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Nonstational Dual Wavelet Frames in Sobolev spaces

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Abstract— In view of the good properties of nonstationary wavelet frames and the better flexibility of wavelets in Sobolev spaces, the nonstationary dual wavelet frames in a pair of dual Sobolev spaces are studied in this paper. We mainly give the oblique extension principle and the mixed extension principle for nonstationary dual wavelet frames in a pair of dual Sobolev spaces $H^s(R^d)$ and $H^{-s}(R^d)$.

Keywords—nonstationary; dual frames; dual Sobolev spaces; extension principle

I. Introduction and Preliminaries

As a redundant wavelet system, wavelet frames are easier to design and provide more flexibilities in applications. Because of this, wavelet frames have been extensively studied in the literature. In particular, wavelet frames obtained from refinable functions are of interest, due to the associated multiresolution structure and fast frame algorithms. Constructions of tight wavelet frames from a refinable function can be done by the unitary extension principle(UEP)[10]; Moreover, the dual wavelet frames can be done by the mixed extension principle(MEP)[11]. Later, more general oblique extension principle(OEP) and mixed oblique extension principle are independently developed by [3,5]. For the stationary case, it is impossible to obtain MRA-based compactly supported tight wavelet frames in $L^2(R)$ whose generators are in $C^\infty(R)$. In recent years, nonstationary spline tight wavelet frames by the OEP have been systematically studied in [1,2]. Particularly, motivated by the work of [4] and equipped with pseudo-splines[6], together with the idea of UEP, [9] constructs nonstationary $C^\infty(R)$ tight wavelet frames in $L^2(R)$ with desirable properties, especially, the symmetric property. Furthermore, it have been proved that such wavelet frames can be used to characterize Sobolev spaces of arbitrary smoothness[8]. Characterization of Sobolev norm and more general Besov norm of a function in terms of its weighted wavelet coefficient sequence has already been studied, using a pair of dual wavelet frames in $L^2(R^d)$, under the assumption that both wavelet frames must have regularity and vanishing moments simultaneously. In [7], the MEP in $L^2(R^d)$ is generalized to a pair of dual Sobolev spaces $H^s(R^d)$ and $H^{-s}(R^d)$. It completely separates the vanishing moments and regularity of two competing requirements for two systems. One can require the analysis system to have vanishing moments to achieve the sparsity, while requiring the synthesis system to have the desired order of regularity for representing functions. In this paper, we will generalize the extension principle in a pair of dual Sobolev spaces $H^s(R^d)$ and $H^{-s}(R^d)$ to the nonstationary case.

For two families of 2π-periodic trigonometric polynomials masks $\widehat{a_j}, j \in N$ and $\widehat{\widetilde{a}_j}, j \in N$, their associated nonstationary refinable functions are defined by

$$\widehat{\phi_{j-1}}(\xi) =: \widehat{a_j}(\frac{\xi}{2})\widehat{\phi_j}(\frac{\xi}{2}) = \prod_{n=1}^{\infty} \widehat{a_{n+j-1}}(2^{-n}\xi), \xi \in R^d, j \in N; \tag{1.1}$$

$$\widehat{\widetilde{\phi}_{j-1}}(\xi) =: \widehat{\widetilde{a}_j}(\frac{\xi}{2})\widehat{\widetilde{\phi}_j}(\frac{\xi}{2}) = \prod_{n=1}^{\infty} \widehat{\widetilde{a}_{n+j-1}}(2^{-n}\xi), \xi \in R^d, j \in N. \tag{1.2}$$

Wavelet functions $\psi_{j-1}^\ell, j \in N$ and $\widetilde{\psi}_{j-1}^\ell, j \in N$ $(\ell = 1, 2, \cdots, L)$ are defined by

$$\widehat{\psi_{j-1}^\ell}(\xi) =: \widehat{b_j^\ell}(\frac{\xi}{2})\widehat{\phi_j}(\frac{\xi}{2}), \quad \widehat{\widetilde{\psi}_{j-1}^\ell}(\xi) =: \widehat{\widetilde{b}_j^\ell}(\frac{\xi}{2})\widehat{\widetilde{\phi}_j}(\frac{\xi}{2}). \tag{1.3}$$

For a real number s, we denote by $H^s(R^d)$ the Sobolev space consisting of all tempered distributions f such that

$$\|f\|^2_{H^s(R^d)} =: \frac{1}{(2\pi)^d}\int_{R^d} |\hat{f}(\xi)|^2(1+\|\xi\|^2)^s d\xi < \infty,$$

where $\|\cdot\|$ denotes the Euclidean norm in R^d. $H^s(R^d)$ is a Hilbert space under the inner product

$$\langle f, g\rangle_{H^s(R^d)} =: \frac{1}{(2\pi)^d}\int_{R^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}(1+\|\xi\|^2)^s d\xi.$$

Moreover, for each $g \in H^{-s}(R^d)$, define linear functional on $H^s(R^d)$ as

$$\langle f, g\rangle =: \frac{1}{(2\pi)^d}\int_{R^d} \hat{f}(\xi)\overline{\hat{g}(\xi)} d\xi, \quad f \in H^s(R^d).$$

The spaces $H^s(R^d)$ and $H^{-s}(R^d)$ form a pair of dual spaces.

