$ = (1, 2, …, *k*)(*k*+1, *k*+2, …, 2*k*) … (76*k*+1, 76*k*+2, …, 132*k*). Let $H = \langle\beta\rangle \cong C_k$. H conjugates G_1 into G_2, G_2 into G_3,…and G_k into G_1. Hence we get the wreath product $(M_{11})wrM_{12})wrC_K \subseteq G$. On the other hand, since $\delta\beta$ = (1, 2, …, *k*, *k*+1, *k*+2, …, 2*k*, …, 131*k*+1, 131*k*+2, …, 132*k*)=σ, then $\sigma \in (M_{11}wrM_{12})wrC_K$. Hence $G =< \sigma, \tau > \cong (M_{11}wrM_{12})wrC_K$.◊

THEOREM 3.3 The wreath product $(L_2(11)\,wrM_{12})\,wr\,S_k$ can be generated using three permutations, the first is of order 132*k*, the second and the third are involutions, for all $k \ge 2$.

Proof : Let σ = (1, 2, …, 132*k*), τ =(*k*, 9*k*)(2*k*, 6*k*)(4*k*, 5*k*)(7*k*, 8*k*)(12*k*, 20*k*, 23*k*, 31*k*)(13*k*, 17*k*)(15*k*, 16*k*)(18*k*, 19*k*)(24*k*, 28*k*)(26*k*, 27*k*)(29*k*, 30*k*)(34*k*, 42*k*, 56*k*, 64*k*)(35*k*, 39*k*)(37*k*, 38*k*)(40*k*, 41*k*)(45*k*, 53*k*)(46*k*, 50*k*)(48*k*, 49*k*)(51*k*, 52*k*)(57*k*, 61*k*)(59*k*, 60*k*)(62*k*, 63*k*)(67*k*, 75*k*)(68*k*, 72*k*)(70*k*, 71*k*) and μ = (1, 2)(*k*+1, *k*+2)(2*k*+1, 2*k*+2) … (131*k*+1, 131*k*+2). Since by Theorem 3.2, $<\sigma,\tau> = (M_{11}wrM_{12})\,wr\,C_k$ and (1, 2, …, *k*)(*k*+1, *k*+2, …, 2*k*) … (131*k*+1, …,132*k*)∈ $(M_{11}wrM_{12})\,wr\,C_k$ then ⟨(1,…,*k*)(*k*+1,…,2*k*)…(131*k*+1,…,132*k*) , μ ⟩ $\cong S_k$. Hence $G = \langle\sigma,\ \tau,\ \mu\rangle \cong (M_{11}wrM_{12})wrS_k$. ◊

COROLLARY 3.4 The wreath product $(M_{11}\ wrM_{12})\,wr\,A_k$ can be generated using three permutations, the first is of order 132*k*, the second is an involution and the third is of order 3, for all odd integers $k \geq 3$.

THEOREM 3.5 The wreath product $(M_{11}\, wr\, M_{12})\, wr\, (S_m\, wr\, C_a)$ can be generated using three permutations, the first is of order 132*k*, the second and the third are involutions, where $k = am$ be any integer with $1 < a < k$.

Proof : Let σ = (1, 2, …, 132*k*), τ =(*k*, 9*k*)(2*k*, 6*k*)(4*k*, 5*k*)(7*k*, 8*k*)(12*k*, 20*k*, 23*k*, 31*k*)(13*k*, 17*k*)(15*k*, 16*k*)(18*k*, 19*k*)(24*k*, 28*k*)(26*k*, 27*k*)(29*k*, 30*k*)(34*k*, 42*k*, 56*k*, 64*k*)(35*k*, 39*k*)(37*k*, 38*k*)(40*k*, 41*k*)(45*k*, 53*k*)(46*k*, 50*k*)(48*k*, 49*k*)(51*k*, 52*k*)(57*k*, 61*k*)(59*k*, 60*k*)(62*k*, 63*k*)(67*k*, 75*k*)(68*k*, 72*k*)(70*k*, 71*k*) and μ = (*k*, *a*)(2*k*, *k*+*a*)(3*k*, 2*k*+*a*) … (132*k*, 131*k*+*a*). Since by Theorem 3.2, $< \sigma, \tau > \cong (M_{11} wr M_{12})\, wr\, C_k$ and (1, …, *k*)(*k*+1, …, 2*k*) … (131*k*+1, … ,132*k*)$\in (M_{11} wr M_{12})\, wr\, C_k$ then $\langle$(1, …, *k*)(*k*+1, …, 2*k*) …(131*k*+1, …, 132*k*, $\mu \rangle \cong (S_m \text{ wr } C_a)$.
Hence $G = \langle \sigma,\ \tau,\ \mu \rangle \cong (M_{11} wr M_{12}) wr (S_m wr C_a)$.◊

THEOREM 3.6 S_{132k+1} and A_{132k+1} can be generated using the wreath product $(M_{11}\, wr M_{12})\, wr\, C_k$ and a transposition in S_{132k+1} for all integers $k > 1$ and an element of order 3 in A_{132k+1} for all odd integers $k > 1$.

