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Mechanical quadrature methods and their extrapolations for solving first kind boundary integral equations of anisotropic Darcy's equation

Xin Luo, Jin Huang and Chuan-Long Wang

Abstract—The mechanical quadrature methods for solving the boundary integral equations of the anisotropic Darcy's equations with Dirichlet conditions in smooth domains are presented. By applying the collectively compact theory, we prove the convergence and stability of approximate solutions. The asymptotic expansions for the error show that the methods converge with the order $O\ (h^3)$, where h is the mesh size. Based on these analysis, extrapolation methods can be introduced to achieve a higher convergence rate $O\ (h^5)$. An a posterior asymptotic error representation is derived in order to construct self-adaptive algorithms. Finally, the numerical experiments show the efficiency of our methods.

Keywords—Darcy's equation, anisotropic, mechanical quadrature methods, extrapolation methods, a posteriori error estimate.

I. INTRODUCTION

CONSIDER the Darcy's equation

$$\sum_{j=1}^{2}\frac{\kappa_j\partial^2 u}{\partial^2 x_j^2}=0,\quad \text{in }\Omega \tag{1}$$

with the Dirichlet boundary condition as follows:

$$u(x)|_{\partial\Omega}=\bar{u},\quad \text{on }\partial\Omega \tag{2}$$

where $\Omega\subset\mathbb{R}^2$ is a two-dimensional bounded region with the boundary $\partial\Omega$, which is a smooth closed curve. As usual, we use $x=(x_1,x_2)\in\mathbb{R}^2$ to denote the Cartesian co-ordinates of the points in the Euclidean space $\mathbb{R}^2$. Here, u is the potential function, parameters κ_1 and κ_2 are positive constants, and $\bar{u}(x)$ is a known function on $\partial\Omega$. The Darcy's equation often plays an important role in porous media flow[1] or in heat conduction[2].

In this paper, we assume that $\kappa_1\neq\kappa_2$. Under this assumption, Eq. (1) is also called anisotropic Darcy's equation. By means of the potential theory, the solutions of the Dirichlet problem (1) can be represented as a single-layer potential of the form

$$u(y)=\int_{\partial\Omega}G_2(x,y)w(x)ds_x,\ y=(y_1,y_2)\text{ in }\Omega, \tag{3}$$

where ds_x denotes the arc length element at a point $x=(x_1,x_2)\in\partial\Omega$, and $G_2(x,y)$ is the fundamental solution of Eq. (3), namely, $G_2(x,y)=-\frac{(\kappa_1\kappa_2)^{-1/2}}{2\pi}\ln(\sum_{j=1}^{2}\frac{(x_j-y_j)^2}{\kappa_j})^{1/2}$. The reformulated problem then becomes the first kind boundary equation (BIE) as follows:

$$\int_{\partial\Omega}G_2(x,y)w(x)ds_x=\bar{u},\ y=(y_1,y_2)\text{ on }\partial\Omega. \tag{4}$$

The flux $w(x)=\sum_{j=1}^{2}\kappa_j\frac{\partial u}{\partial x_j}\nu_{xj}$ is an unknown to be sought, where ν_{xj} is the direction cosine of the normal ν to the boundary $\partial\Omega$ with respect to x_j. Eq. (4) is the weakly singular BIE system of the first kind, whose solution exists and is unique as long as $C_{\partial\Omega}\neq 1$ [6], where C_Γ is the logarithmic capacity (i.e., the transfinite diameter). As soon as $w(x)$ is solved from (4), $u(y)$ $(y\in\Omega)$ can be calculated by (3).

The kernels of (4) have singularities at the points $x=y$, which degrade the rate of convergence. Several numerical methods have been proposed to overcome this difficulty, such as the Galerkin method (GM)[13], the collocation method (CM)[7], and the quadrature method[5]. However, these methods do not provide a good accuracy in the solution near the singular points. For example, the accuracy of Galerkin methods[13] is only $O\ (h^\tau)$ $(0<\tau<2)$ and the accuracy of collocation methods[7] is even lower.