Denote $N_0 =: N \cup \{0\}$. For given $\phi_0, \psi_j^\ell (j \in N_0,\ \ell = 1, 2, \cdots, L) \in H^s(R^d)$, a properly normalized wavelet system $X^s(\phi_0;\ \{\psi_j^\ell\}_{j\in N_0,\ \ell\in\{1,2,\cdots,L\}})$ in $H^s(R^d)$ is defined as

$$\{\phi_0(\cdot - k) :\ k \in Z^d\} \cup \{\psi_{j;j,k}^{\ell,s} :\ j \in N_0,\ \ell = 1, 2, \cdots, L\}$$

with $\psi_{j;j,k}^{\ell,s} =: 2^{-js}\psi_{j;j,k}^\ell = 2^{j(\frac{d}{2}-s)}\psi_j^\ell(2^j \cdot -k)$. We say that $X^s(\phi_0;\ \{\psi_j^\ell\}_{j\in N_0,\ \ell\in\{1,2,\cdots,L\}})$ is a nonstationary wavelet frame in $H^s(R^d)$ if there exist positive constants C_1 and C_2 such that for any $f \in H^s(R^d)$,

$$C_1\|f\|^2_{H^s(R^d)} \le \sum_{k\in Z^d} |\langle f, \phi_{0;0,k}\rangle_{H^s(R^d)}|^2 +$$

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$$\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in Z^d}|\langle f,\psi^{\ell,s}_{j;j,k}\rangle_{H^s(R^d)}|^2\le C_2\|f\|^2_{H^s(R^d)}.$$

It is called a Bessel sequence if the right-side inequality holds. Furthermore, $X^s(\phi_0;\ \{\psi^\ell_j\}_{j\in N_0,\ \ell\in\{1,2,\cdots,L\}})$ and $X^{-s}(\widetilde{\phi}_0;\ \{\widetilde{\psi}^\ell_j\}_{j\in N_0,\ \ell\in\{1,2,\cdots,L\}})$ is a pair of nonstationary dual wavelet frames in $(H^s(R^d),\ H^{-s}(R^d))$ if the following two conditions are satisfied:
(i) $X^s(\phi_0;\ \{\psi^\ell_j\}_{j\in N_0,\ \ell\in\{1,2,\cdots,L\}})$ is a nonstationary wavelet frame in $H^s(R^d)$ and $X^{-s}(\widetilde{\phi}_0;\ \{\widetilde{\psi}^\ell_j\}_{j\in N_0,\ \ell\in\{1,2,\cdots,L\}})$ is a nonstationary wavelet frame in $H^{-s}(R^d)$;
(ii) For any $f\in H^s(R^d)$ and $g\in H^{-s}(R^d)$, there have

$$\langle f,g\rangle=\sum_{k\in Z^d}\langle f,\widetilde{\phi}_{0;0,k}\rangle\langle\phi_{0;0,k},g\rangle+$$

$$\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in Z^d}\langle f,\widetilde{\psi}^{\ell,-s}_{j;j,k}\rangle\langle\psi^{\ell,s}_{j;j,k},g\rangle.$$

II. Extension principle

For a $2\pi-$ periodic trigonometric polynomial $\hat{a}$ in $d-$ variables, we denote $deg(\hat{a})$ the smallest nonnegative integer such that its Fourier coefficients vanish outside $[-deg(\hat{a}),deg(\hat{a})]^d$.

Lemma 2.1[9] Let $\widehat{a_j},\ j\in N$ be $2\pi-$ periodic trigonometric polynomials such that $\sup_{j\in N}\|\widehat{a_j}\|_{L^\infty(R^d)}<\infty$. If $\sum_{j=1}^{\infty}2^{-j}deg(\widehat{a_j})<\infty$ and $\sum_{j=1}^{\infty}|\widehat{a_j}(0)-1|<\infty$ hold, then the infinite product in (1.1) converges uniformly on every compact set of R^d and all $\phi_j,\ j\in N_0$ are well-defined compactly supported tempered distributions.

For two functions $f,g:\ R^d\to C$, define

$$[f,g]_t(\xi)=:\sum_{k\in Z^d}f(\xi+2k\pi)\overline{g(\xi+2k\pi)}(1+\|\xi+2k\pi\|^2)^t.$$

Furthermore, for our use, we define $\nu(\phi)=:\sup\{s\in R:\ [\hat{\phi}_j,\hat{\phi}_j]_s\le M,\ j\in N_0\}$.

The following lemma can be obtained by modifying the Theorem 2.3 of [7]:

Lemma 2.2 Let $\phi_j(j\in N_0)\in H^s(R^d),\ s\in R$ satisfy $[\hat{\phi}_j,\hat{\phi}_j]_t\le M$ for some $t>s$ and all $j\in N_0$. Define $\widehat{\psi_{j-1}}(2\xi)=:\widehat{b_j}(\xi)\widehat{\phi_j}(\xi),\ \xi\in R^d$, where $\widehat{b_j}(\xi)(j\in N)$ are $2\pi-$ periodic measurable functions in $d-$ variables. Assume that there exists a nonnegative number $\alpha>-s$ and a positive constant C independent of j such that

$$|\widehat{b_j}(\xi)|\le C\min(1,\ \|\xi\|^\alpha),\ \xi\in R^d.$$

Then $X^s(\phi_0;\psi_j,j\in N_0)$ is a nonstationary Bessel wavelet sequence in $H^s(R^d)$.

