Existence of solution for boundary value problems of differential equations with delay

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Abstract—In this paper , by using fixed point theorem , upper and lower solution's method and monotone iterative technique , we prove the existence of maximum and minimum solutions of differential equations with delay , which improved and generalize the result of related paper.

Keywords—Banach space , boundary value problem , differential equation , delay .

I. INTRODUCTION

THE theory of differential equation with delay is emerging as an important area of investigation since it is much richer than the corresponding theory of concerning equation without delay . Many mathematical models described by differential equation with delay are arising from neural networks ecological mathematics and automatic control areas, it has become the focus of science research . The character of the equation with delay is that the rate of change is not only depend on the current state but also the state of past time, Recently, some existence results concerning the problem of differential equation with delay have been obtained ([1-4]) , in thesis [4], by using fixed point theorem, upper and lower solution's method and monotone iterative technique, Ms Yang proved the existence of solution for the equation :

$$\begin{cases} x'(t) = f(t, x(t), x_t), \\ \alpha x(0) = x(2\pi), \quad \alpha = 1, \end{cases}$$

where $f : [0, 2\pi] \times C[0, 2\pi] \times C[-\tau, 0] \to R$ is continuous, $x \in C[[0, 2\pi], R]$, $x_t(s) = x(t + s), s \in [-\tau, 0], t \in [0, 2\pi]$ and $x_t \in C[-\tau, 0]$, $0 < \tau < 2\pi$. Motivated by the work of Yang , in this paper we study the following second-order differential equation :

$$\begin{cases} x'(t) = f(t, x(t), x_t), \\ \alpha x(0) = x(2\pi), \quad \alpha > 1, \end{cases} \tag{1}$$

if $1 > \alpha > 0$, let $s = 2\pi - t$, then it can be switched to the case of equation (1). For convenience sake , we list some preliminary lemmas .

Lemma 1.1 Let X be a complete space , if A is a contract mapping in X , then A has only one fixed point in X .

Lemma 1.2 Let $E = C[[-\tau, 2\pi], R] \bigcap C^1[0, 2\pi], R]$, $E_0 = \{x \in C[[-\tau, 2\pi], R] : x(\theta) = x(0), \theta \in [-\tau, 0]\}$, $\| x \|_0 = \max_{t \in [-\tau, 2\pi]} | x(t) |$, if $\alpha \times N \times \int_0^{2\pi} \int_{t-\tau}^t k^*(t, s) ds dt < 1$, where $N \geq 0, k^*(t, s) = k(t, s)e^{M(t-s)}, M > 0, k \in C[[0, 2\pi] \times [-\tau, 2\pi], R^+]$, then $m \in E \bigcap E_0, m' \leq -Mm -$

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$N \int_{t-\tau}^t k(t, s) m(s) ds, \alpha m(0) \leq m(2\pi) \Rightarrow m(t) \leq 0, t \in [0, 2\pi].$

Proof : From $m' \leq -Mm - N \int_{t-\tau}^t k(t, s) m(s) ds$,i.e.

$m' + Mm \leq -N \int_{t-\tau}^t k(t, s) m(s) ds$, we can get

$(me^{Mt})' \leq -N \int_{t-\tau}^t k^*(t, s) m(s) e^{Ms} ds.$

Let $v(t) = m(t) e^{Mt}$, then

$$\begin{cases} v' \leq -N \int_{t-\tau}^t k^*(t, s) v(s) ds, \\ v(2\pi) \geq \alpha v(0), \end{cases} \tag{2}$$

if we want to certify $m(t) \leq 0, t \in [0, 2\pi]$, we only need to prove $v(t) \leq 0, t \in [0, 2\pi]$, otherwise we can just suppose $\max_{t \in [0, 2\pi]} v(t) = v(\tau_1)$, because of $v(2\pi) \geq \alpha v(0)$, obviously $\tau_1 \neq 0$, so $\tau_1 \in (0, 2\pi]$, let $\min_{t \in [0, 2\pi]} v(t) = v(\tau_0)$, it is easy to see $v(\tau_0) < 0$, otherwise by(2) we can see $v(t)$ is increasing in $[0, 2\pi]$, this is a contradiction to $v(2\pi) > \alpha v(0))$.

