Almost periodic sequence solutions of a discrete cooperation system with feedback controls

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Abstract—In this paper, we consider the almost periodic solutions of a discrete cooperation system with feedback controls. Assuming that the coefficients in the system are almost periodic sequences, we obtain the existence and uniqueness of the almost periodic solution which is uniformly asymptotically stable.

Keywords—Discrete cooperation model; Almost periodic solution; Feedback control; Lyapunov function.

I. INTRODUCTION

In Ref. [1], Cui and Chen studied the following continuous cooperation model:

\[
\begin{align*}
\dot{u} &= r_1(t)u \left(1 - \frac{u}{a_1(t) + b_1(t)v} - c_1(t)u\right), \\
\dot{v} &= r_2(t)v \left(1 - \frac{v}{a_2(t) + b_2(t)u} - c_2(t)v\right),
\end{align*}
\]

(1)

where \(r_i(t), a_i(t), b_i(t), c_i(t) (i = 1, 2)\) are continuous functions bounded above and below by positive constants. They investigated the asymptotic behavior of system (1). Also, under the assumption that \(r_1(t), a_i(t), b_i(t), c_i(t) (i = 1, 2)\) are all continuous \(T\)-periodic functions, they obtained sufficient conditions which guarantee the existence of a unique globally asymptotically stable strictly positive periodic solution of system (1).

Since many authors (see for e.g. [2], [3]) have argued that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations, then, discrete-time models can provide efficient computational types of continuous models for numerical simulations. It is reasonable to study discrete-time population models governed by difference equations.

In Ref. [4], Bai, Fan and Wang studied the existence of periodic solutions of the following discrete cooperation system:

\[
\begin{align*}
x_1(k + 1) &= x_1(k) \exp \left\{ r_1(k) \left[ 1 - \frac{x_1(k)}{a_1(k) + b_1(k)x_2(k)} - c_1(k)x_1(k) \right] \right\}, \\
x_2(k + 1) &= x_2(k) \exp \left\{ r_2(k) \left[ 1 - \frac{x_2(k)}{a_2(k) + b_2(k)x_1(k)} - c_2(k)x_2(k) \right] \right\},
\end{align*}
\]

(2)

Feedback control is the basic mechanism by which systems, whether mechanical, electrical, or biological, maintain their equilibrium or homeostasis. During the last decade, a series of mathematical models have been established to describe the dynamics of feedback control systems ([5], [6], [7], [8]).

In this paper, we are concerned with the following discrete cooperation system with feedback controls:

\[
\begin{align*}
x_1(k + 1) &= x_1(k) \exp \left\{ r_1(k) \left[ 1 - \frac{x_1(k)}{a_1(k) + b_1(k)x_2(k)} - c_1(k)x_1(k) \right] - d_1(k)u_1(k) \right\}, \\
x_2(k + 1) &= x_2(k) \exp \left\{ r_2(k) \left[ 1 - \frac{x_2(k)}{a_2(k) + b_2(k)x_1(k)} - c_2(k)x_2(k) \right] - d_2(k)u_2(k) \right\},
\end{align*}
\]

where \(x_i(k) (i = 1, 2)\) is the density of cooperation species \(i\) at the \(r\)th generation, \(r_i(k)\) denotes the intrinsic growth rate of species \(i\) and \(u_i(k) (i = 1, 2)\) is the control variables (see [1,2] and the references cited therein). Under the assumptions of almost periodicity of coefficients of system (2), we will discuss the existence and uniqueness of almost periodic solutions for system (2).

For any bounded sequence \(\{r(k)\}\), we denote

\[
r^u = \sup_{k \in N} \{r(k)\}, \quad r^l = \inf_{k \in N} \{r(k)\}.
\]

Throughout this paper, we assume that

\[
\{r_i(k)\}, \{a_i(k)\}, \{b_i(k)\}, \{c_i(k)\}, \{d_i(k)\}, \{f_i(n)\} \text{ and } \{g_i(n)\} (i = 1, 2) \text{ are bounded non-negative almost periodic sequences such that}
\]

\[
\begin{align*}
0 < r_i^l \leq r_i(k) \leq r_i^u, \\
0 < a_i^l \leq a_i(k) \leq a_i^u, \\
0 < b_i^l \leq b_i(k) \leq b_i^u, \\
0 < c_i^l \leq c_i(k) \leq c_i^u, \\
0 < d_i^l \leq d_i(k) \leq d_i^u, \\
0 < f_i^l \leq f_i(k) \leq f_i^u < 1, \\
0 < g_i^l \leq g_i(k) \leq g_i^u.
\end{align*}
\]

(3)
By the biological meaning, we focus our discussion on the positive solutions of the system (2). Hence, it is assumed that the initial conditions of (2) are of the form
\[ x_i(0) > 0, \quad u_i(0) > 0, \quad i = 1, 2. \tag{4} \]

One can easily show that the solutions of (2) with the initial condition (4) are defined and remain positive for all \( k \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\} \).

## II. Preliminaries

In this section, we will introduce two definitions and some useful lemmas.