Lemma 2.3[8] $X^s(\phi_0;\ \{\psi^\ell_j\}_{j\in N_0,\ \ell\in\{1,2,\cdots,L\}})$ is a frame in $H^s(R^d)$ with $C_1,C_2>0$ if and only if

$$C_1\|g\|^2_{H^{-s}(R^d)}\le\sum_{k\in Z^d}|\langle g,\phi_{0;0,k}\rangle|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in Z^d}|\langle g,\psi^{\ell,s}_{j;j,k}\rangle|^2$$
$$\le C_2\|g\|^2_{H^{-s}(R^d)},\ g\in H^{-s}(R^d).$$

The following result is the OEP for nonstationary dual wavelet frames in Sobolev spaces:

Theorem 2.1 Let $\widehat{a_j},\widehat{b^\ell_j}(j\in N,\ \ell=1,2,\cdots,L)$ and $\widehat{\widetilde{a}_j},\widehat{\widetilde{b}^\ell_j}(j\in N,\ \ell=1,2,\cdots,L)$ be $2\pi-$ periodic trigonometric polynomials in $d-$ variables, which satisfy the conditions of Lemma 2.1. Suppose that $\widehat{\phi_j}$, $\widehat{\widetilde{\phi}_j}(j\in N_0)$ and $\widehat{\psi^\ell_j}$, $\widehat{\widetilde{\psi}^\ell_j}(j\in N_0)$ are defined as in (1.1), (1.2) and (1.3), $\Theta_j,\ j\in N$ are $2\pi-$period trigonometric polynomial satisfying $\Theta_j(0)=1$ and $|\Theta_j(\xi)|\le C_0$ for all $j\in N$. Moreover, the following identity holds

$$(1)\ \Theta_j(2\xi)\overline{\widehat{a_j}(\xi)}\widehat{\widetilde{a}}_j(\xi+\gamma\pi)+\sum_{\ell=1}^{L}\overline{\widehat{b^\ell_j}(\xi)}\widehat{\widetilde{b}^\ell_j}(\xi+\gamma\pi)=\Theta_{j+1}(\xi);$$

$$\Theta_j(2\xi)\overline{\widehat{a_j}(\xi)}\widehat{\widetilde{a}}_j(\xi+\gamma\pi)+\sum_{\ell=1}^{L}\overline{\widehat{b^\ell_j}(\xi)}\widehat{\widetilde{b}^\ell_j}(\xi+\gamma\pi)=0,\gamma\in\{0,1\}^d\backslash\{0\}$$

(2) for a real number $s\in R$ satisfying

$$\nu(\phi)>s\quad and\quad\nu(\widetilde{\phi})>-s,$$

there exist nonnegative numbers α and $\widetilde{\alpha}$ with $\alpha>-s$ and $\widetilde{\alpha}>s$, such that the following conditions hold for constants C,C' independent of j:

$$|\widehat{b^\ell_j}(\xi)|\le C\min(1,\ \|\xi\|^\alpha)$$

$$|\widehat{\widetilde{b}^\ell_j}(\xi)|\le C'\min(1,\ \|\xi\|^{\widetilde{\alpha}}),\ \ell=1,2,\cdots,L.$$

Let $\widehat{\eta_j}(\xi)=:\Theta_{j+1}(\xi)\widehat{\widetilde{\phi}_j}(\xi).X^s(\phi_0;\ \{\psi^\ell_j\}_{j\in N_0,\ \ell\in\{1,2,\cdots,L\}})$ and $X^{-s}(\eta_0;\ \{\widetilde{\psi}^\ell_j\}_{j\in N_0,\ \ell\in\{1,2,\cdots,L\}})$ is a pair of nonstationary dual wavelet frames in $(H^s(R^d),\ H^{-s}(R^d))$. Furthermore, there are positive constants C_1 and C_2 such that

$$C_1\|g\|^2_{H^{-s}(R^d)}\le\sum_{k\in Z^d}|\langle g,\phi_{0;0,k}\rangle|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in Z^d}|\langle g,\psi^{\ell,s}_{j;j,k}\rangle|^2$$
$$\le C_2\|g\|^2_{H^{-s}(R^d)},\ g\in H^{-s}(R^d);$$

$$C_2^{-1}\|f\|^2_{H^s(R^d)}\le\sum_{k\in Z^d}|\langle f,\eta_{0;0,k}\rangle|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in Z^d}|\langle f,\widetilde{\psi}^{\ell,-s}_{j;j,k}\rangle|^2$$
$$\le C_1^{-1}\|f\|^2_{H^s(R^d)},\ f\in H^s(R^d).\qquad(2.1)$$

Let $\Theta_j(\xi)\equiv1$, we obtain the MEP for nonstationary dual wavelet frames in Sobolev spaces:

Corollary 2.1 Let $\widehat{a_j},\widehat{b^\ell_j}(j\in N,\ \ell=1,2,\cdots,L)$, $\widehat{\widetilde{a}_j},\widehat{\widetilde{b}^\ell_j}(j\in N,\ \ell=1,2,\cdots,L)$, $\widehat{\phi_j}$, $\widehat{\widetilde{\phi}_j}(j\in N_0)$ and $\widehat{\psi^\ell_j}$, $\widehat{\widetilde{\psi}^\ell_j}(j\in N_0)$ be defined as in Theorem 2.1. Suppose that

$$\overline{\widehat{a_j}(\xi)}\widehat{\widetilde{a}}_j(\xi+\gamma\pi)+\sum_{\ell=1}^{L}\overline{\widehat{b^\ell_j}(\xi)}\widehat{\widetilde{b}^\ell_j}(\xi+\gamma\pi)=\delta_\gamma,\quad\gamma\in\{0,1\}^d;$$

If the condition (2) of Theorem 2.1 are satisfied, then $X^s(\phi_0;\ \{\psi^\ell_j\}_{j\in N_0,\ \ell\in\{1,2,\cdots,L\}})$ and $X^{-s}(\widetilde{\phi}_0;\ \{\widetilde{\psi}^\ell_j\}_{j\in N_0,\ \ell\in\{1,2,\cdots,L\}})$ is a pair of nonstationary dual wavelet frames in $(H^s(R^d),\ H^{-s}(R^d))$.