Suppose $v(\tau_0) = -\lambda, \lambda > 0$, we consider following two different cases:
case 1: if $\tau_0 \in [0, \tau_1]$, then we have

$$\begin{aligned} v(\tau_1) =& v(\tau_0) + \int_{\tau_0}^{\tau_1} v'(t) dt \\ \leq& -\lambda - N \int_{\tau_0}^{\tau_1} \int_{t-\tau}^t k^*(t, s) v(s) ds dt \\ \leq& -\lambda + N\lambda \int_{\tau_0}^{\tau_1} \int_{t-\tau}^t k^*(t, s) ds dt \\ \leq& \lambda(-1 + N \int_0^{2\pi} \int_{t-\tau}^t k^*(t, s) ds dt), \end{aligned}$$

this is contradiction to $v(\tau_1) > 0$.
case 2 : $\tau_0 \in (\tau_1, 2\pi]$, by virtue of

$$v(2\pi) = v(\tau_0) + \int_{\tau_0}^{2\pi} v'(t) dt = -\lambda + \int_{\tau_0}^{2\pi} v'(t) dt,$$

$$v(0) = v(\tau_1) - \int_0^{\tau_1} v'(t) dt \text{ , and } v(2\pi) \geq \alpha v(0) \text{ ,}$$

we can get $-\lambda + \int_{\tau_0}^{2\pi} v'(t) dt \geq \alpha v(\tau_1) - \alpha \int_0^{\tau_1} v'(t) dt$,
i.e.

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$$\alpha v(\tau_1) \leq -\lambda + \int_{\tau_0}^{2\pi} v^{'}(t)dt + \alpha \int_0^{\tau_1} v^{'}(t)dt$$

$$\leq -\lambda + N\lambda \int_{\tau_0}^{2\pi} \int_{t-\tau}^{t} k^*(t,s)dsdt$$

$$+ \alpha N\lambda \int_0^{\tau_1} \int_{t-\tau}^{t} k^*(t,s)dsdt$$

$$\leq -\lambda + N\lambda\alpha \int_0^{2\pi} \int_{t-\tau}^{t} k^*(t,s)dsdt$$

$$= \lambda(-1 + N\alpha \int_0^{2\pi} \int_{t-\tau}^{t} k^*(t,s)dsdt) \leq 0,$$

this is contradiction to $v(\tau_1) > 0$. Summarizing the two cases above, we can get $\max_{t \in [0,2\pi]} v(t) = v(\tau_1) \leq 0$, hence $v(t) \leq 0$.

Lemma 1.3 Suppose E, E_0, M, N and k are all same to lemma 2 , $\beta, \gamma \in E$, if $(1 + \frac{1}{e^{2\pi M}\alpha - 1})N \int_0^{2\pi} \int_{t-\tau}^{t} k^*(t,s)dsdt < 1$, then for all $\eta \in [\beta, \gamma] = \{x \in E : \beta(t) \leq x(t) \leq \gamma(t), \forall t \in [-\tau, 2\pi]\}$, the following boundary value problem

$$\begin{cases} u'(t) = f(t, \eta, \eta_t) - M(u - \eta) \\ \qquad -N \int_{t-\tau}^{t} k(t,s)[u(s) - \eta(s)]ds, & (3) \\ \alpha u(0) = u(2\pi), \quad \alpha > 1, \end{cases}$$

has a unique solution in $E \bigcap E_0$.

