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Numerical solution of Riccati differential equations by using hybrid functions and tau method

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Abstract—A numerical method for Riccati equation is presented in this work. The method is based on the replacement of unknown functions through a truncated series of hybrid of block-pulse functions and Chebyshev polynomials. The operational matrices of derivative and product of hybrid functions are presented. These matrices together with the tau method are then utilized to transform the differential equation into a system of algebraic equations. Corresponding numerical examples are presented to demonstrate the accuracy of the proposed method.

Keywords—Hybrid functions, Riccati differential equation, Block-pulse, Chebyshev polynomials, Tau method, operational matrix.

I. Introduction

In this paper, a numerical method using hybrid of block-pulse functions and Chebyshev polynomials is presented for the following Riccati differential equation

$$u'(x) = p(x) + q(x)u(x) + r(x)u^2(x), 0 \leq x \leq X \quad (1)$$

with initial value

$$u(0) = a. \quad (2)$$

These kinds of differential equations are a class of nonlinear differential equations of much importance, and play a significant role in many fields of applied science[1]. For example, as is known, a one-dimensional static Schrödinger equation is closely related to a Riccati differential equation. Solitary wave solution of a nonlinear partial differential equation can be expressed as polynomial in two elementary functions satisfying a projective Riccati equation [2]. The Riccati differential equation is also one of the central objects of present day control theory. In fact, in the theory of control systems, the qualitative control problem has received considerable research interest. This problem is regarded as an extension of the classical result of on controllability and stability of linear systems which is relevant to such differential equations [3] . Riccati differential equations also play predominant roles in other control theory problems such as dynamic games, linear systems with Markovian jumps, and stochastic control. Another application is found in Kalman filtering systems [4] such as orbiting satellites, seasonal phenomena like river flows, and econometric models, etc. Thus, the solution methods for these equations are of great importance to engineers and scientists. Although many important differential equations can be solved by well known analytical techniques, a greater number of physically significant differential equations can not be solved. Therefore, one has to go for numerical techniques or approximate approaches for getting its solution. Recently, Adomian decomposition method(ADM) has been proposed for solving Riccati differential equations in [5], [6].Abbasbandy solved Riccati differential equations using He's variational iteration method(VIM), homopoty perturbation method(HPM) and iterated He's homotopy perturbation method and compared the accuracy of the obtained solution with the derived by Adomian decomposition method[7], [8], [9]. Gülsu applied Taylor matrix method(TMM) to solve Riccati differential equations[10]. Furthermore, Legendre wavelet method was used to solved quadratic Riccati differential equations in[11]. But few papers reported application of hybrid function to solve the Riccati differential equation.

In this paper, we introduce a new numerical method to solve Riccati differential equations. The method consists of reducing the differential equations to a set of algebraic equations by expanding the solution as hybrid functions with unknown coefficients. The operational matrices of derivative and product of hybrid functions are given. These matrices together with the tau method are then utilized to evaluate the unknown coefficients and find approximate solutions for $u(x)$.

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II. Properties of hybrid functions

A. Hybrid functions of block-pulse and Chebyshev polynomials

Hybrid functions $h_{nm}(x), n = 1, 2, \cdots, N, m = 0, 1, 2, \cdots, M-1$, are defined on the interval$[0, 1)$ as

$$h_{nm}(x) = \begin{cases} T(2Nx - 2n + 1), & x \in [\frac{n-1}{N}, \frac{n}{N}); \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

Here,$T_m(x)$are the well-known Chebyshev polynomials of order m which satisfy following recursion formula:

$$T_0(x) = 1, T_1(x) = x,$$

$$T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x).$$

The derivative of Chebyshev polynomials is a linear combination of lower order Chebyshev polynomials, in fact [13]

$$\begin{cases} T'_m(x) = 2m \sum_{k=1}^{m-1} T_k(x), & m \text{ even}; \\ T'_m(x) = 2m \sum_{k=1}^{m-1} T_k(x) + mT_0(x), & m \text{ odd}. \end{cases} \quad (4)$$