For proving the theorems,we give the following lemmas:

Lemma 2.4 If the condition of item (1) in Theorem 2.1 is satisfied, then for any $f \in H^s(R^d)$ and $g \in H^{-s}(R^d)$,

$$\sum_{\ell=1}^{L}\sum_{k\in Z^d}\langle f,\widetilde{\psi}^{\ell}_{j;j,k}\rangle\langle\psi^{\ell}_{j;j,k},g\rangle =$$

$$\sum_{k\in Z^d}\langle f,\eta_{j+1;j+1,k}\rangle\langle\phi_{j+1;j+1,k},g\rangle-\sum_{k\in Z^d}\langle f,\eta_{j;j,k}\rangle\langle\phi_{j;j,k},g\rangle.$$

Proof For simplicity, we only prove the case for $d=1$, the general case can be proved similarly. For any $h\in H^s(R)$ and $h\in H^{-s}(R)$, by Plancherel formula and Parseval identity,

$$\sum_{k\in Z}\langle f,\widetilde{h}(\cdot-k)\rangle\langle h(\cdot-k),g\rangle=\frac{1}{(2\pi)}\int_{-\pi}^{\pi}[\hat{f},\hat{\widetilde{h}}]_0(\xi)[\hat{h},\hat{g}]_0(\xi)d\xi\ .$$

Since $\langle f,\eta_{j;j,k}\rangle = 2^{-\frac{j}{2}}\langle f(2^{-j}\cdot),\eta_j(\cdot-k)\rangle$, $\langle\phi_{j;j,k},g\rangle = 2^{-\frac{j}{2}}\langle\phi_j(\cdot-k),g(2^{-j}\cdot)\rangle$, then

$$\sum_{k\in Z}\langle f,\eta_{j;j,k}\rangle\langle\phi_{j;j,k},g\rangle$$

$$=2^{-j}\sum_{k\in Z}\langle f(2^{-j}\cdot),\eta_j(\cdot-k)\rangle\langle\phi_j(\cdot-k),g(2^{-j}\cdot)\rangle$$

$$=\frac{2^{-j}}{2\pi}\int_{-\pi}^{\pi}[(f(2^{-j}\cdot))^{\wedge},\widehat{\eta_j}]_0(\xi)\cdot[\widehat{\phi_j},(g(2^{-j}\cdot))^{\wedge}]_0(\xi)d\xi$$

$$=\frac{2^j}{2\pi}\int_{-\pi}^{\pi}[\sum_m\widehat{f}(2^j\xi+2^j\cdot 2m\pi)\overline{\widehat{\eta_j}(\xi+2m\pi)}]\cdot$$

$$[\sum_n\widehat{\phi_j}(\xi+2n\pi)\overline{\widehat{g}(2^j\xi+2^j\cdot 2n\pi)}]d\xi.$$

Due to

$$\widehat{\phi_j}(\xi)=\widehat{a_{j+1}}(\frac{\xi}{2})\widehat{\phi_{j+1}}(\frac{\xi}{2})$$

$$\widehat{\eta_j}(\xi)=\Theta_{j+1}(\xi)\widehat{\widetilde{a}_{j+1}}(\frac{\xi}{2})\widehat{\widetilde{\phi}_{j+1}}(\frac{\xi}{2}),$$

we obtain

$$[\sum_m\widehat{f}(2^j\xi+2^j\cdot 2m\pi)\overline{\widehat{\eta_j}(\xi+2m\pi)}]\times$$

$$[\sum_n\widehat{\phi_j}(\xi+2n\pi)\overline{\widehat{g}(2^j\xi+2^j\cdot 2n\pi)}]$$

$$=[\sum_m\widehat{f}(2^j\xi+2^j\cdot 2m\pi)\overline{\widehat{\widetilde{\phi}_{j+1}}(\frac{\xi}{2}+m\pi)\Theta_{j+1}(\xi)\widehat{\widetilde{a}_{j+1}}(\frac{\xi}{2}+m\pi)}]$$

$$\times[\sum_n\widehat{\phi_{j+1}}(\frac{\xi}{2}+n\pi)\widehat{a_{j+1}}(\frac{\xi}{2}+n\pi)\overline{\widehat{g}(2^j\xi+2^j\cdot 2n\pi)}]$$

$$=[\sum_m\widehat{f}(2^j\xi+2^{j+1}\cdot 2m\pi)\overline{\widehat{\widetilde{\phi}_{j+1}}(\frac{\xi}{2}+2m\pi)}]\times$$

$$[\sum_n\widehat{\phi_{j+1}}(\frac{\xi}{2}+2n\pi)\overline{\widehat{g}(2^j\xi+2^{j+1}\cdot 2n\pi)}]\times$$

$$\overline{\Theta_{j+1}(\xi)\widehat{\widetilde{a}_{j+1}}(\frac{\xi}{2})\widehat{a_{j+1}}(\frac{\xi}{2})}$$

$$+[\sum_m\widehat{f}(2^j\xi+2^{j+1}\cdot 2m\pi)\overline{\widehat{\widetilde{\phi}_{j+1}}(\frac{\xi}{2}+2m\pi)}]\times$$

$$[\sum_n\widehat{\phi_{j+1}}(\frac{\xi}{2}+2n\pi+\pi)\overline{\widehat{g}(2^j\xi+2^{j+1}\cdot 2n\pi+2^{j+1}\pi)}]\times$$

