

# Instability of a nonlinear differential equation of fifth order with variable delay

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**Abstract**—In this paper, we study the instability of the zero solution to a nonlinear differential equation with variable delay. By using the Lyapunov functional approach, some sufficient conditions for instability of the zero solution are obtained.

**Keywords**—Instability, Lyapunov-Krasovskii functional, delay differential equation, fifth order.

## I. INTRODUCTION

Study on the instability of fifth order non-linear differential equations without delay is not a new topic. Significant results in this direction have been obtained by Ezeilo ([2]-[4]), Li and Duan [7], Li and Yu [8], Sadek [9], Sun and Hou [10], Tiryaki [11], Tunç ([12], [13]), Tunç and Erdogan [19], Tunç and Karta [20] and Tunç and Şevli [21] on instability for some fifth order nonlinear differential equations without delay. Throughout all of these papers, based on Krasovskii's properties (see Krasovskii [6]), the Lyapunov's second (or direct) method has been used as a basic tool to prove the results established therein. This method is one of the powerful and fruitful techniques that has over the years, gained increasing significance in studying qualitative behavior of solutions of differential equations. However, to the best of our knowledge from the literature, an author has considered instability of the solutions of fifth order non-linear differential equations with varying time delays (see Tunç [14]-[18]). Thus, it is worthwhile to continue to investigate the instability of the solutions of fifth order non-linear differential equations with varying time delays in this case.

It is worth mentioning that in 1979, Ezeilo [3] proved an instability theorem to the fifth order nonlinear differential equation without delay

$$x^{(5)} + a_1 x^{(4)} + a_2 x''' + g(x')x'' + h(x, x', x'', x''', x^{(4)}) + f(x) = 0. \quad (1)$$

In this paper, instead of Eq. (1), we consider nonlinear differential equation of the fifth order with variable deviating argument  $\tau(t)$  :

$$\begin{aligned} &x^{(5)} + a_1 x^{(4)} + a_2 x''' + g(x')x'' \\ &+ h(x(t - \tau(t)), x'(t - \tau(t)), \dots, x^{(4)}(t - \tau(t))) \\ &+ f(x(t - \tau(t))) = 0. \end{aligned} \quad (2)$$

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We write Eq. (2) in system form as follows

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= w, \\ w' &= u, \\ u' &= -a_1 u - a_2 w - g(y)z - h(x(t - \tau(t)), \dots, u(t - \tau(t))) \\ &- f(x) + \int_{t-\tau(t)}^t f'(x(s))y(s)ds, \end{aligned} \quad (3)$$

where  $a_1$  and  $a_2$  are some positive constants,  $\tau(t)$  is variable delay, the primes in Eq. (2) denote differentiation with respect to  $t$ ,  $t \in \mathbb{R}^+ = [0, \infty)$ ;  $g$ ,  $h$  and  $f$  are continuous functions in their arguments on  $\mathbb{R}$ ,  $\mathbb{R}^5$  and  $\mathbb{R}$ , respectively, and with  $f(0) = 0$ . The continuity of these functions is a sufficient condition for the existence of the solution of Eq. (2) (see [1], pp.14). It is also assumed that  $g$ ,  $h$  and  $f$  satisfy a Lipschitz condition in their respective arguments so that the uniqueness of solutions of Eq. (2) is guaranteed (see [1], pp.15). We assume in what follows that  $f$  is also differentiable, and  $x(t)$ ,  $y(t)$ ,  $z(t)$ ,  $w(t)$  and  $u(t)$  are abbreviated as  $x$ ,  $y$ ,  $z$ ,  $w$  and  $u$ , respectively.

The motivation for the current paper comes from the works of Ezeilo [3] and Tunç ([14]-[18]). Our results extend and improve the results obtained by Ezeilo [3] for the instability of the zero solution of Eq. (2). Furthermore our result complements existing results on qualitative behavior of solutions of fifth order nonlinear differential equations.

In the following theorem, we give basic idea of the method about the instability of solutions of ordinary differential equations. The following theorem, due to the Russian mathematician N. G. Cetaev's (see LaSalle and Lefschetz [5]).