proof: Let $\sigma(t) = f(t, \eta, \eta_t) + M\eta + N \int_{t-\tau}^{t} k(t,s)\eta(s)ds$, first we show that problem (3) is equivalent to the following integral equation's solution :

$$u(t) =$$
$$\frac{1}{e^{2M\pi}\alpha - 1} \int_0^{2\pi} (\sigma(s) - N \int_{s-\tau}^{s} k(s,\xi)u(\xi)d\xi)e^{M(s-t)}ds$$
$$+ \int_0^t (\sigma(s) - N \int_{s-\tau}^{s} k(s,\xi)u(\xi)d\xi)e^{M(s-t)}ds. \quad (4)$$

According to equation (3), $u'(s) =$
$$f(s, \eta, \eta_s) - M(u - \eta) - N \int_{s-\tau}^{s} k(s,\xi)[u(\xi) - \eta(\xi)]d\xi$$
$$= \sigma(s) - Mu(s) - N \int_{s-\tau}^{s} k(s,\xi)u(\xi)d\xi, \text{ so}$$
$$u'(s) + Mu(s) = \sigma(s) - N \int_{s-\tau}^{s} k(s,\xi)u(\xi)d\xi, \text{ which imply that}$$
$$e^{Ms}(u'(s) + Mu(s)) = e^{Ms}(\sigma(s) - N \int_{s-\tau}^{s} k(s,\xi)u(\xi)d\xi),$$
i.e.
$$(e^{Ms}u(s))^{'} = e^{Ms}(\sigma(s) - N \int_{s-\tau}^{s} k(s,\xi)u(\xi)d\xi),$$
by taking integral from 0 to t, we can get
$$e^{Mt}u(t) - u(0) = \int_0^t e^{Ms}(\sigma(s) - N \int_{s-\tau}^{s} k(s,\xi)u(\xi)d\xi)ds,$$
let $t = 2\pi$, we can get $e^{2M\pi}u(2\pi) - u(0) =$
$$\int_0^{2\pi} e^{Ms}(\sigma(s) - N \int_{s-\tau}^{s} k(s,\xi)u(\xi)d\xi)ds \text{ , by virtue of}$$
$u(2\pi) = \alpha u(0)$, we can get $(\alpha e^{2M\pi} - 1)u(0) =$

$$\int_0^{2\pi} e^{Ms}(\sigma(s) - N \int_{s-\tau}^{s} k(s,\xi)u(\xi)d\xi)ds, \text{ so } u(0) =$$
$$\frac{1}{\alpha e^{2M\pi} - 1}(\int_0^{2\pi} e^{Ms}(\sigma(s) \quad - \quad N \int_{s-\tau}^{s} k(s,\xi)u(\xi)d\xi)ds)$$
and $e^{Mt}u(t) = \int_0^t e^{Ms}(\sigma(s) - N \int_{s-\tau}^{s} k(s,\xi)u(\xi)d\xi)ds +$
$$\frac{1}{\alpha e^{2M\pi} - 1}(\int_0^{2\pi} e^{Ms}(\sigma(s) - N \int_{s-\tau}^{s} k(s,\xi)u(\xi)d\xi)ds),$$
which imply that (4) hold.
The operator A is defined by $(Au)(t) =$

$$\begin{cases} \frac{1}{e^{2M\pi}\alpha - 1} \int_0^{2\pi} (\sigma(s) - N \int_{s-\tau}^{s} k(s,\xi)u(\xi)d\xi)e^{M(s-t)}ds \\ + \int_0^t (\sigma(s) - N \int_{s-\tau}^{s} k(s,\xi)u(\xi)d\xi)e^{M(s-t)}ds, t \in [0, 2\pi], \\ \\ (Au)(0), t \in [-\tau, 0], \quad (5) \end{cases}$$

therefore $|Au - Av| =$

$$|\frac{1}{e^{2M\pi}\alpha - 1} \int_0^{2\pi} (N \int_{s-\tau}^{s} k(s,\xi)(u(\xi) - v(\xi))d\xi)e^{M(s-t)}ds$$
$$+ \int_0^t (N \int_{s-\tau}^{s} k(s,\xi)(u(\xi) - v(\xi))d\xi)e^{M(s-t)}ds|$$
$$\leq (1 + \frac{1}{e^{2\pi M}\alpha - 1})N \int_0^{2\pi} \int_{t-\tau}^{t} k^*(t,s)dsdt) \mid u - v \mid$$
$$< \mid u - v \mid, t \in [0, 2\pi].$$