$$\overline{\Theta_{j+1}(\xi)\widehat{\widetilde{a}_{j+1}}(\frac{\xi}{2}+\pi)}\widehat{a_{j+1}}(\frac{\xi}{2})$$

$$+[\sum_m\widehat{f}(2^j\xi+2^{j+1}\cdot 2m\pi+2^{j+1}\pi)\overline{\widehat{\widetilde{\phi}_{j+1}}(\frac{\xi}{2}+2m\pi+\pi)}]\times$$

$$[\sum_n\widehat{\phi_{j+1}}(\frac{\xi}{2}+2n\pi)\overline{\widehat{g}(2^j\xi+2^{j+1}\cdot 2n\pi)}]\times$$

$$\overline{\Theta_{j+1}(\xi)\widehat{\widetilde{a}_{j+1}}(\frac{\xi}{2}+\pi)}\widehat{a_{j+1}}(\frac{\xi}{2})$$

$$+[\sum_m\widehat{f}(2^j\xi+2^{j+1}\cdot 2m\pi+2^{j+1}\pi)\overline{\widehat{\widetilde{\phi}_{j+1}}(\frac{\xi}{2}+2m\pi+\pi)}]\times$$

$$[\sum_n\widehat{\phi_{j+1}}(\frac{\xi}{2}+2n\pi+\pi)\overline{\widehat{g}(2^j\xi+2^{j+1}\cdot 2n\pi+2^{j+1}\pi)}]\times$$

$$\overline{\Theta_{j+1}(\xi)\widehat{\widetilde{a}_{j+1}}(\frac{\xi}{2}+\pi)}\widehat{a_{j+1}}(\frac{\xi}{2}+\pi).$$

By the item (1) of Theorem 2.1, one can know

$$\overline{\Theta_{j+1}(\xi)\widehat{\widetilde{a}_{j+1}}(\frac{\xi}{2})}\widehat{a_{j+1}}(\frac{\xi}{2})=\overline{\Theta_{j+2}(\frac{\xi}{2})}-\sum_{\ell=1}^{L}\overline{\widehat{\widetilde{b}^{\ell}_{j+1}}(\frac{\xi}{2})}\widehat{b^{\ell}_{j+1}}(\frac{\xi}{2})$$

$$\overline{\Theta_{j+1}(\xi)\widehat{\widetilde{a}_{j+1}}(\frac{\xi}{2}+\pi)}\widehat{a_{j+1}}(\frac{\xi}{2}+\pi)$$

$$=\overline{\Theta_{j+2}(\frac{\xi}{2}+\pi)}-\sum_{\ell=1}^{L}\overline{\widehat{\widetilde{b}^{\ell}_{j+1}}(\frac{\xi}{2}+\pi)}\widehat{b^{\ell}_{j+1}}(\frac{\xi}{2}+\pi)$$

$$\overline{\Theta_{j+1}(\xi)\widehat{\widetilde{a}_{j+1}}(\frac{\xi}{2})}\widehat{a_{j+1}}(\frac{\xi}{2}+\pi)=-\sum_{\ell=1}^{L}\overline{\widehat{\widetilde{b}^{\ell}_{j+1}}(\frac{\xi}{2})}\widehat{b^{\ell}_{j+1}}(\frac{\xi}{2}+\pi)$$

$$\overline{\Theta_{j+1}(\xi)\widehat{\widetilde{a}_{j+1}}(\frac{\xi}{2}+\pi)}\widehat{a_{j+1}}(\frac{\xi}{2})=-\sum_{\ell=1}^{L}\overline{\widehat{\widetilde{b}^{\ell}_{j+1}}(\frac{\xi}{2}+\pi)}\widehat{b^{\ell}_{j+1}}(\frac{\xi}{2}).$$

Furthermore,

$$[\sum_m\widehat{f}(2^j\xi+2^j\cdot 2m\pi)\overline{\widehat{\eta_j}(\xi+2m\pi)}]\times$$

$$[\sum_n\widehat{\phi_j}(\xi+2n\pi)\overline{\widehat{g}(2^j\xi+2^j\cdot 2n\pi)}]$$

$$=[\sum_m\widehat{f}(2^j\xi+2^{j+1}\cdot 2m\pi)\overline{\Theta_{j+2}(\frac{\xi}{2})\widehat{\widetilde{\phi}_{j+1}}(\frac{\xi}{2}+2m\pi)}]\times$$

$$[\sum_n\widehat{\phi_{j+1}}(\frac{\xi}{2}+2n\pi)\overline{\widehat{g}(2^j\xi+2^{j+1}\cdot 2n\pi)}]+$$

$$[\sum_m\widehat{f}(2^j\xi+2^{j+1}\cdot 2m\pi+2^{j+1}\pi)\overline{\Theta_{j+2}(\frac{\xi}{2}+\pi)\widehat{\widetilde{\phi}_{j+1}}(\frac{\xi}{2}+2m\pi+\pi)}]$$

$$\times[\sum_n\widehat{\phi_{j+1}}(\frac{\xi}{2}+2n\pi+\pi)\overline{\widehat{g}(2^j\xi+2^{j+1}\cdot 2n\pi+2^{j+1}\pi)}]$$

$$-\sum_{\ell=1}^{L}[\sum_m\widehat{f}(2^j\xi+2^j\cdot 2m\pi)\overline{\widehat{\widetilde{\phi}_{j+1}}(\frac{\xi}{2}+m\pi)\widehat{\widetilde{b}^{\ell}_{j+1}}(\frac{\xi}{2}+m\pi)}]\times$$

$$[\sum_n \widehat{\phi_{j+1}}(\frac{\xi}{2}+n\pi)\widehat{b^\ell_{j+1}}(\frac{\xi}{2}+n\pi)\overline{\widehat{g}(2^j\xi+2^j\cdot 2n\pi)}].$$