In this article, the mechanical quadrature methods (MQMs) are proposed to calculate weakly singular integrals by Sidi's quadrature rules[3], and the extrapolation methods (EMs)[4] are applied to improve the accuracy of solutios. Once discrete equations on some coarse meshes are solved in parallel, the accuracy of numerical solutions can be greatly improved by EMs.

This paper is organized as follows: in Section II, we present the MQMs, and prove the convergence and stability of MQMs. in Section III, we construct the EMs, and we provide the asymptotic expansion of errors and an a posterior error estimate. Two numerical examples are provided to verify the theoretical results in Section IV, and some useful conclusions are listed in Section V.

II. MECHANICAL QUADRATURE METHODS

A. Existence and Convergence of MQMs Solutions

Assume that the smoothed boundary $\partial\Omega$ has the following parametrization

$$x=(x_1(s),x_2(s)):[0,2\pi]\to\partial\Omega.$$

with $[(x_1'(s))^2+(x_2'(s))^2]^{1/2}>0$. Let $z(s)=(\kappa_1^{-1/2}x_1(s),\ \kappa_2^{-1/2}x_2(s))$. Then the integral equation (4) can

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be split into a singularity part and a compact perturbation part

$$(A+B)v = f, \tag{5}$$

where $v(t) = |x'(t)|w(x(t))$, $f = (\kappa_1\kappa_2)^{1/2}\bar{u}(s)$ and

$$(Av)(s) = \int_0^{2\pi} a(s,t)v(t)dt, \tag{6}$$

with $a(s,t) == -\frac{1}{2\pi}\ln\left|2e^{-1/2}\sin\frac{s-t}{2}\right|$,

$$(Bv)(s) = \int_0^{2\pi} b(s,t)v(t)dt, \tag{7}$$

with $b(s,t) = -\frac{1}{2\pi}\ln|z(s)-z(t)|$. A is an isometry operator from H^t to H^{t+1} for any number t, that is, $\|Av\|_{t+1} = \|v\|_t$[12].

Let $h = \frac{2\pi}{n}$ $(n \in N)$ be the mesh width and $t_j = jh$ $(j = 0,1,\cdots,n)$ be nodes. For the integral operators B with periodic kernels, we can construct the Nyström approximation by the midpoint or the trapezoidal rule [8], which has the error bounds $O\ (h^{2l})$, $l \in N$. For the logarithmically singular operators A, by the Sidi's quadrature formula [3], we can construct the Fredholm approximation

$$(A^h v)(s_i) = -\frac{h}{2\pi}\left[\sum_{j=0,i\neq j}^{n} \ln\left|2e^{-\frac{1}{2}}\sin\frac{(i-j)h}{2}\right| v(t_j)\right] - \frac{h}{2\pi}\ln\left|2\pi e^{-\frac{1}{2}}\frac{h}{2\pi}\right| v(s_i), \quad i = 0,\cdots,n, \tag{8}$$

which has the following error bounds[12]:

$$E_n(A) = \frac{-2}{\pi}\sum_{\mu=1}^{2l-1}\frac{\zeta'(-2\mu)}{(2\mu)!}[v(s)]^{(2\mu)}\Big|_{s=s_i} h^{2\mu+1} + O(h^{2l}), \tag{9}$$

where $E_n(A) = (A^h v)(s_i) - (Av)(s_i)$, and $\zeta(z)$ is a Riemann function.

