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Transient Thermal Stresses of Functionally Graded Thick Hollow Cylinder under the Green-Lindsay Model

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***Abstract*—**The transient thermoelastic response of thick hollow cylinder made of functionally graded material under thermal loading is studied. The generalized coupled thermoelasticity based on the Green-Lindsay model is used. The thermal and mechanical properties of the functionally graded material are assumed to be varied in the radial direction according to a power law variation as a function of the volume fractions of the constituents. The thermal and elastic governing equations are solved by using Galerkin finite element method. All the finite element calculations were done by using commercial finite element program FlexPDE. The transient temperature, radial displacement, and thermal stresses distribution through the radial direction of the cylinder are plotted.

***Keywords*—**Finite element method; Thermal stresses; Green-Lindsay theory; Functionally graded material.

I. Introduction

FUNCTIONALLY graded material (FGM) is nonhomogeneous material classified by its graded structure. The thermal and mechanical properties of the FGM are spatially varying designed to optimize the performance under thermal and mechanical loading.

FGM is designed to reduce the thermal stresses induced in the mechanical elements that are subjected to high temperatures. The FGM usually contains ceramic and metal constituents and the composition is varied as a function of the volume fractions of these constituents.

Numerous theoretical studies investigated the thermoelastic behavior in different structural components made of FGMs. Fukui et al. [1] investigated the effect of the graded composition on the thermal stresses for thick – walled tubes of FGM under uniform thermal loading. Jin and Noda [2] investigated the transient thermal stress intensity factors of the functionally graded finite space with an internal crack. Obata and Noda [3] studied the thermal stresses in functionally graded hollow spheres and cylinders with uniformly heated boundaries by using a perturbation method. Zimmerman and Lutz [4] studied the thermal stresses inside a spherical body with uniformly heated boundary by assuming the Young's modulus and the thermal expansion coefficients varying linearly in the radial direction. Ootao and Tanigawa [5] obtained the 3-D thermal stresses inside a functionally graded rectangular plate subjected to a partial heating. Wang, et. al. [6] investigated the performance of a graded plate that has some nonparallel cracks due to dynamic thermal loading. Cheng and Batra [7] presented a closed form solution for the thermomechanical deformation of a functionally graded elliptic plate rigidly clamped at the edge. Tarn [8] obtained exact solutions for functionally graded anisotropic cylinders subjected to thermal and mechanical loads. Shabana and Noda [9] investigated the thermoelastoplastic stresses in a full functionally graded plate subjected to a thermal load. Fujimoto and Noda [10] discussed the growing of two cracks in a functionally graded plate subjected to thermal shock. Darabseh and Bani Salameh [11] used parabolic heat conduction model to investigate the transient thermal stresses in a functionally graded cylinder with different thermal boundary conditions.

The objective of this paper is to study the dynamic thermoelastic response of FGM hollow circular cylinder considering the Green-Lindsay (GL) model. The thermal and mechanical properties of the FGM considered in this paper are assumed to be independent of temperature and varied in the radial direction using a power law variation as a function of the volume fractions of the material constituents. The coupled transient governing equations for the temperature and displacement fields in a FGM cylinder considering the GL theory are derived and solved. These equations are solved numerically by using Galerkin finite element method.

II. Governing Equations

The transient heat conduction equation for FGM as deduced from the generalized thermoelastic theories [12] without heat generation is

$$\begin{aligned}&(kT_{,i})_{,i}+\tau_2(k\dot{T}_{,i})_{,i}=\\&\rho c_E(\dot{T}+\tau_0\ddot{T})+(3\lambda+2\mu)\alpha T_\infty(\dot{\varepsilon}_{kk}+\tau_3\ddot{\varepsilon}_{kk})\end{aligned} \tag{1}$$

where k is the thermal conductivity, ρ the density, c_E the specific heat at constant strain, T the absolute temperature, ε_{ij} the strain tensor, λ and μ the Lame constants, α the thermal expansion coefficient, τ_0 the thermal relaxation time for temperature gradient (heat flux), τ_2 the thermal retardation time for temperature gradient (heat flux), T_∞ the initial temperature at which the cylinder is stress-free, and τ_3 the time

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parameter that describes the effect of acceleration of strain on temperature. The comma denotes partial differentiation with respect to a variable, and a repeated subscript implies summation. The dot denotes differentiation with respect to time.