B. Function approximation

A function $u(x)$ defined over the interval 0 to 1 may be expanded as

$$u(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}h_{nm}(x) \quad (5)$$

where

$$c_{nm}=(u(x),h_{nm}(x))$$

in which $(.,.)$denotes the inner product. If $u(x)$ in (5) is truncated, then (5) can be written as

$$u(x)\approx\sum_{n=1}^{N}\sum_{m=0}^{M-1}c_{nm}h_{nm}(x)=C^TH(x)=H^T(x)C \quad (6)$$

where C and $H(x)$ are $MN\times 1$matrices given by

$$\begin{aligned}C=[&c_{10},c_{11},\cdots,c_{1M-1},c_{20},c_{21},\cdots,\\&c_{2M-1},\cdots,c_{N0},c_{NM-1}]^T\end{aligned} \quad (7)$$

$$\begin{aligned}H(x)=[&h_{10}(x),\cdots,h_{1M-1}(x),h_{20}(x),\cdots,\\&h_{2M-1}(x),\cdots,h_{N0},\cdots,h_{NM-1}(x)]^T.\end{aligned} \quad (8)$$

In (7) and (8), $c_{nm}, n=1,2,\cdots,N, m=0,1,2,\cdots,M-1$ are the coefficients expansions of the function $u(x)$in the nth subinterval$[\frac{n-1}{N},\frac{n}{N})$ and $h_{nm}(x), n=1,2,\cdots,N, m=0,1,2,\cdots,M-1$ are defined in (3).

C. Operational matrix of derivative

In the following we introduce a new method for deriving operational matrix of derivative for hybrid function.

Theorem 1: The derivative of the vector $H(x)$ defined in (8) can be expressed by

$$dH(x)/dx=DH(x) \quad (9)$$

where $D$is the $MN\times MN$matrix of derivative as follow

$$D=\begin{pmatrix}F&0&\cdots&0\\0&F&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&F\end{pmatrix}$$

in which $F$is the $M\times M$matrix. And its (i,j) element is defined as follow

$$F(i,j)=\begin{cases}2N\cdot 2(i-1), & i\text{ odd}, j=2,4,6,\cdots,i-1;\\2N\cdot(i-1), & i\text{ even}, j=1;\\2N\cdot 2(i-1), & i\text{ even}, j=3,5,7,\cdots,i-1;\\0, & \text{otherwise.}\end{cases}$$

proof:

In the interval $[\frac{n-1}{N},\frac{n}{N}], n=1,2,\cdots,N$,we can know

$$h_{nm}(x)=T_m(2Nx-2n+1), m=0,1,2,\cdots,M-1 \quad (10)$$

By differentiation with respect to x in (10) we have

$$h'_{nm}(x)=T'_m(2Nx-2n+1)$$

Applying(4) we get

$$h'_{nm}(x)=\begin{cases}4mN\sum_{k=1}^{m-1}T_k(2Nx-2n+1), & m\text{ even};\\ \begin{array}{c}4mN\sum_{k=1}^{m-1}T_k(2Nx-2n+1)+\\2NmT_0(2Nx-2n+1)\end{array}, & m\text{ odd}.\end{cases} \quad (11)$$

This function $h'_{nm}(x), m=0,1,2,\cdots,M-1$ is zero outside the interval$[\frac{n-1}{N},\frac{n}{N}]$, hence its hybrid functions expansion only have those elements of basis hybrid functions in $H(x)$ that are nonzero in the interval $[\frac{n-1}{N},\frac{n}{N}]$, i.e. $h_{nm}(x), m=0,1,2,\cdots,M-1$. So, its hybrid functions expansion has the following form

$$h'_{nm}(x)=\begin{cases}2m\cdot 2N\sum_{k=1}^{m-1}h_{nk}(x), & m\ \text{even};\\2m\cdot 2N\sum_{k=2}^{m-1}h_{nk}(x)+2Nmh_{n0}(x), & m\ \text{odd}.\end{cases}$$