Consider the discrete approximation of (5)

$$(A^h + B^h)v^h = f^h, \tag{10}$$

where $v^h = (v_0^h, v_1^h, \cdots, v_n^h)^T$, $A^h = [a(s_i,t_j)]_{i,j=0}^n$, $B^h = [b_0(s_i,t_j]_{i,j=0}^n$, and $f^h = (f(x(t_0)),\cdots,f(x(t_n)))^T$ with $f(x(s_i)) = \bar{u}(x(s_i))$. Obviously, (10) is a linear equation system with n unknowns. Once v^h is solved from (10), the solution of (3) $u(y)$ $(y \in \Omega)$ can be computed by

$$u^h = -\frac{h}{2\pi\sqrt{\kappa_1\kappa_2}}\sum_{j=0}^{n}\left[\ln\left(\sum_{j=1}^{2}\frac{(x_j-y_j)^2}{\kappa_j}\right)^{1/2}\right]\left|x'(s_j)\right|v^h(s_j) \tag{11}$$

From (8), we have $A^h = [-\frac{1}{2\pi}h\ln|2e^{-\frac{1}{2}}\sin\frac{jh}{2}|]_{j=1}^n$, and we known A^h is a symmetric circulant matrix.

Lemma 2.1. [14](1) The eigenvalues λ_i of A^h are positive, and

$$\frac{1}{4n\pi} \le \lambda_i \le c\ (i = 0,\cdots,n-1), \tag{12}$$

where c is a constant independent of h.

(2) A^h is invertible and $\|(A^h)^{-1}\| = O\ (n)$, where $\|\cdot\|$ denotes the spectral norm.

We define some special operators in order to discuss the existence and convergence of numerical approximations. Let $V^h = \text{span}\{e_i(s), i = 0,1,\cdots,n\} \subset C[0,2\pi)$ be a piecewise linear function subspace with nodes $\{s_j\}_{j=0}^n$, where $e_i(s)$ is the basis function satisfying $e_i(s_j) = \delta_{ij}$. Define a prolongation operator $I^h : \mathbb{R}^n \to V^h$ satisfying

$$I^h v = \sum_{i=0}^{n} v_i e_i(s), \quad \forall v = (v_0,\cdots,v_n) \in \mathbb{R}^n,$$

and a restricted operator $R^h : C[0,2\pi) \to \mathbb{R}^n$ satisfying

$$R^h v = (v(s_0),\cdots,v(s_n)) \in \mathbb{R}^n, \quad \forall v \in C[0.2\pi).$$

Lemma 2.2.[14] The operator sequence $\Big\{I^h(A^h)^{-1}R^hA^h : C^3[0,2\pi) \to C[0,2\pi)\Big\}$ is uniformly bounded and convergent to the embedding operator I.

Corollary 2.3. For an integral operator B with a periodic smooth kernel $b(s,t)$, defining the Nyström approximation

$$(B^h v)(s) = h\sum_{j=0}^{n} b(s,t_j)v(s_i), \quad s \in [0,2\pi),$$

and assume that (4) is uniquely solvable. we have

$$I^h(A^h)^{-1}R^hB^h \overset{c.c}{\to} (A)^{-1}B \text{ in } C[0,2\pi) \to C[0,2\pi), \tag{13}$$

where $\overset{c.c}{\to}$ denotes the collectively compact convergence, and $E^h + I^h(A^h)^{-1}R^hB^h$ is invertible, and the inverse operator is uniformly bounded.

Proof: Because the kernel $b(s,t)$ of the operator B and its derivatives of higher order are continuous [11], and we have $\|I^h(A^h)^{-1}R^hB^h\|_{0,0} \le \|I^h(A^h)^{-1}R^hA^h\|_{0,3}\|(A^h)^{-1}B^h\|_{3,0}$. From the literatures [9,10], we know $(A^h)^{-1}B^h$ is collectively and compactly convergent to $A^{-1}B \in L(C[0,2\pi), C^3[0,2\pi))$, and there exists a constant M_0 such that

$$\|(A^h)^{-1}B^h\|_{3,0} \le M_0,\ \|I^h(A^h)^{-1}R^hA^h\|_{0,3} \le M_0,$$

where $\|\cdot\|_{n_2,n_1}$ is the norm of the linear bounded operator space $L(C^{n_1}[0,2\pi), C^{n_2}[0,2\pi))$. Applying the results in the literature [10], the operator sequence $\{(A^h)^{-1}B^h : C[0,2\pi), C^3[0,2\pi)\}$ must be collectively compactly convergent to $(A^h)^{-1}B$. Hence, the proof of Corollary 2.3 is completed. ■