For plane strain model with radially symmetric deformation, the only nonzero component of the displacement is the radial displacement component *u*. The two nonzero strain components are

$$\varepsilon_{rr} = \partial u / \partial r \qquad \varepsilon_{\phi\phi} = u / r \tag{2}$$

The temperature is assumed to be symmetrical about the axis and independent of the axial coordinate *z*. The material parameters k , ρ , and c_E are functions of the radial position *r*. Then, the heat conduction equation (1) is reduced to

$$\left(1+\tau_2 \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 T}{\partial r^2}+\frac{\partial T}{\partial r}\left(\frac{1}{r}+\frac{1}{k}\frac{dk}{dr}\right)\right)= \frac{\rho c_E}{k}\left(\dot{T}+\tau_0\ddot{T}\right)+\frac{3\lambda+2\mu}{k}\alpha T_\infty\left(\dot{\varepsilon}_{kk}+\tau_3\ddot{\varepsilon}_{kk}\right) \tag{3}$$

where $\varepsilon_{kk} = \varepsilon_{rr} + \varepsilon_{\phi\phi} = \partial u / \partial r + u / r = (1/r)\partial(ru)/\partial r$.

The equation of motion for an isotropic FGM without body forces is given by:

$$\frac{\partial \sigma_{rr}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\phi\phi}}{r}=\rho\frac{\partial^2 u}{\partial t^2} \tag{4}$$

The stress-strain relations for an isotropic FGM is given by

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - (3\lambda+2\mu)\alpha\left(T-T_\infty+\tau_1\dot{T}\right)\delta_{ij} \tag{5}$$

where σ_{ij} is the stress tensor, τ_1 the time parameter that describes the effect of temperature gradient on displacement and stress, and δ_{ij} the Kronecker's delta.

Combining (2) and (5) yields the following stress-displacement relations:

$$\begin{aligned}
\sigma_{rr} &= (\lambda+2\mu)\frac{\partial u}{\partial r}+\lambda\frac{u}{r}-(3\lambda+2\mu)\alpha\left(T-T_\infty+\tau_1\dot{T}\right)\\
\sigma_{\phi\phi} &= (\lambda+2\mu)\frac{u}{r}+\lambda\frac{\partial u}{\partial r}-(3\lambda+2\mu)\alpha\left(T-T_\infty+\tau_1\dot{T}\right)\\
\sigma_{zz} &= \lambda\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)-(3\lambda+2\mu)\alpha\left(T-T_\infty+\tau_1\dot{T}\right)
\end{aligned} \tag{6}$$

The Von Mises stress in the FGM cylinder based on the plane strain assumption is given by

$$\sigma_{V.M.} = \sqrt{\sigma_{rr}{}^2+\sigma_{\phi\phi}{}^2+\sigma_{zz}{}^2-\sigma_{rr}\sigma_{\phi\phi}-\sigma_{\phi\phi}\sigma_{zz}-\sigma_{rr}\sigma_{zz}} \tag{7}$$

The material parameters λ , μ , and α are functions of the radial position *r*. Substituting (6) into (4) leads to the following equation of motion

$$\begin{aligned}
&(\lambda+2\mu)\frac{\partial^2 u}{\partial r^2}+\left(\frac{d(\lambda+2\mu)}{dr}+\frac{\lambda+2\mu}{r}\right)\frac{\partial u}{\partial r}+\\
&\left(\frac{d\lambda}{dr}-\frac{\lambda+2\mu}{r}\right)\frac{u}{r}-\left(\frac{d(3\lambda+2\mu)}{dr}\alpha+(3\lambda+2\mu)\frac{d\alpha}{dr}\right)\left(T-T_\infty+\tau_1\dot{T}\right)\\
&-(3\lambda+2\mu)\alpha\left(\frac{\partial T}{\partial r}+\tau_1\frac{\partial \dot{T}}{\partial r}\right)=\rho\frac{\partial^2 u}{\partial t^2}
\end{aligned} \tag{8}$$