**Theorem A** (Instability Theorem of Cetaev's). Let  $\Omega$  be a neighborhood of the origin. Let there be given a function  $V(x)$  and region  $\Omega_1$  in  $\Omega$  with the following properties:

- (i)  $V(x)$  has continuous first partial derivatives in  $\Omega_1$ .
- (ii)  $V(x)$  and  $\dot{V}(x)$  are positive in  $\Omega_1$ .
- (iii) At the boundary points of  $\Omega_1$  inside  $\Omega$ ,  $V(x) = 0$ .
- (iv) The origin is a boundary point of  $\Omega_1$ .

Under these conditions the origin is unstable.

Let  $r \geq 0$  be given, and let  $C = C([-r, 0], \mathbb{R}^n)$  with

$$\|\phi\| = \max_{-r \leq s \leq 0} |\phi(s)|, \phi \in C.$$

For  $H > 0$  define  $C_H \subset C$  by

$$C_H = \{\phi \in C : \|\phi\| < H\}.$$

If  $x : [-r, A) \rightarrow \mathbb{R}^n$  is continuous,  $0 < A \leq \infty$ , then, for each  $t$  in  $[0, A)$ ,  $x_t$  in  $C$  is defined by

$$x_t(s) = x(t+s), -r \leq s \leq 0, t \geq 0.$$

Let  $G$  be an open subset of  $C$  and consider the general autonomous delay differential system with finite delay

$$\dot{x} = F(x_t), x_t = x(t+\theta), -r \leq \theta \leq 0, t \geq 0,$$

where  $F : G \rightarrow \mathbb{R}^n$  is a continuous and maps closed and bounded sets into bounded sets. It follows from these conditions on  $F$  that each initial value problem

$$\dot{x} = F(x_t), x_0 = \phi \in G$$

has a unique solution defined on some interval  $[0, A)$ ,  $0 < A \leq \infty$ . This solution will be denoted by  $x(\phi)(\cdot)$  so that  $x_0(\phi) = \phi$ .

**Definition.** The zero solution,  $x = 0$ , of  $\dot{x} = F(x_t)$  is stable if for each  $\varepsilon > 0$  there exists  $\delta = \delta(\varepsilon) > 0$  such that  $\|\phi\| < \delta$  implies that  $|x(\phi)(t)| < \varepsilon$  for all  $t \geq 0$ . The zero solution is said to be unstable if it is not stable.

## II. MAIN RESULTS

Our main result is the following theorem.

**Theorem.** In addition to the basic assumptions imposed on the functions  $g, h$  and  $f$  that appear in Eq. (2), we assume that there exist non-zero constants  $a_2, a_4, a_5, M_0$  and  $M$  such that the following conditions hold:

$$a_4 > \frac{1}{4}a_2^2, f(0) = 0, f(x) \neq 0 \quad \text{when } x \neq 0, |f'(x)| \leq |a_5|,$$

$$h(x(t-\tau(t)), 0, \dots, u(t-\tau(t))) = 0,$$

$$yh(x(t-\tau(t)), \dots, u(t-\tau(t))) \geq a_4y^2.$$

If

$$\sup_{0 \leq t < \infty} \tau(t) < \frac{1}{4M_0M|a_5|},$$

then the zero solution of Eq. (2) is unstable for all arbitrary  $a_1$ .

**Proof.** We define a Lyapunov functional  $V = V(x_t, y_t, z_t, w_t, u_t)$  by

$$\begin{aligned} V &= \frac{1}{2}M[a_1z^2 - 2 \int_0^y g(\eta)\eta d\eta] \\ &+ [yz - M\{a_1yw + yu - zw + \int_0^x f(s)ds\}] \\ &- \lambda \int_{-\tau(t)}^0 \int_{t+s}^t y^2(\theta)d\theta ds, \end{aligned} \quad (4)$$

where  $s$  is a real variable such that the integral  $\int_{-\tau(t)}^0 \int_{t+s}^t y^2(\theta)d\theta ds$  is non-negative, and  $\lambda$  is a positive constant which will be determined later in the proof.

It is clear that

$$V(0, 0, \varepsilon^2, \varepsilon, 0) = M\left(\frac{1}{2}a_1\varepsilon^4 + \varepsilon^3\right) > 0$$

for all sufficiently small  $\varepsilon > 0$ , which verifies the property  $(K_1)$  of Krasovskii [6].