Replacing $(Q^h)^{-1} = (A^h)^{-1}$, A^h, and B^h by $(\hat{Q}^h)^{-1} = I^h(Q^h)^{-1}R^h$, $\hat{A}^h = I^h(A^h)R^h$ and $\hat{B}^h = I^hB^hR^h$, respectively, we obtain the operator

$$\hat{G}^h = I^h(Q^h)^{-1}R^h(A^h+B^h)R^h.$$

Consider the operator equation

$$(E^h + \hat{G}^h)\hat{v}^h = \hat{f}^h, \tag{14}$$

with $\hat{f}^h = I^h(Q^h)^{-1}R^hf^h$. Obviously, if $\hat{v}^h = I^hv^h$ is a solution of (14), then $R^h\hat{v}^h$ must be a solution of

$$(Q^h)^{-1}(A^h+B^h)v^h = (Q^h)^{-1}f^h. \tag{15}$$

Conversely, if v^h is a solution of (10), then $\hat{v}^h$ must be a solution of (14). The following theorem shows there exists a unique solution $\hat{v}^h$ in (14) such that converges to v.

Theorem 2.4.[9] The operator sequence $\{\hat{G}^h\}$ is collectively compactly convergent to $G = (Q^h)^{-1}(A^h + B^h)$ in $C[0, 2\pi)$, *i.e.*,

$$\hat{G}^h \overset{c.c}{\to} G. \quad (16)$$

Corollary 2.5. Assuming that (4) has a unique solution and h is sufficiently small, then there exists a unique solution $\hat{v}^h$ in (14), and $\hat{v}^h$ has the following error bound under the norm of $C[0, 2\pi)$:

$$\|\hat{v}^h - v\| \le \|(I+G)^{-1}\| \frac{\|(\hat{G}^h - G)\hat{f}\| + \|(\hat{G}^h - G)\hat{G}^h v\|}{1 - \|(I+\hat{G}^h)^{-1}(\hat{G}^h - G)\hat{G}^h\|}. \quad (17)$$

B. Stability Analysis for MQMs

For the stability of MQMs, we have the following theorem.

Theorem 2.6. Assume that $\partial\Omega$ satisfy $C_{\partial\Omega} \neq 1$, and $\partial\Omega$ be smooth curves. Suppose that A^h and B^h are the discrete matrices defined by (6) and (7), respectively. Then the eigenvalues λ_i $(i = 1, \cdots, n)$ of discrete matrix $K^h = A^h + B^h$ satisfy

$$\check{c} \ge |\lambda_i| \ge \hat{c}h, \quad i = 1, \cdots, n,$$

where $\check{c}$ and $\hat{c}$ are two positive constants independent of $h(= 1/n)$ is the mesh step size of a curved edge $\partial\Omega$, and there exists the bound of condition number

$$Cond(K^h) = \frac{|\lambda_{\max}(K^h)|}{|\lambda_{\min}(K^h)|} = O(h^{-1}). \quad (18)$$

where $\lambda_i(K^h)$ are the eigenvalues of $K^h = A^h + B^h$, and $Cond$ is the traditional 2-norm condition number.