When $\tau_1 = \tau_2 = 0$, the generalized thermoelastic model is reduced to Lord-Shulman (LS) theory. When $\tau_1 = 0$, and $\tau_3 = \tau_0$, the generalized thermoelastic model is reduced to the extended LS theory. When $\tau_2 = \tau_3 = 0$, the generalized thermoelastic model is reduced to Green-Lindsay (GL) theory in which $\tau_1 \geq \tau_0 \geq 0$. When $\tau_0 = \tau_1 = \tau_2 = \tau_3 = 0$, the generalized thermoelastic model is reduced to the classical linear dynamic theory of thermoelasticity

III. MATERIAL PROPERTIES

The FGM compose of two different materials gradually spatially varying. One material is usually ceramic to resist the extreme thermal loading from the high-temperature environment due to its low thermal conductivity. The other material is a metal alloy to maintain the rigidity of the structure and to prevent damage due to thermal stresses initiated by high temperature gradient in a very short period of time.

For a composite material consists of two phases, the effective value of a material property (*P*) of the composite is computed based on the rule of mixtures by:

$$P = P_m V_m + P_c V_c \tag{9}$$

where V_m, and V_c are the volume fractions of the metal and ceramic, respectively. The volume fractions satisfy the relation $V_m + V_c = 1$. The volume fraction of the metal V_m using the power law is defined as

$$V_m = V_{m,i} + \left(V_{m,o} - V_{m,i}\right)\left(\frac{r-r_i}{r_o-r_i}\right)^n \tag{10}$$

where r_i is the inner radius of the cylinder, r_o is the outer radius of the cylinder, $V_{m,i}$ and $V_{m,o}$ are the volume fractions of the metal constituent on the inner and outer surfaces, respectively, and *n* is the power law exponent that represents the graded distribution along the radial direction. The cylinder has a linear variation for *n* = 1. As the parameter *n* increases, the cylinder becomes rich in ceramic when $V_{m,i} = 0$ and becomes rich in metal when $V_{m,i} = 1$. Conversely, as *n* decreases, the cylinder becomes rich in metal when $V_{m,i} = 0$ and becomes rich in ceramic when $V_{m,i} = 1$.

The FGM cylinder considered in this paper is made of Zirconia (ZrO2) and Titanium (Ti-6AL-4V) with negligible porosity. The mechanical and thermal properties of these materials at room temperature are shown in Table I [10]. The material properties are assumed to be independent of temperature.

The properties of FGM are calculated by substituting the volume fractions of the material constituents into the rule of mixtures (9). These properties are expressed as follows:

$$k(r) = k_m\left(\left(1-\frac{k_c}{k_m}\right)V_m+\frac{k_c}{k_m}\right) \tag{11}$$

$$\rho(r) = \rho_m\left(\left(1-\frac{\rho_c}{\rho_m}\right)V_m+\frac{\rho_c}{\rho_m}\right) \tag{12}$$

$$c_E(r) = c_{Em}\left(\left(1-\frac{c_{Ec}}{c_{Em}}\right)V_m+\frac{c_{Ec}}{c_{Em}}\right) \tag{13}$$

$$E(r) = E_m\left(\left(1-\frac{E_c}{E_m}\right)V_m+\frac{E_c}{E_m}\right) \tag{14}$$

$$\nu(r)=\nu_m\left(\left(1-\frac{\nu_c}{\nu_m}\right)V_m+\frac{\nu_c}{\nu_m}\right) \tag{15}$$

$$\alpha(r)=\alpha_m\left(\left(1-\frac{\alpha_c}{\alpha_m}\right)V_m+\frac{\alpha_c}{\alpha_m}\right) \tag{16}$$

Lame's constants are function of Young's modulus, E and Poisson's ratio, ν, as follows:

$$\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)}$$
$$\mu=\frac{E}{2(1+\nu)} \tag{17}$$
$$\lambda+2\mu=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$
$$3\lambda+2\mu=\frac{E}{1-2\nu}$$

TABLE I
MATERIAL PROPERTIES OF TITANIUM AND ZIRCONIA AT ROOM TEMPERATURE

	k, $W/(mK)$	E, GPa	α 1/K	ν	ρ, kg/m^3	c_E, $J/(kgK)$
Ti-6AL-4V	18.1	66.2	10.3×10^{-6}	0.321	4.42×10^{3}	808.3
ZrO_2	2.036	117.0	7.11×10^{-6}	0.333	5.6×10^{3}	615.6