So,we can get

$$F(i,j)=\begin{cases}2N\cdot 2(i-1), & i\text{ odd}, j=2,4,6,\cdots,i-1;\\2N\cdot(i-1), & i\text{ even}, j=1;\\2N\cdot 2(i-1), & i\text{ even}, j=3,5,7,\cdots,i-1;\\0, & \text{otherwise.}\end{cases}$$

D. The product operational matrix of the hybrid of block-pulse functions and Chebyshev polynomials

The following property of the product of two hybrid function vectors will also be used. Let

$$H(x)H^T(x)C=\widetilde{C}H(x) \quad (12)$$

where

$$C=(c_{10},c_{11},\cdots c_{1M-1},\cdots,c_{N0}\cdots c_{NM-1})^T$$
$$\widetilde{C}=diag(\widetilde{C}_1,\widetilde{C}_2,\cdots,\widetilde{C}_N)$$

is a $MN\times MN$ product operational matrix.And,$\widetilde{C}_i, i=1,2,3,\cdots,N$ are $M\times M$matrices given in[12]

III. Solution of Riccati differential equation

Consider (1), we approximate $p(x),q(x),r(x)$ by the way mentioned in Section3 as

$$p(x)=P^TH(x),$$
$$q(x)=Q^TH(x),$$
$$r(x)=R^TH(x).$$

Now, we assume that

$$u(x)=C^TH(x) \quad (13)$$

By using (9) we have

$$u'(x)=C^TDH(x)$$

With substituting in (1) we have

$$\begin{aligned}C^TDH(x)=&P^TH(x)+Q^TH(x)\cdot H^T(x)C\\&+R^TH(x)\cdot C^TH(x)\cdot H^T(x)\cdot C\end{aligned}$$

Applying (12) we get

$$C^TDH(x)=P^TH(x)+Q^T\widetilde{C}H(x)+C^T\widetilde{C}\widetilde{R}H(x). \quad (14)$$

The residual $R(x)$for (1)can be written as

$$R(x) = C^T DH(x) - P^T H(x) - Q^T \widetilde{C} H(x) - C^T \widetilde{C} \widetilde{R} H(x). \quad (15)$$

As in a typical tau method, we generate $NM - 1$ equations by applying

$$\int_0^1 h_j(x) R(x) = 0, j = 1, 2, \cdots, NM - 1$$

where $h_j(x) = h_{nm}(x)$defined in (3) and

$$j = (n-1)M + (m+1), n = 1, 2, \cdots, N, M = 0, 1, 2, \cdots, M-1.$$

Also, by substituting initial condition (2), we have

$$u(0) = C^T H(0) = a. \quad (16)$$

Equation (15)and (16) generate NM set of nonlinear equations. These equations can be solved for unknown coefficients of the vector C.

IV. Accuracy of solution

We can easily verify the accuracy of the method. Given that the truncated hybrid function in (6) is an approximate solution of (1), it must have approximately satisfied these equations. Thus, for each $x_i \in [0, X]$

$$E(x_i) = C^T DH(x_i) - p(x_i) - q(x_i) \cdot C^T H(x_i) \\ -r(x_i) \cdot C^T \widetilde{C} \cdot H(x_i) \approx 0$$

If max $E(x_i) = 10^{-k}$(k is any positive integer) is prescribed, then the truncation limit N, M are increased until the difference at each of the points x_i becomes smaller than the prescribed.

Proposition 4.1: Let $u(x) \in H^k(-1, 1)$ (Sobolev space) $u_N(x) = \sum_{i=0}^N a_i T_i(x)$ be the best approximation polynomial of in L^2_ω-norm. Thus, the truncation error is:

$$\|u(x) - u_N(x)\|_{L^2_\omega[-1,1]} \leq C_0 N^{-k} \|u(x)\|_{H^k(-1,1)}$$

where C_0 is a positive constant, which is dependent on the selected norm and independent of $y(x)$ and N (proof [14]).