So A is a contraction mapping , by lemma 1.2 we can see lemma 1.3 holds.

lemma 1.4 Let $\beta, \gamma \in E \bigcap E_0, \beta(t) < \gamma(t), t \in [-\tau, 2\pi]$, suppose the following conditions are satisfied:
$(H_0) : \beta^{'} \leq f(t, \beta, \beta_t), \alpha\beta(0) \leq \beta(2\pi); \gamma^{'} \geq f(t, \gamma, \gamma_t), \alpha\gamma(0) \geq \gamma(2\pi),$
(H_1) : for all $v(t)$ and $u(t)$, assume $\beta(t) \leq v(t) \leq u(t) \leq \gamma(t), t \in [0, 2\pi], f(t, u, u_t) - f(t, v, v_t) \geq -M(u - v) - N \int_{t-\tau}^{t} k(t,s)(u - v)ds$, where M, N, k are same to lemma 1.2,
$(H_2) : \quad (1 + \frac{1}{e^{2\pi M}\alpha - 1})N \int_0^{2\pi} \int_{t-\tau}^{t} k^*(t,s)dsdt + \alpha N \int_0^{2\pi} \int_{t-\tau}^{t} k^*(t,s)dsdt < 1,$
for all $\eta \in [\beta, \gamma]$, we define $A : A\eta = u$ (where u is a unique solution for (4), the Operator A has the following properties $(1)\beta \leq A\beta, \gamma \geq A\gamma; (2)A$ is increasing in $[\beta, \gamma]$.

Proof : (1) Let $\beta_1 = A\beta, m = \beta - \beta_1$, then $m^{'} = \beta^{'} - \beta_1^{'} \leq -Mm - N \int_{t-\tau}^{t} k(t,s)m(s)ds, \alpha m(0) \leq m(2\pi)$, by lemma 1.2, we can see $m \leq 0$, i.e. $\beta \leq A\beta$. Similarly we can get $\gamma \geq A\gamma$.

(2) For all $\eta_1, \eta_2 \in [\beta, \gamma], \eta_1 < \eta_2$, let $u_i = A\eta_i, (i = 1, 2)$ $m = u_1 - u_2$, by (H_1) we can get $m^{'} =$
$$f(t, \eta_1, \eta_{1t}) - M(u_1 - \eta_1) - N \int_{t-\tau}^{t} k(t,s)(u_1(s) - \eta_1(s))ds$$
$$- f(t, \eta_2, \eta_{2t}) + M(u_2 - \eta_2) + N \int_{t-\tau}^{t} k(t,s)(u_2(s) - \eta_2(s))ds$$
$$= -Mm + [f(t, \eta_1, \eta_{1t}) - f(t, \eta_2, \eta_{2t})] + M(\eta_1 - \eta_2)$$
$$- N \int_{t-\tau}^{t} k(t,s)m(s)ds + N \int_{t-\tau}^{t} k(t,s)(\eta_1 - \eta_2)ds$$

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$$\leq -Mm - N \int_{t-\tau}^{t} k(t,s)m(s)ds,$$

considering $\alpha m(0) = m(2\pi)$, by lemma 1.2 , we can get $m \leq 0$, so A is increasing in $[\beta, \gamma]$.