IV. MATHEMATICAL MODEL

Initially, the temperature of the cylinder is assumed to be uniform and equal to the ambient temperature, T_∞. To represent a thermal shock, temperature at the inner surface of the cylinder is suddenly elevated to a new value, T_i, while the outer surface temperature, T_0 is maintained at the reference temperature, T_∞. Therefore, the initial and boundary conditions for the thermal field are written as

$$T(r,0)=T_\infty,\ \frac{\partial T}{\partial t}(r,0)=0,\ T(r_i,t)=T_i,\ T(r_o,t)=T_0 \tag{18}$$

The thermal stresses are free, $\sigma_{ij}=0$, at the initial temperature T_∞. The traction free boundary conditions at the inner and outer surfaces of the cylinder are assumed. Therefore, the initial and boundary conditions for the thermoelastic field are written as

$$u(r,0)=0,\qquad \frac{\partial u}{\partial t}(r,0)=0$$
$$\sigma_{rr}(r_i,t)=0,\qquad \sigma_{rr}(r_0,t)=0 \tag{19}$$

The governing equations can be written in a more suitable form by using the following dimensionless variables

$$\theta=\frac{T-T_\infty}{T_i-T_o}\qquad R=\frac{r}{r_i}\qquad \zeta=\frac{t}{t_o}\qquad t_o=\frac{\rho_m c_{Em}}{k_m}r_i^2$$

$$\beta_i=\frac{\tau_i}{t_o}(i=0,1,2,3)\qquad k^*=\frac{k}{k_m}\qquad \rho^*=\frac{\rho}{\rho_m}\qquad c_E^*=\frac{c_E}{c_{Em}}$$

$$E^*=\frac{E}{E_m}\qquad \alpha^*=\frac{\alpha}{\alpha_m}\qquad U=\frac{u}{r_i\alpha_m(T_i-T_o)}\qquad R^*=\frac{r_o}{r_i}$$

$$\lambda^*=\frac{\lambda}{E_m}\qquad \mu^*=\frac{\mu}{E_m}\qquad \delta_1=\frac{E_m\alpha_m^2T_\infty}{\rho_m c_{Em}}\qquad \delta_2=\frac{k_m}{t_0E_mc_{Em}}$$

$$S_i=\frac{\sigma_{ii}}{E_m\alpha_m(T_i-T_o)}(i=r,\phi,z)$$

where δ_1 represents the non-dimensional thermoelastic coupling constant.

By using these dimensionless variables, the following non-dimensional governing equations obtained

$$\left(1+\beta_2\frac{\partial}{\partial\zeta}\right)\left(\frac{\partial^2\theta}{\partial R^2}+\frac{\partial\theta}{\partial R}\left(\frac{1}{R}+\frac{1}{k^*}\frac{dk^*}{dR}\right)\right)= \tag{20}$$
$$\frac{\rho^*c_E^*}{k^*}\left(\frac{\partial\theta}{\partial\zeta}+\beta_0\frac{\partial^2\theta}{\partial\zeta^2}\right)+\delta_1\frac{(3\lambda^*+2\mu^*)\alpha^*}{k^*}\left(\frac{\partial}{\partial\zeta}+\beta_3\frac{\partial^2}{\partial\zeta^2}\right)\left(\frac{1}{R}\frac{\partial}{\partial R}(RU)\right)$$