Proof: Because there exists unique solution in (5) based on $C_{\partial\Omega} \neq 1$, we have $\lambda_i((A^h)^{-1}B^h) \neq -1$. From Lemma 2.2 and Theorem 2.4, the operator $(A^h)^{-1}B^h$ is compact operator. Based on the properties of compact operator [8], we get

$$c_1 \le |\lambda_i(I + (A^h)^{-1}B^h)| \le c_2, i = 1, ..., \sum_{j=1}^{n},$$

where c_1 and c_2 are two positive constants independent of h and h $(= 1/n$ is the mesh step size of a curved edge $\partial\Omega$. From Lemma 2.1 and the literature [11], we have $\check{c} \ge |\lambda_i(A^h(I + (A^h)^{-1}B^h))| \ge \hat{c}h$. The proof of Theorem 2.6 is completed. ■

III. EXTRAPOLATION METHODS

Theorem 3.1. If there exists a unique solution in (4), f, f^h are computed by (5) and (10) respectively, $x_i \in C^6[0, 2\pi)$ $(i = 1, 2)$and $f(s) \in C^5[0, 2\pi)$, then there exists a function $\varrho \in C^5[0, 2\pi)$ independent of h such that

$$(v - \hat{v}^h)\big|_{s=s_i} = h^3 \varrho\big|_{s=s_i} + O(h^5). \quad (19)$$

Proof: By the midpoint trapezoidal rule, the asymptotic expansion holds [9,12]

$$(f - f^h)\big|_{s=s_i} = h^3 I^h R^h \psi_1\big|_{s=s_i} + O(h^5),$$

with $\psi_1 = -\xi^{'}(-2)f''(t)/\pi$. Using (8) and (9), we can obtain

$$(c_0 A_0^h + B_0^h)R^h(\hat{v}^h - v)\big|_{s=s_i} = h^3 I^h R^h \psi\big|_{s=s_i} + O(h^5), \quad (20)$$

where $\psi_2 = c_0\xi^{'}(-2)v''(t)/\pi$, and $\psi = \psi_1 + \psi_2$. From Theorem 2.4, we have

$$(E^h + \hat{G}^h)(v - \hat{v}^h)\big|_{s=s_i} = h^3(\hat{Q}^h)^{-1} I^h R^h \psi\big|_{s=s_i} + O(h^5). \quad (21)$$

Define the auxiliary equation

$$(E + G)\varrho = Q^{-1}\psi, \quad (22)$$

and its approximate equation

$$(E^h + \hat{G}^h)\varrho^h = (\hat{Q}^h)^{-1} I^h R^h \psi. \quad (23)$$

Substituting (23) into (22) yields

$$(E^h + \hat{G}^h)(v - \hat{v}^h - h^3\varrho^h)\big|_{s=s_i} = O(h^5). \quad (24)$$

Since $(E^h + \hat{G}^h)^{-1}$ is uniformly bounded by Theorem 2.4, we obtain

$$(v - \hat{v}^h - h^3\varrho^h)\big|_{s=s_i} = O(h^5). \quad (25)$$

Replacing ϱ^h in (25) with ϱ and applying Theorem 2.4, we complete the proof of Theorem 3.1. ■

The asymptotic expansion (19) implies that the extrapolation methods (EMs) can be applied to solve (4). Moreover, the high order $O(h^5)$ of accuracy can be obtained on coarse grids of $\partial\Omega$ in parallel. The related work on EMs can be find in the literature [12] and can be described as follows:

Step 1. Choose h and $\frac{h}{2}$, and solve (10) in parallel, where $v^h(s_i)$ and $v^{\frac{h}{2}}(s_i)$ are their solutions.

Step 2. Computing the solutions at the coarse grid points by (11)

$$u^*(s_i) = \frac{1}{7}[8u^{\frac{h}{2}}(s_i) - u^h(s_i)], \quad (26)$$

An a posteriori estimate can be obtained by (19) and (11)

$$\left|u(s_i) - u^{\frac{h}{2}}(s_i)\right| \le \frac{8}{7}\left|u^{\frac{h}{2}}(s_i) - u^h(s_i)\right| + O(h^5). \quad (27)$$

IV. NUMERICAL EXAMPLES

In this section, to verify theoretical results in this paper, we present two numerical examples for the anisotropic Darcy's equations by MQMs and EMs.