II. CONCLUSION

Theorem 2.1 If conditions in lemma 1.4 are all satisfied, then there exist monotone sequence $\{\beta_n\}$ and $\{\gamma_n\}, \beta_0 = \beta, \gamma_0 = \gamma$, which $\lim_{n\to\infty} \beta_n(t) = \rho(t), \lim_{n\to\infty} \gamma_n(t) = r(t)$. Moreover, they are convergence to the maximal and minimal solution of (1) in $[0, 2\pi]$ uniformly , i.e. $\rho, r \in E \bigcap E_0, \rho(\theta) = \rho(0), r(\theta) = r(0), \theta \in [-\tau, 0], \rho, r$ are all solutions of equation (1), and for any solution $x(t)$, we have $\rho(t) \leq x(t) \leq r(t), t \in [0, 2\pi]$.

Proof: According to lemma 1.3 , for $\forall \eta \in [\beta, \gamma]$, equation (4) has a unique solution $u \in E \bigcap E_0$, by lemma 1.4 , there exist monotone sequence $\{\beta_n\}$ and $\{\gamma_n\}$ generated by mapping $A\eta = u$, where $\beta_{n+1} = A\beta_n, \gamma_{n+1} = A\gamma_n, (\beta_0 = \beta, \gamma_0 = \gamma)$ such that in $[-\tau, 2\pi]$, $\beta \leq \beta_1 \leq \beta_2 \leq ... \leq \beta_n \leq \gamma_n \leq \gamma_{n-1} \leq ...\gamma_2 \leq \gamma_1 \leq \gamma$, $\beta_n, \gamma_n \in E \bigcap E_0, n = 0, 1, 2, ...$, and in $[0, 2\pi]$ we have

$$\begin{cases} \beta'_n = f(t, \beta_{n-1}, \beta_{n-1,t}) - M(\beta_n - \beta_{n-1}) \\ -N \int_{t-\tau}^{t} k(t,s)[\beta_n(s) - \beta_{n-1}(s)]ds, \quad (6) \\ \alpha\beta_n(0) = \beta_n(2\pi), \quad \alpha > 1, \end{cases}$$

$$\begin{cases} \gamma'_n = f(t, \gamma_{n-1}, \gamma_{n-1,t}) - M(\gamma_n - \gamma_{n-1}) \\ -N \int_{t-\tau}^{t} k(t,s)[\gamma_n(s) - \gamma_{n-1}(s)]ds, \quad (7) \\ \alpha\gamma_n(0) = \gamma_n(2\pi), \quad \alpha > 1, \end{cases}$$

by virtue of $\beta_n \in [\beta, \gamma]$ and the continuity of f, we know that $\{f(t, \beta_{n-1}, \beta_{n-1,t})\}$ is bounded uniformly in $[0, 2\pi]$, so by equation (6), it is easy to see β'_n is bounded uniformly in $[0, 2\pi]$, so β_n is bounded uniformly and equicontinuous on $[0, 2\pi]$, consequently according to Ascoli-Arzela theorem, there exists a subsequence $\{\beta_{n_i}\}$ in $[0, 2\pi]$ which is convergence uniformly, by the monotonicity , there exists $\rho \in C[0, 2\pi]$ such that $\lim_{n\to\infty} \beta_n(t) = \rho(t)$ hold uniformly in $[0, 2\pi]$, because $\beta_n \in E_0$, we can extend ρ to be a continuous function in $[-\tau.2\pi]$ such that $\rho(\theta) = \rho(0), \theta \in [-\tau, 0]$, It follow from equation (6) that ρ is a solution of equation (1).

According to $\gamma_n \in [\beta, \gamma]$ and equation (7) , similarly we can prove there exist $r(t) \in E_0, r(\theta) = r(0), \theta \in [-\tau, 0]$ such that $r(t)$ is a solution of equation (1).

By using mathematical induction method , it follows from lemma 1.2 which for any solution of equation (1), we all have $\beta_n(t) \leq x(t) \leq \gamma_n(t), t \in [0, 2\pi]$, let $n \to \infty$, so $\rho(t) \leq x(t) \leq r(t), t \in [0, 2\pi]$, i.e. ρ, r is minimal and maximal solution of equation (1) respectively.

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