The third item equals

$$\sum_{\ell=1}^{L}[\sum_m \widehat{f}(2^j\xi+2^j\cdot 2m\pi)\overline{\widehat{\widetilde{\psi}^\ell_j}(\xi+2m\pi)}]\times$$
$$[\sum_n \widehat{\psi^\ell_j}(\xi+2n\pi)\overline{\widehat{g}(2^j\xi+2^j\cdot 2n\pi)}].$$

Therefore, we have the following result by integrating

$$\frac{2^j}{2\pi}\int_{-\pi}^{\pi}\sum_{\ell=1}^{L}[\sum_m \widehat{f}(2^j\xi+2^j\cdot 2m\pi)\overline{\widehat{\widetilde{\psi}^\ell_j}(\xi+2m\pi)}]\times$$
$$[\sum_n \widehat{\psi^\ell_j}(\xi+2n\pi)\overline{\widehat{g}(2^j\xi+2^j\cdot 2n\pi)}]d\xi$$
$$=\sum_{\ell=1}^{L}\sum_{k\in Z}\langle f,\widetilde{\psi}^\ell_{j;j,k}\rangle\langle\psi^\ell_{j;j,k},g\rangle.$$

Integrating for the first and the second item on $[-\pi,\pi]$ and using the $2\pi-$period, we obtain

$$\frac{2^{j+1}}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[\sum_m \widehat{f}(2^{j+1}\xi+2^{j+1}\cdot 2m\pi)\overline{\Theta_{j+2}(\xi)\widehat{\widetilde{\phi}_{j+1}}(\xi+2m\pi)}]$$
$$\cdot[\sum_n \widehat{\phi_{j+1}}(\xi+2n\pi)\overline{\widehat{g}(2^{j+1}\xi+2^{j+1}\cdot 2n\pi)}]d\xi$$
$$+\frac{2^{j+1}}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}[\sum_m \widehat{f}(2^{j+1}\xi+2^{j+1}\cdot 2m\pi)\overline{\Theta_{j+2}(\xi)\widehat{\widetilde{\phi}_{j+1}}(\xi+2m\pi)}]$$
$$\cdot[\sum_n \widehat{\phi_{j+1}}(\xi+2n\pi)\overline{\widehat{g}(2^{j+1}\xi+2^{j+1}\cdot 2n\pi)}]d\xi$$
$$=\frac{2^{j+1}}{2\pi}\int_{-\pi}^{\pi}[\sum_m \widehat{f}(2^{j+1}\xi+2^{j+1}\cdot 2m\pi)\overline{\Theta_{j+2}(\xi)\widehat{\widetilde{\phi}_{j+1}}(\xi+2m\pi)}]$$
$$\cdot[\sum_n \widehat{\phi_{j+1}}(\xi+2n\pi)\overline{\widehat{g}(2^{j+1}\xi+2^{j+1}\cdot 2n\pi)}]d\xi$$
$$=\sum_{k\in Z}\langle f,\eta_{j+1;j+1,k}\rangle\langle\phi_{j+1;j+1,k},g\rangle.$$

The final equality follows from

$$\Theta_{j+2}(\xi)\widehat{\widetilde{\phi}_{j+1}}(\xi+2m\pi)$$
$$=\Theta_{j+2}(\xi+2m\pi)\widehat{\widetilde{\phi}_{j+1}}(\xi+2m\pi)=\widehat{\eta_{j+1}}(\xi+2m\pi).$$

Finally, we obtain the desired result.

Lemma 2.5 Suppose that the $2\pi-$ periodic trigonometric polynomials $a_j,\ j\in N$ satisfies the conditions of Lemma 2.1 and define $\phi_j,\ j\in N_0$ by (1.1), then

$$\lim_{n\to+\infty}\widehat{\phi}_n(2^{-n}\xi)=1.$$

Proof By Lemma 2.1, $\phi_j,\ j\in N_0$ are well defined, which means that for all $j\in N$,

$$\lim_{n\to+\infty}\prod_{k=1}^{n}\widehat{a}_{k+j-1}(2^{-k}\xi)=\widehat{\phi}_{j-1}(\xi)=\prod_{k=1}^{\infty}\widehat{a}_{k+j-1}(2^{-k}\xi).$$

In particular, $\lim_{n\to+\infty}\prod_{k=1}^{n}\widehat{a}_k(2^{-k}\xi)=\widehat{\phi}_0(\xi)=\prod_{k=1}^{\infty}\widehat{a}_k(2^{-k}\xi)$. Note that

$$\widehat{\phi}_n(\xi)=\prod_{k=1}^{\infty}\widehat{a}_{n+k}(2^{-k}\xi).$$

Therefore,

$$\widehat{\phi}_n(2^{-n}\xi)=\prod_{k=1}^{\infty}\widehat{a}_{n+k}(2^{-(n+k)}\xi)$$
$$=\prod_{\ell=n+1}^{\infty}\widehat{a}_\ell(2^{-\ell}\xi)=\frac{\prod_{\ell=1}^{\infty}\widehat{a}_\ell(2^{-\ell}\xi)}{\prod_{\ell=1}^{n}\widehat{a}_\ell(2^{-\ell}\xi)}.$$