The following two examples can be regarded as the models of steady state heat conduction in the smooth domains which are materials possessing zoned orthotropic thermal conductivity in the Darcy's equations. Here, we assume that the heat generation is absent.

Example 1. We consider the two-dimensional anisotropic Darcy's equation in $\Omega = \{(x_1, x_2) : x_1^2 + x_2^2 \le 1\}$. The Dirichlet boundary condition is given by $u(x_1, x_2) = -10x_1^2 + 2x_2^2 + 3x_1x_2 + 4x_1 + 5x_2 + 6$,where u represents the temperature. The thermal conductivity coefficients used are $\kappa_1 = 1.0$ and $\kappa_2 = 5.0$.

Tables I, and II, accord with our error analysis, where the numbers of nodes n is $2\pi/h$. $e^h(t) = |u(t) - u^h(t)|$ gives the absolute error at a given point $(\cos t, \sin t)(t = 0, \pi/2)$; $r^h(t) = e^h(t)/e^{h/2}(t)$ shows the convergence ratio by MQMs; $e_E^h(t) = |u(t) - u^*(t)|$ shows the error at the given point after using the EMs once; and $r_E^h(t) = e_E^h(t)/e_E^{h/2}(t)$ shows the convergence ratio by EMs. From the numerical results we can see that $Cond|_{2^{k+1}}/Cond|_{2^k} \approx 2$ $(k = 5, \cdots, 8)$,

TABLE I
THE CONDITION NUMBER FOR EXAMPLE 1.

2^n	2^5	2^6	2^7	2^8	2^9
$\|\lambda_{\min}\|$	0.272	0.136	0.068	0.034	0.017
$\|\lambda_{\max}\|$	24.23	24.24	24.23	24.24	24.24
Cond	89.05	1.781E+2	3.562E+2	7.124E+2	1.425E+3

TABLE II
ERROR e^h, ERROR RATIO r^h, AND ERROR e_E^h OF u FOR EXAMPLE 1.

t	$t=0$		$t=\frac{\pi}{2}$	
n	e^h; r^h	e_E^h; r_E^h	e^h; r^h	e_E^h; r_E^h
2^3	7.0E-2; $2^{2.9}$	1.7E-5; $2^{2.7}$	4.2E-3; $2^{2.7}$	1.6E-4; $2^{8.7}$
2^4	8.8E-3; $2^{3.0}$	2.6E-6; $2^{4.8}$	6.7E-4; $2^{3.0}$	3.8E-7; $2^{4.8}$
2^5	1.2E-3; $2^{2.9}$	9.0E-8; $2^{4.9}$	8.3E-5; $2^{3.0}$	1.4E-8; $2^{5.0}$
2^6	1.4E-4; $2^{3.0}$	2.9E-9; $2^{5.0}$	1.0E-5; $2^{3.0}$	4E-10; $2^{5.0}$
2^7	1.7E-5; $2^{3.0}$	3E-11; $2^{5.0}$	1.3E-6; $2^{2.9}$	1E-11; $2^{5.0}$
2^8	2.1E-6; $2^{3.0}$	9E-12; $--$	1.6E-7; $2^{3.0}$	4E-13; $--$
2^9	2.7E-7; $--$		2.0E-8; $--$	

TABLE III
THE CONDITION NUMBER FOR EXAMPLE 2.