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R(\lambda^*+2\mu^*)\frac{\partial U}{\partial R}\right)+\left(\frac{d\lambda^*}{dR}-\frac{\lambda^*+2\mu^*}{R}\right)\frac{U}{R} \tag{21}$$
$$-\frac{\partial}{\partial R}\left((3\lambda^*+2\mu^*)\alpha^*\left(\theta+\beta_1\frac{\partial\theta}{\partial\zeta}\right)\right)=\delta_2\rho^*\frac{\partial^2U}{\partial\zeta^2}$$

$$S_r=(\lambda^*+2\mu^*)\frac{\partial U}{\partial R}+\lambda^*\frac{U}{R}-(3\lambda^*+2\mu^*)\alpha^*\left(\theta+\beta_1\frac{\partial\theta}{\partial\zeta}\right)$$
$$S_\phi=(\lambda^*+2\mu^*)\frac{U}{R}+\lambda^*\frac{\partial U}{\partial R}-(3\lambda^*+2\mu^*)\alpha^*\left(\theta+\beta_1\frac{\partial\theta}{\partial\zeta}\right) \tag{22}$$
$$S_z=\lambda^*\left(\frac{\partial U}{\partial R}+\frac{U}{R}\right)-(3\lambda^*+2\mu^*)\alpha^*\left(\theta+\beta_1\frac{\partial\theta}{\partial\zeta}\right)$$

Then the dimensionless initial and boundary conditions are given by:

$$\theta(R,0)=\frac{\partial\theta}{\partial\zeta}(R,0)=0,\qquad U(R,0)=\frac{\partial U}{\partial\zeta}(R,0)=0 \tag{23}$$
$$\theta(1,\zeta)=\frac{T_i-T_\infty}{T_i-T_o},\qquad \theta(R^*,\zeta)=\frac{T_o-T_\infty}{T_i-T_o},\qquad S_r(1,\zeta)=S_r(R^*,\zeta)=0$$

V. NUMERICAL RESULTS AND DISCUSSION

The FGM hollow cylinder considered in this paper has an outer radius of r_o =150 mm and inner radius of r_i =50 mm. The inner surface temperature is suddenly increased to T_i = 1800 K and then kept constant, while the outer surface temperature is maintained at the reference temperature, that is, T_o = 300 K. Also, the cylinder is assumed to be traction free at the inner and outer surfaces. The generalized coupled thermoelasticity based on the Green-Lindsay (GL) model is considered in this paper.

When $\tau_2=\tau_3=0$, the generalized thermoelastic model is reduced to Green-Lindsay (GL) theory in which $\tau_1\geq\tau_0\geq0$. The numerical values of the relaxation times ($\tau_1=\tau_0$) are assumed to be 10-10 s.

The solution of the heat conduction equation (20) and the equation of motion (21) are obtained by implementing the Galerkin finite element method. All the finite element calculations were done with a commercial finite element package FlexPDE.

Figures 1-5 show the variations at different times of the temperature, radial displacement, radial stress, hoop stress and axial stress, respectively, along the radial direction of the cylinder, for the case of $V_{m,i} = 0$, $V_{m,o} = 1$ and $n = 1$. This means that the inner and outer surfaces of the cylinder are fully ceramic and fully metal, respectively. The axial and hoop stresses are compressive at the inner surface of the cylinder and are tensile at the outer surface as shown in Figs. 4-5. The compressive stress has a maximum value on the inner surface at the beginning of the heating process and then decreases suddenly with distance from the inner surface. As shown from the figures, the extreme stress gradients occur at the heated inner surface. The thermal stress distributions approach the steady state distribution as the time increases.

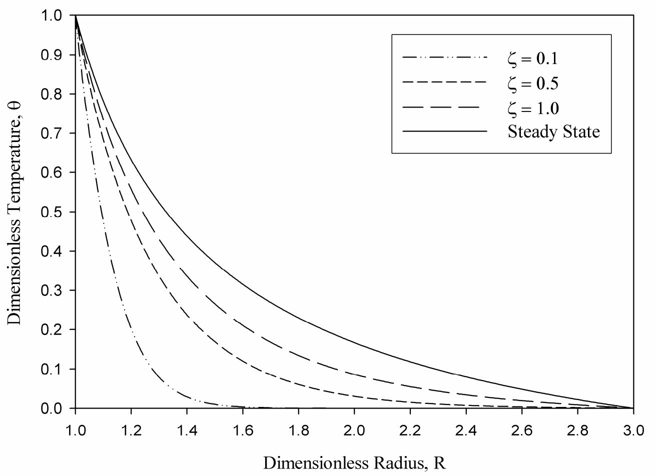

Fig. 1 Transient distribution of the dimensionless temperature along the dimensionless radius for $V_{m,i} = 0$, $V_{m,o} = 1$ and $n = 1$.