Theorem 2: Let $u(x) \in H^k(0, 1), I_n = [\frac{n-1}{N}, \frac{n}{N}]$then

$$\|u(x) - u_{NM}(x)\|_{L^2_\omega[0,1]} \leq C_0 N^{-k} \max_{0 \leq n \leq N} \|u(x)\|_{H^k(I_n)}$$

By using of Proposition 4.1 it is obvious[15].

V. Numerical examples

In this section, we applied the method proposed in this paper to solve three test problems. To show the efficiency of the present method for our problem in comparison with the exact solution we evaluate our absolute error defined by

$$E_{NM}(x) = |u(x) - u_{NM}(x)|,$$

where $u(x)$ is the exact solution, and $u_{NM}(x)$ is the approximate solution.

Example1. Let us first consider the Riccati differential equation

$$\begin{cases} u'(x) = 1 - u^2(x), & 0 \leq x \leq 1; \\ u(0) = 0. \end{cases}$$

TABLE I
Absolute error in $u(x)$ for different values of N, M for Example1

x	$N=1, M=6$	$N=2, M=6$	$N=3, M=6$
0	0	0	0
0.1	1.5904e-004	1.8804e-006	4.1401e-007
0.2	1.8375e-004	4.4248e-006	6.0186e-007
0.3	1.6514e-004	6.9937e-006	7.3747e-007
0.4	1.4753e-004	9.6184e-006	1.7323e-007
0.5	1.3908e-004	1.2314e-006	6.8524e-007
0.6	1.2918e-004	2.5904e-006	7.9810e-007
0.7	1.0992e-004	4.4563e-006	9.2621e-007
0.8	9.0561e-005	6.7831e-006	2.8318e-007
0.9	9.0893e-005	8.3251e-006	6.6469e-007
1.0	9.9844e-005	9.5567e-006	7.2660e-007

TABLE II
Comparison of the absolute error of Example2

x	Euler method	TMM($N=6$)	Proposed method
0.1	1.31046e-002	2.94100e-006	5.2700e-004
0.2	1.85470e-002	1.48756e-004	5.2820e-004
0.3	2.04833e-002	1.97190e-003	4.0012e-004
0.4	2.07262e-001	1.11989e-002	3.2721e-004
0.5	2.01354e-002	4.14865e-002	2.7498e-004

The exact solution is

$$u(x) = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

Table 1 shows the numerical results for Example 1 with $N = 1, 2, 3, M = 6$.

Example2. As the second example consider Riccati differential equation[10]:

$$\begin{cases} u'(x) = u(x) - 2u^2(x), \\ u(0) = 1. \end{cases}$$

We use hybrid function method for solving this equation with $N = 1, M = 6$. Comparison are made between Taylor matrix method and Euler method and proposed method in Table2. The results reveal that the proposed method is very effective.

Example3. As the third example consider Riccati differential equation[5], [7], [8], [9]:

$$\begin{cases} u'(x) = 1 + 2u(x) - u^2(x), & 0 \leq x \leq 4 \\ u(0) = 0. \end{cases}$$

The exact solution of this problem is

$$u(x) = 1 + \sqrt{2} tanh(\sqrt{2} + \frac{log(-1 + \sqrt{2})/(1 + \sqrt{2})}{2})$$

We use hybrid function method for solving this equation with$N = 4, M = 10$ and the solution is obtained in the interval [0,4]. Fig.1 shows the exact solution versus approximate solution obtained from the proposed method, ADM,10 iterations, VIM, 3 iterations, and HPM, 5 iterations in the interval [0,1.5]. As this figure shows ADM,VIM and HPM are very much inaccurate for solving the nonlinear Riccati differential equation, especially out of the interval [0,1].

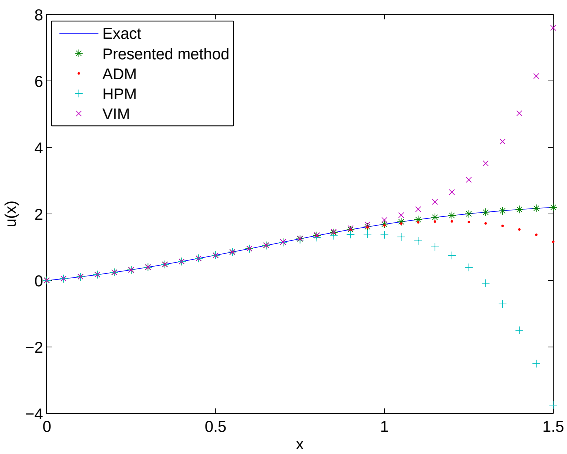

Fig. 1. The solutions solved by different methods in Example3.