Obviously, we have

$$\lim_{n\to+\infty}\widehat{\phi}_n(2^{-n}\xi)=1.$$

III. Proof of Theorem

Proof of Theorem 2.1: Since $\nu(\phi)>s$, we can take t such that $\nu(\phi)>t>s$, then $[\widehat{\phi_j},\widehat{\phi_j}]_t\le M\ (j\in N_0)$ and $\phi_j\in H^s(R^d)$. By Lemma 2.2, $X^s(\phi_0;\ \{\psi^\ell_j\}_{j\in N_0,\ \ell\in\{1,2,\cdots,L\}})$ is a nonstationary Bessel sequence in $H^s(R^d)$; Similarly, $X^{-s}(\eta_0;\ \{\widetilde{\psi}^\ell_j\}_{j\in N_0,\ \ell\in\{1,2,\cdots,L\}})$ is a nonstationary Bessel sequence in $H^{-s}(R^d)$, which is then equivalent to

$$\sum_{k\in Z^d}|\langle g,\phi_{0;0,k}\rangle|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in Z^d}|\langle g,\psi^{\ell,s}_{j;j,k}\rangle|^2$$
$$\le C_2\|g\|^2_{H^{-s}(R^d)},\ g\in H^{-s}(R^d);$$
$$\sum_{k\in Z^d}|\langle f,\eta_{0;0,k}\rangle|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in Z^d}|\langle f,\widetilde{\psi}^{\ell,-s}_{j;j,k}\rangle|^2$$
$$\le\frac{1}{C_1}\|f\|^2_{H^s(R^d)},\ f\in H^s(R^d).$$

By Lemma 2.3, in order to show the frame property, we only need to show the left sides of (2.1).

Let $B(R^d)$ denote the set of all tempered distributions f such that $\hat{f}$ is compactly supported and $\hat{f}\in L^\infty(R^d)$, then $B(R^d)\subseteq H^\nu(R^d)$ for any $\nu\in R$. By Lemma 2.4, we obtain

$$\sum_{\ell=1}^{L}\sum_{k\in Z^d}\langle f,\widetilde{\psi}^\ell_{j;j,k}\rangle\langle\psi^\ell_{j;j,k},g\rangle$$
$$=\sum_{k\in Z^d}\langle f,\eta_{j+1;j+1,k}\rangle\langle\phi_{j+1;j+1,k},g\rangle-\sum_{k\in Z^d}\langle f,\eta_{j;j,k}\rangle\langle\phi_{j;j,k},g\rangle.$$

Therefore, we have

$$\sum_{k\in Z^d}\langle f,\eta_{0;0,k}\rangle\langle\phi_{0;0,k},g\rangle+\sum_{\ell=1}^{L}\sum_{j=0}^{n-1}\sum_{k\in Z^d}\langle f,\widetilde{\psi}^{\ell,-s}_{j;j,k}\rangle\langle\psi^{\ell,s}_{j;j,k},g\rangle$$
$$=\sum_{k\in Z^d}\langle f,\eta_{n;n,k}\rangle\langle\phi_{n;n,k},g\rangle,\ f,g\in B(R^d).$$

At the end of the proof, we will show that

$$\langle f,g\rangle=\lim_{n\to\infty}\sum_{k\in Z^d}\langle f,\eta_{n;n,k}\rangle\langle\phi_{n;n,k},g\rangle,\ f,g\in B(R^d).\quad(*)$$

Then

$$\langle f,g\rangle=\sum_{k\in Z^d}\langle f,\eta_{0;0,k}\rangle\langle\phi_{0;0,k},g\rangle+$$

$$\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in Z^d}\langle f,\widetilde{\psi}^{\ell,-s}_{j;j,k}\rangle\langle\psi^{\ell,s}_{j;j,k},g\rangle,\ f,g\in B(R^d),$$

with the series converging absolutely. Moreover, by Holder inequality,

$$|\langle f,g\rangle|^2\le(\sum_{k\in Z^d}|\langle f,\eta_{0;0,k}\rangle|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in Z^d}|\langle f,\widetilde{\psi}^{\ell,-s}_{j;j,k}\rangle|^2)$$

$$\cdot(\sum_{k\in Z^d}|\langle\phi_{0;0,k},g\rangle|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in Z^d}|\langle\psi^{\ell,s}_{j;j,k},g\rangle|^2).$$

Therefore, we have

$$|\langle f,g\rangle|^2\le\frac{1}{C_1}\|f\|^2_{H^s(R^d)}(\sum_{k\in Z^d}|\langle\phi_{0;0,k},g\rangle|^2+$$

$$\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in Z^d}|\langle\psi^{\ell,s}_{j;j,k},g\rangle|^2),f,g\in B(R^d).$$

Furthermore,

$$C_1\sup_{f\in B(R^d)\backslash\{0\}}\frac{|\langle f,g\rangle|^2}{\|f\|^2_{H^s(R^d)}}$$

$$\le\sum_{k\in Z^d}|\langle\phi_{0;0,k},g\rangle|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in Z^d}|\langle\psi^{\ell,s}_{j;j,k},g\rangle|^2,\ g\in B(R^d).$$

Since $B(R^d)$ is dense in $H^s(R^d)$, then

$$C_1\|g\|^2_{H^{-s}(R^d)}\le\sum_{k\in Z^d}|\langle\phi_{0;0,k},g\rangle|^2+\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in Z^d}|\langle\psi^{\ell,s}_{j;j,k},g\rangle|^2$$
$$\le C_2\|g\|^2_{H^{-s}(R^d)},\ g\in B(R^d).$$

Since $B(R^d)$ is dense in $H^{-s}(R^d)$, the above inequality holds for all $g\in H^{-s}(R^d)$, that is, $X^s(\phi_0;\ \{\psi^\ell_j\}_{j\in N_0,\ \ell\in\{1,2,\cdots,L\}})$ is a nonstationary wavelet frame in $H^s(R^d)$; Similarly, we know $X^{-s}(\eta_0;\ \{\widetilde{\psi}^\ell_j\}_{j\in N_0,\ \ell\in\{1,2,\cdots,L\}})$ is a nonstationary wavelet frame in $H^{-s}(R^d)$. Moreover, they are a pair of dual wavelet frames in $(H^s(R^d),\ H^{-s}(R^d))$.