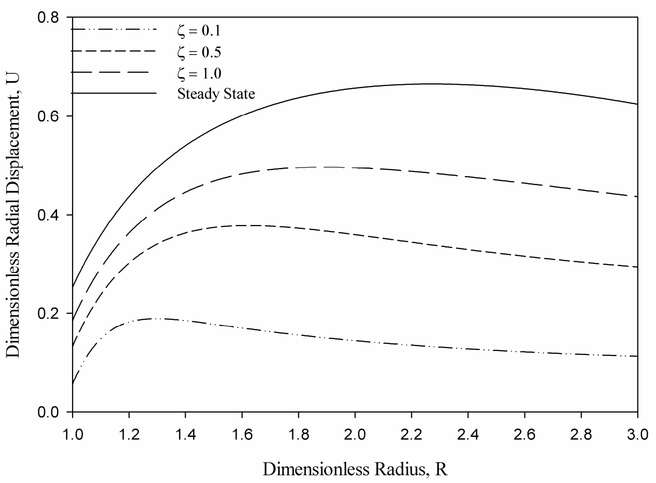

Fig. 2 Transient distribution of the dimensionless radial displacement along the dimensionless radius for $V_{m,i} = 0$, $V_{m,o} = 1$ and $n = 1$.

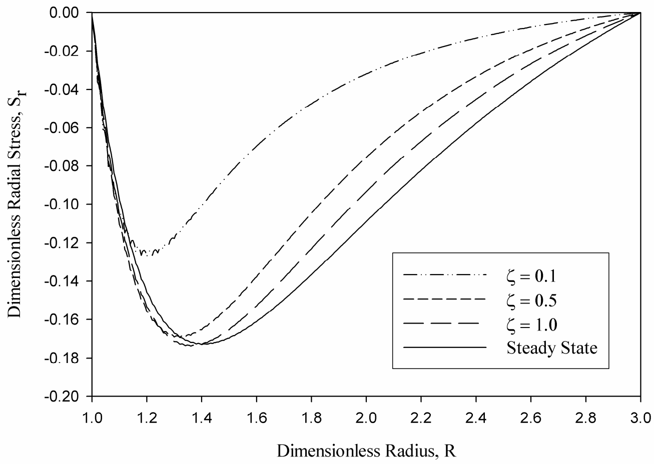

Fig. 3 Transient distribution of the dimensionless radial stress along the dimensionless radius for $V_{m,i} = 0$, $V_{m,o} = 1$ and $n = 1$.

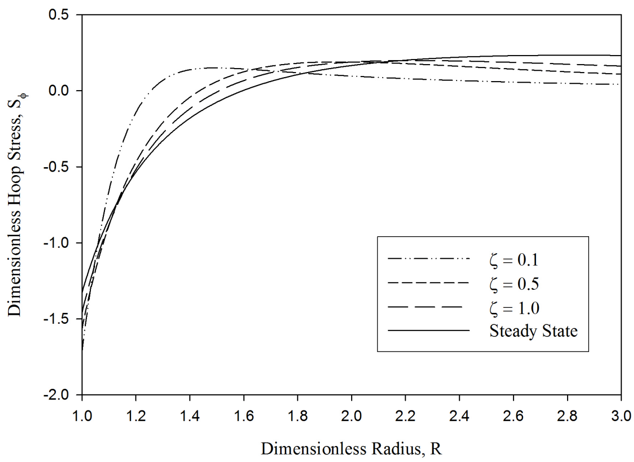

Fig. 4 Transient distribution of the dimensionless hoop stress along the dimensionless radius for $V_{m,i} = 0$, $V_{m,o} = 1$ and $n = 1$.