VI. Conclusion

In this paper we have presented a numerical method to solve the Riccati differential equation using the hybrid block-pulse functions and Chebyshev polynomials. The properties of hybrid functions and the collocation method are used to reduce the equation to the solution of algebraic equations. The accuracy of the proposed method, other mentioned methods and exact solution are compared in Table2 and Fig.1. It is clearly seen that our numerical solutions are good agreement with the exact. Consequently, hybrid function method is very simple easy to implement and is able to approximate the solution more accurate in a bigger interval when compared to other discussed methods. The advantages of hybrid functions are that the values of N and M are adjustable as well as being able to yield more accurate numerical solutions. Also hybrid functions have good advantage in dealing with piecewise continuous functions.

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References

[1] W.T.Reid, *Riccati Differential Equations*, New York, USA: Academic Press,1972.
[2] J. F. Carinena, G. Marmo , A. M. Perelomov, and M. F. Ranada, *Related operators and exact solutions of Schrödinger equations*, International Journal of Modern Physics A 13(1998) 4913-4929.
[3] S. Bittanti, P. Colaneri and G.O. Guardabassi, *Periodic solutions of periodic Riccati equations*. IEEE Trans. Autom. Control, 29 (7) (1984) 665-667.
[4] B.D.O. Anderson,J.B. Moore, *Optimal Filtering*, Prentice-Hall, Englewood Cliffs, NJ (1979).
[5] M.A.El-Tawil, A.A.Bahnasawi, A.Abdel-Naby, *Solving Riccati differential equation using Adomians decomposition method*, Appl. math. comput. 157(2004)503-514.
[6] P. Y. Tsai, C.K. Chen, *An approximate analytic solutionof the nonlinear Riccati differential equation*, J.Franklin Inst. 347(2011)1850-1862.
[7] S.Abbasbandy, *Homotopy perturbation method for quadratic Riccati differential equation and comparison with Adomians decomposition method*, Appl. math. comput. 172(2006)485-490.
[8] S.Abbasbandy, *Iterated He's homotopy perturbation method for quadratic Riccati equation*, Appl. math. comput. 175(2006)581-589.
[9] S.Abbasbandy, *A new application of He's variational iteration method for quardratic Riccati differential equation by using Adomian's polynomials*, J. Comput. Appl. Math. 207(2007)59-63.
[10] M. Gülsu, M.Sezer, *On the solution of the Riccati equation by the Taylor matrix method*, Appl. math. comput. 176(2006)414-421.
[11] F. Mohammadi, M.M. Hosseini, *A comparative study of numerical methods for quadratic Riccati differential equations*, J.Franklin Inst. 348(2011)156-164.
[12] C. Yang, *Numerical Solution of Nonlinear Fredholm Integrodifferential Equations of Fractional Order by Using Hybrid of Block-Pulse Functions and Chebyshev Polynomials*, Mathematical Problems in Engineering, vol. 2011, Article ID 341989, 2011. doi:10.1155/2011/341989
[13] M. Sezer, M. Kaynak, *Chebyshev polynomials solutions of linear differential equations*, Int. J. Math. Educ. Sci. Technol. 27(4) (1996) 607-618.
[14] C. Canuto, M.Y. Hussaini, A.Quarteroni, T.A. Zhang, *Spectral Methods on Fluid Dynamics*, Springer-Verlag, New York, (1988).
[15] K.Maleknejad, M.Tavassoli Kajani, *Solving linear integro-differential equations system by Galerkin methods with hybrid functions*, Appl. Math. Comput. 159 , 603-612(2004).

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