Using the Lyapunov functional  $V$  and (3), the time derivative of  $V$  yields that

$$\begin{aligned} \dot{V} &= Myh(x(t-\tau(t)), \dots, u(t-\tau(t))) + Mw^2 \\ &+ (a_2M + 1)yw + z^2 - My \int_{t-\tau(t)}^t f'(x(s))y(s)ds \\ &- \lambda\tau(t)y^2 + \lambda(1 - \tau'(t)) \int_{t-\tau(t)}^t y^2(s)ds. \end{aligned} \quad (5)$$

Using the assumptions of the theorem and applying the estimate  $2|mn| \leq m^2 + n^2$ , we get the following estimates for some terms included in (5):

$$yh(x(t-\tau(t)), \dots, u(t-\tau(t))) \geq a_4y^2,$$

$$\begin{aligned} &-My \int_{t-\tau(t)}^t f'(x(s))y(s)ds \\ &\geq -M|y| \int_{t-\tau(t)}^t |f'(x(s))| |y(s)| ds \\ &\geq -\frac{1}{2}M|a_5|\tau(t)y^2 - \frac{1}{2}M|a_5| \int_{t-\tau(t)}^t y^2(s)ds. \end{aligned}$$

Then, we have

$$\begin{aligned} \dot{V} &\geq Ma_2y^2 + Mw^2 + (a_2M + 1)yw + z^2 \\ &- \frac{1}{2}M|a_5|\tau(t)y^2 - \lambda\tau(t)y^2 + \{\lambda(1 - \tau'(t)) - \frac{1}{2}M|a_5|\} \\ &\times \int_{t-\tau(t)}^t y^2(s)ds \\ &= M\left[w + \frac{1}{2}M^{-1}(Ma_2 + 1)y\right]^2 \\ &+ \frac{1}{4}M^{-1}\{(4a_4 - a_2^2)M^2 - 2Ma_2 - 1\}y^2 + z^2 \\ &- \frac{1}{2}M|a_5|\tau(t)y^2 - \lambda\tau(t)y^2 + \{\lambda(1 - \tau'(t)) - \frac{1}{2}M|a_5|\} \\ &\times \int_{t-\tau(t)}^t y^2(s)ds. \end{aligned}$$

Let  $1 - \tau'(t) \geq \frac{1}{2}$  and  $\lambda = \frac{1}{2}|a_5|M$ . Hence

$$\begin{aligned} \dot{V} &\geq M\left[w + \frac{1}{2}M^{-1}(Ma_2 + 1)y\right]^2 \\ &+ \frac{1}{4}M^{-1}\{(4a_4 - a_2^2)M^2 - 2Ma_2 - 1\}y^2 \\ &+ z^2 - M|a_5|\tau(t)y^2. \end{aligned}$$

It follows that the coefficient  $(4a_4 - a_2^2)$  of  $M^2$  is positive. We can also assert that

$$(4a_4 - a_2^2)M^2 - 2Ma_2 - 1 \geq 1$$

when  $M > M_0$  for sufficiently large  $M_0 \equiv M_0(a_2, a_4) > 0$  so that

$$\dot{V} \geq M_0 \left[ w + \frac{1}{2} M_0^{-1} (M_0 a_2 + 1) y \right]^2 + \left\{ \frac{1}{4} M_0^{-1} - M |a_5| \tau(t) \right\} y^2 + z^2.$$

If

$$\sup_{0 \leq t < \infty} \tau(t) < \frac{1}{4M_0M|a_5|},$$

then

$$\dot{V} \geq M_0 \left[ w + \frac{1}{2} M_0^{-1} (M_0 a_2 + 1) y \right]^2 + \delta y^2 + z^2 > 0 \quad (6)$$

for some positive constant  $\delta$ , which verifies the property  $(K_2)$  of Krasovskii [6].

Indeed, because of (6),  $\dot{V} = 0$  necessarily implies that

$$y = 0 = z = w. \quad (7)$$

This fact in turn leads to

$$x(t) = \xi \quad (\text{constant}) \quad u = w' = 0, \quad u' = 0. \quad (8)$$

The substitutions of (7) and (8) in (3) gives that  $f(\xi) = 0$  which by the assumptions  $f(0) = 0$ ,  $f(x) \neq 0$  when  $x \neq 0$  and  $h(x(t - \tau(t)), 0, \dots, u(t - \tau(t))) = 0$ , implies that  $\xi = 0$ . Thus  $\dot{V} = 0$  implies necessarily that

$$x = 0 = y = z = w = u,$$

which verifies the remaining property  $(K_3)$  of Krasovskii [6]. By this discussion, we conclude that the zero solution of Eq. (2) is unstable. The theorem is established.

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