Proof: Let σ = (1, 2, …, 132*k*), τ =(*k*, 9*k*)(2*k*, 6*k*)(4*k*, 5*k*)(7*k*, 8*k*)(12*k*, 20*k*, 23*k*, 31*k*)(13*k*, 17*k*)(15*k*, 16*k*)(18*k*, 19*k*)(24*k*, 28*k*)(26*k*, 27*k*)(29*k*, 30*k*)(34*k*, 42*k*, 56*k*, 64*k*)(35*k*, 39*k*)(37*k*, 38*k*)(40*k*, 41*k*)(45*k*, 53*k*)(46*k*, 50*k*)(48*k*, 49*k*)(51*k*, 52*k*)(57*k*, 61*k*)(59*k*, 60*k*)(62*k*, 63*k*)(67*k*, 75*k*)(68*k*, 72*k*)(70*k*, 71*k*), μ =(132*k*+1,1) and μ' =(1,*k*, 132*k*+1) be four permutations, of order 132*k*, 2, 2 and 3 respectively. Let $H = \langle \sigma,\ \tau \rangle$. By theorem 3.2 $H \cong (M_{11}\, wr\, M_{12})\, wr\, C_k$.

Case 1: Let $G = \langle \sigma,\ \tau,\ \mu \rangle$. Let $\alpha = \sigma\mu$, then $\alpha = (1,2,...,132k,132k+1)$ which is a cycle of order $132k+1$.

By theorem 2.4 $G < \sigma, \tau, \mu' > \cong < \alpha, \mu > \cong\ S_{132k+1}$.

Case 2: Let $G = \langle \sigma,\ \tau, \mu' \rangle$

By theorem 2.5 $< \sigma, \mu' > \cong A_{132K+1}$. Since τ is an even permutation, then $G \cong A_{132K+1}$.

THEOREM 3.7 S_{132k+1} and A_{132k+1} can be generated using the wreath product $L_2(11)\, wr\, M_{12}$ and an element of order $k+1$ in S_{132k+1} and A_{132k+1} for all integers $k \geq 1$.

Proof: Let $G = \langle \sigma,\ \tau,\ \mu \rangle$, where, σ =(1, 2, 3, …, 132)(132(*k*-(*k*-1))+1, …, 132(*k*-(*k*-1))+132) … (132(*k*-1)+1, …, 132(*k*-1)+132), τ =(1, 9)(2, 6)(4, 5)(7, 8)(12, 20, 23, 31)(13, 17)(15, 16)(18, 19)(24, 28)(26, 27)(29, 30)(34, 42, 56, 64)(35, 39)(37, 38)(40, 41)(45, 53)(46, 50)(48, 49)(51, 52)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71)(73, 74) … (132(*k*-1)+1, 132(*k*-1)+9) … (132(*k*-1)+73, 132(*k*-1)+74), and μ =(132, 154, …,132*k*, 132*k*+1), where *k*-*i* >0, be three permutations of order 132, 4 and *k*+1 respectively. Let $H = \langle \sigma,\ \tau \rangle$. Define the mapping θ as follows;

θ(12(*k*- *i*)+*j*) = *j* $\forall\ 1 \leq i \leq k$, $\forall\ 1 \leq j \leq 12$

Hence $H =< \sigma, \tau > \cong M_{11}\, wr\, M_{12}$. Let $\alpha = \mu\sigma$ it is easy to show that $\alpha = (1,2,3,...,132k+1)$, which is a cycle of order $132k+1$. Let $\mu' = \mu^{\sigma} = (1,133,...,132(k-1)+1,132k+1)$ and $\beta = [\mu,\ \mu'] = (1,132,132k+1)$.

Since $h.c.f\,(1,132,132k+1)$, then by theorem 2.3 $G = \langle \sigma, \tau, \mu \rangle \cong \langle \alpha, \beta \rangle$ S_{132k+1} or A_{132K+1} depending on whether *k* is an odd or an even integer respectively. ◊

REFERENCES

[1] A. M. Hammas and I. R. Al-Amri, Symmetric generating set of the alternating groups , JKAU: Educ. Sci., 7 (1994), 3-7.
[2] B. H. Shafee, Symmetric generating set of the groups and using th the wreath product , Far East Journal of Math. Sci. (FJMS), 28(3) (2008), 707-711.
[3] B. H. Shafee and I. R. Al-Amri, On the Structure of Some Groups Containing , International Journal of Algebra, vol.6, 2012,no.17, 857-862.
[4] I.R. Al-Amri, Computational methods in permutation groups , ph.D Thesis, University of St. Andrews, September 1992.
[5] I.R. Al-Amri, and A.M. Hammas, Symmetric generating set of Groups and , JKAU: Sci., 7 (1995), 111-115.
[6] J. H. Conway and others, Atlas of Finite Groups, Oxford Univ. Press, New York, 1985.