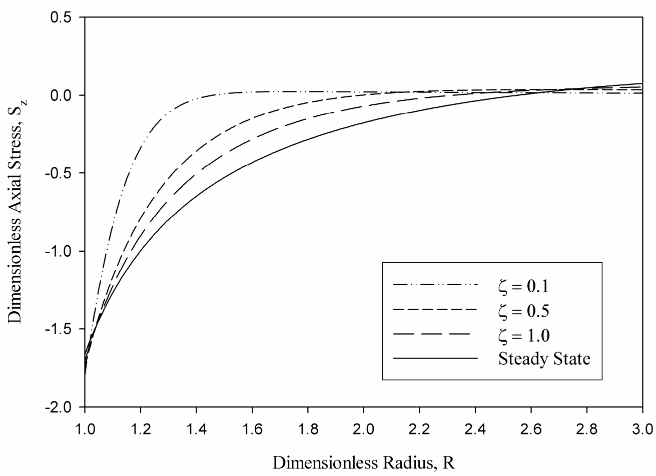

Fig. 5 Transient distribution of the dimensionless axial stress along the dimensionless radius for $V_{m,i} = 0$, $V_{m,o} = 1$ and $n = 1$.

VI. CONCLUSION

The dynamic thermoelastic response of a hollow thick cylinder made of functionally graded material subjected to thermal loading is investigated. The generalized coupled thermoelasticity theory based on the Green-Lindsay model is considered. The material composition of the cylinder is graded through the radial direction of the cylinder. The thermal and mechanical properties of the FGM cylinder are obtained by using the rule of mixtures scheme, where a power law distribution is assumed for the volume fraction of the material constituents.

The FG cylinder is assumed to be symmetrically loaded and one-dimensional transient analysis of isotropic linear thermoelastic FG cylinder under thermal loading is investigated. The heat conduction equation and the equation of motion are solved numerically by using the Galerkin finite element method. The transient temperature, radial displacement, radial stress, hoop stress and axial stress distributions through the radial direction of the cylinder are shown.

REFERENCES

[1] Y. Fukui, N. Yamanaka, and K. Wakashima, "The stresses and strains in a thick-walled tube for functionally graded material under uniform thermal loading," *JSME International Journal Series A* ,vol. 36, pp. 156-162, 1993.

[2] Z. H. Jin, and N. Noda, "Transient thermal stress intensity factors for a crack in a semi-infinite plate of a functionally gradient material," *International Journal of Solids and Structures*, vol. 31, pp. 203-218, 1994.

[3] Y. Obata, and N. Noda, "Steady thermal stresses in a hollow circular cylinder and a hollow sphere of a functionally gradient material," *Journal of Thermal Stresses*, vol. 17, pp. 471-487, 1994.

[4] M. P. Lutz, and R. W. Zimmerman, "Thermal stresses and effective thermal expansion coefficient of a functionally gradient sphere," *Journal of Thermal Stresses*, vol. 19, pp. 39-54, 1996.

[5] Y. Ootao, and Y. Tanigawa, "Three-dimensional transient thermal stresses of functionally graded rectangular plate due to partial heating," *Journal Thermal Stresses*, vol. 55, pp. 22-35, 1999.

[6] B. L. Wang , J. C. Han, and S. Y. Du, "Crack problem for functionally graded materials under transient thermal loading," *Journal of Thermal Stresses*, vol. 23, pp. 143-168, 2000.

[7] Z. Q. Cheng, and R. C. Batra, "Three-dimensional thermoelastic deformation of a functionally graded elliptic plate," *Composites Part B: Engineering*, vol. 31, pp. 97-106, 2000.

[8] J. Q. Tarn, "Exact solutions for functionally graded anisotropic cylinders subjected to thermal and mechanical loads," *International Journal of Solids and Structures*, vol. 38, pp. 8189-8206, 2001.

[9] Y. M. Shabana, and N. Noda, "Combined microscopic analysis of thermoelastoplastic stresses of functionally graded material plate," *Journal of Thermal Stresses*, vol. 24, pp. 799-815, 2001.

[10] T. Fujimoto, and N. Noda, "Two crack growths in a functionally graded plate under thermal shock," *Journal of Thermal Stresses*, vol. 24, pp. 847-862, 2001.

[11] T. T. Darabseh, and K. Bani Salameh, "Numerical solution of transient thermal stresses in a functionally graded cylinder," in *3rd WSEAS International Conference on Engineering Mechanics, Structures, Engineering Geology*, Corfu Island, Greece, 2010, pp.89-96.

[12] A. E. Green, and K. A. Lindsay, "Thermoelasticity," *Journal of Elasticity*, vol. 2, pp. 1-7, 1972.