Now, it remains to prove $(*)$:

$$\sum_{k\in Z^d}\langle f,\eta_{n;n,k}\rangle\langle\phi_{n;n,k},g\rangle$$

$$=\frac{2^{nd}}{(2\pi)^d}\int_{[-\pi,\pi]^d}[\hat{f}(2^n\cdot),\widehat{\eta_n}]_0(\xi)[\widehat{\phi_n},\hat{g}(2^n\cdot)]_0(\xi)d\xi$$

$$=\frac{1}{(2\pi)^d}\int_{R^d}\widehat{\phi_n}(2^{-n}\xi)\overline{\hat{g}(\xi)}[\hat{f}(2^n\cdot),\widehat{\eta_n}]_0(2^{-n}\xi)d\xi.$$

Since $f,g\in B(R^d)$, there exists a positive number N such that $\hat{f}(\xi)=\hat{g}(\xi)=0$ for all $\xi\notin[-N,N]^d$. For $n>\log_2(\frac{N}{\pi})$, it is easy to show that $\overline{\hat{g}(\xi)}\hat{f}(\xi+2k\pi2^n)=0$ for all $k\in Z^d\backslash\{0\}$ and $\xi\in R^d$. Therefore,

$$\widehat{\phi_n}(2^{-n}\xi)\overline{\hat{g}(\xi)}[\hat{f}(2^n\cdot),\widehat{\eta_n}]_0(2^{-n}\xi)$$

$$=\widehat{\phi_n}(2^{-n}\xi)\overline{\hat{g}(\xi)}\sum_{k\in Z^d}\hat{f}(\xi+2k\pi2^n)\overline{\widehat{\eta_n}(2^{-n}\xi+2k\pi)}$$

$$=\widehat{\phi_n}(2^{-n}\xi)\overline{\hat{g}(\xi)}\hat{f}(\xi)\overline{\widehat{\eta_n}(2^{-n}\xi)}=\widehat{\phi_n}(2^{-n}\xi)\overline{\widehat{\eta_n}(2^{-n}\xi)}\hat{f}(\xi)\overline{\hat{g}(\xi)}.$$

Since $[\widehat{\phi_j},\widehat{\phi_j}]_s\le M_1$ and

$$[\widehat{\eta_j},\widehat{\eta_j}]_{-s}=|\Theta_{j+1}(\xi)|^2[\widehat{\widetilde{\phi}_j},\widehat{\widetilde{\phi}_j}]_{-s}\le C_0^2[\widehat{\widetilde{\phi}_j},\widehat{\widetilde{\phi}_j}]_{-s}\le M_2$$

for all $j\in N_0$, then we obtain

$$|\widehat{\eta_j}(\xi)\overline{\widehat{\phi_j}(\xi)}|\le[|\widehat{\eta_j}|,|\widehat{\phi_j}|]_0(\xi)$$
$$\le([\widehat{\eta_j},\widehat{\eta_j}]_{-s})^{\frac{1}{2}}([\widehat{\phi_j},\widehat{\phi_j}]_s)^{\frac{1}{2}}\le\sqrt{M_1M_2}<\infty.$$

Therefore, for $n>\log_2(\frac{N}{\pi})$, we have

$$|\widehat{\phi_n}(2^{-n}\xi)\overline{\hat{g}(\xi)}[\hat{f}(2^n\cdot),\widehat{\eta_n}]_0(2^{-n}\xi)|$$
$$=|\widehat{\phi_n}(2^{-n}\xi)\overline{\widehat{\eta_n}(2^{-n}\xi)}\hat{f}(\xi)\overline{\hat{g}(\xi)}|\le\sqrt{M_1M_2}|\hat{f}(\xi)\overline{\hat{g}(\xi)}|.$$

Since $f,g\in B(R^d)$, we know $\hat{f}\bar{\hat{g}}\in L^1(R^d)$. Note that $\widehat{\eta_n}(2^{-n}\xi)=\Theta_{n+1}(2^{-n}\xi)\widehat{\widetilde{\phi}_n}(2^{-n}\xi)$, by the Lebesgue dominated convergence theorem and Lemma 2.5, then

$$\lim_{n\to\infty}\sum_{k\in Z^d}\langle f,\eta_{n;n,k}\rangle\langle\phi_{n;n,k},g\rangle$$

$$=\frac{1}{(2\pi)^d}\int_{R^d}\lim_{n\to\infty}\widehat{\phi_n}(2^{-n}\xi)\overline{\hat{g}(\xi)}[\hat{f}(2^n\cdot),\widehat{\eta_n}]_0(2^{-n}\xi)d\xi$$
$$=\frac{1}{(2\pi)^d}\int_{R^d}\lim_{n\to\infty}\widehat{\phi_n}(2^{-n}\xi)\overline{\widehat{\eta_n}(2^{-n}\xi)}\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi$$
$$=\frac{1}{(2\pi)^d}\int_{R^d}\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi=\langle f,g\rangle.$$

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