2^n	2^5	2^6	2^7	2^8	2^9
$\|\lambda_{\min}\|$	0.272	0.136	0.068	0.034	0.017
$\|\lambda_{\max}\|$	18.68	18.67	18.67	18.67	18.67
Cond	68.58	1.372E+2	2.743E+2	5.487E+2	1.097E+3

TABLE IV
ERROR e^h, ERROR RATIO r^h, AND ERROR e_E^h OF u FOR EXAMPLE 2.

t	$t=0$		$t=\frac{\pi}{2}$	
n	e^h ; r^h	e_E^h; r_E^h	e^h; r^h	e_E^h; r_E^h
2^3	5.3E-3; $2^{1.2}$	1.9E-3; 2^{11}	1.4E-2; $2^{3.6}$	7.2E-4; 2^{10}
2^4	2.3E-3; $2^{2.9}$	5.7E-7; $2^{4.6}$	1.1E-3; $2^{3.0}$	4.8E-7; $2^{5.4}$
2^5	2.9E-4; $2^{3.0}$	2.3E-8; $2^{4.9}$	1.4E-4; $2^{3.0}$	1.2E-8; $2^{5.0}$
2^6	3.6E-5; $2^{3.0}$	8E-10; $2^{4.9}$	1.8E-5; $2^{2.9}$	4E-10; $2^{5.0}$
2^7	4.5E-6; $2^{3.0}$	3E-11; $2^{4.1}$	2.2E-6; $2^{3.0}$	1E-11; $2^{5.1}$
2^8	5.6E-7; $2^{3.0}$	2E-12; $--$	2.8E-7; $2^{3.0}$	5E-13; $--$
2^9	7.0E-8; $--$		3.5E-8; $--$	

$log_2(r^h(t)) \approx 3$ $(t = 0, \pi/2)$, and $log_2(r_E^h(t)) \approx 5$ $(t = 0, \pi/2)$, which are consistent with Theorem 2.6, Theorem 3.1 and (27).

Example 2. Let $\partial\Omega$ denote the boundary of the oblate circle, where $\partial\Omega = \{(x_1, x_2) : \ x_1^2/a^2 + x_2^2/b^2 = 1, \ a = 1, \ b = 1/3\}$. The Dirichlet boundary condition is given by $u(x_1,\ x_2) = x_1^2 - 2x_2^2 + x_1x_2/3 + x_1 + x_2/3 + 1$. parameter values used are $\kappa_1 = 1.0$ and $\kappa_2 = 0.5$

Tables III, and IV, accord with our error analysis, where the numbers of nodes n is $2\pi/h$. $e^h(t) = |u(t) - u^h(t)|$ gives the absolute error at a given point $(\text{cos}t, \frac{1}{3}\text{sin}t)(t = 0, \pi/2)$; $r^h(t) = e^h(t)/e^{h/2}(t)$ shows the convergence ratio by MQMs; $e_E^h(t) = |u(t) - u^*(t)|$ shows the error at the given point after using the EMs once; and $r_E^h(t) = e_E^h(t)/e_E^{h/2}(t)$ shows the convergence ratio by EMs. From the numerical results we can see that $Cond|_{2^{k+1}}/Cond|_{2^k} \approx 2$ $(k = 5, \cdots, 8)$, $log_2(r^h(t)) \approx 3$ $(t = 0, \pi/2)$, and $log_2(r_E^h(t)) \approx 5$ $(t = 0, \pi/2)$, which are consistent with Theorem 2.6, Theorem 3.1 and (27) perfectly.

V. CONCLUSIONS

To close this paper, a few concluding remarks can be made.

1. The MQMs proposed in this paper has the following advantages: (a) the quadrature formula is simple and easier to implement; (b) the optimal convergence rate is $O\ (h^3)$.

2. This paper reflects on the excellent stability for singularity problems with condition number in the order of $O\ (h^{-1})$ for MQMs. Having a small condition number is significant to singularity solutions for the first kind BIEs. This is a remarkable advantage of MQMs, which other existing numerical methods, such as the GM and EM, do not possess.

3. The convergence rate is $O\ (h^5)$ after extrapolation once, which is a significant improvement in accuracy.

In this paper we discussed the MQMs and EMs only for problems with a smooth closed boundary $\partial\Omega$. The discussion of problems with a nonsmooth boundary will be presented in a separate paper.

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