**Definition 1.** [9] A sequence \( x : \mathbb{Z} \rightarrow \mathbb{R}^k \) is called an almost periodic sequence if the \( \varepsilon \)-translation set of \( x \)
\[ E(\varepsilon, x) := \{ \tau \in \mathbb{Z} : |x(k + \tau) - x(k)| < \varepsilon \} \]
for all \( k \in \mathbb{Z} \) is a relatively dense set in \( \mathbb{Z} \) for all \( \varepsilon > 0 \), that is, for any given \( \varepsilon > 0 \), there exists an integer \( L(\varepsilon) > 0 \) such that each discrete interval of length \( L(\varepsilon) \) contains an integer \( \tau = \tau(\varepsilon) \in E(\varepsilon, x) \) such that
\[ |x(k + \tau) - x(k)| < \varepsilon \]
for all \( k \in \mathbb{Z} \), \( \tau \) is called the \( \varepsilon \)-translation number of \( x(k) \).

**Definition 2.** [9] Let \( f : \mathbb{Z} \times D \rightarrow \mathbb{R}^k \), where \( D \) is an open set in \( \mathbb{R}^k \), \( f(k, x) \) is said to be almost periodic in \( k \)
uniformly for \( x \in D \), or uniformly almost periodic for short, if for any \( \varepsilon > 0 \) and any compact set \( S \in D \), there exists a positive integer \( L(\varepsilon, S) \) such that any interval of length \( L(\varepsilon, S) \)
contains an integer \( \tau = \tau(\varepsilon, S) \) for which
\[ |f(k + \tau, x) - f(k, x)| < \varepsilon \]
for all \( k \in \mathbb{Z} \) and \( x \in S \). \( \tau \) is called the \( \varepsilon \)-translation number of \( f(k, x) \).

**Lemma 1.** [9] If \( \{ x(k) \} \) is an almost periodic sequence, then \( \{ x(k) \} \) is bounded.

**Lemma 2.** [9] \( \{ x(k) \} \) is an almost periodic sequence if and only if for sequence \( \{ b_k \} \subset \mathbb{Z} \), there exists a subsequence \( \{ b_k \} \subset \mathbb{Z} \), such that \( x(k+b_k) \) converges uniformly on \( x \in S \) as \( k \rightarrow \infty \). Furthermore, the limit sequence is also an almost periodic sequence.

**Lemma 3.** [9] Let \( k \in \mathbb{N}_0^r = \{k_0, k_0 + 1, \ldots, k_0 + r \} \), \( r \geq 0 \). For any fixed \( k \), \( g(k, r) \) is a non-decreasing function with respect to \( r \), and for \( k \geq k_0 \), the following inequalities hold: \( g(k+1) \leq g(k, y(k)), u(k+1) \leq g(k, u(k)) \). If \( y(k_0) \leq u(k_0) \), then \( y(k) \leq u(k) \) for all \( k \geq k_0 \).

Now let us consider the following single species discrete model:
\[ N(k+1) = N(k) \exp[a(k) - b(k)N(k)], \]
where \( \{a(k)\} \) and \( \{b(k)\} \) are strictly positive sequences of real numbers defined for \( k \in \mathbb{N} = \{0, 1, 2, \ldots\} \) and \( 0 < a' < a^*, 0 < b' < b^* \).

**Lemma 4.** [10] If \( \{ x(k) \} \) with initial condition \( x(0) > 0 \) and for all \( k \in N \) satisfies
\[ x(k+1) \leq x(k) \exp[a(k) - b(k)x(k)], \]
then
\[ \lim_{k \rightarrow +\infty} \sup_{x(k)} x(k) \leq M, \]
where \( a(k) \) and \( b(k) \) are nonnegative sequences with positive bounded below, \( M = \frac{1}{b^*} \exp[a^* - 1] \).

**Lemma 5.** [10] If \( \{ x(k) \} \) satisfies
\[ x(k+1) \geq x(k) \exp[a(k) - b(k)x(k)], k \geq N_0 \]
and
\[ \lim_{k \rightarrow +\infty} \sup_{x(k)} x(k) \leq M, \]
then
\[ \lim_{k \rightarrow +\infty} \inf_{x(k)} x(k) \geq m, \]
where \( a(k) \) and \( b(k) \) are nonnegative sequences with positive bounded below, \( m = \frac{1}{b^*} \exp[a^* - b^* M] \).

## III. Persistence

In this section, we establish a persistence result for model (2).

**Theorem 1.** Assume that (3) and (4) hold, furthermore, \( (H_1) \quad r_i - d_i^u u^*_i > 0, i = 1, 2 \)
is satisfied. Then for any positive solution \( \{x_1(k), x_2(k), u_1(k), u_2(k)\} \) of (2), we have
\[ x_{i*} \leq \lim_{k \rightarrow +\infty} \inf_{x(k)} x(k) \leq \lim_{k \rightarrow +\infty} \sup_{x(k)} x(k) \leq x^*_i, \]
\[ u_{i*} \leq \lim_{k \rightarrow +\infty} \inf_{u(k)} u(k) \leq \lim_{k \rightarrow +\infty} \sup_{u(k)} u(k) \leq u^*_i, \]
\[ \tag{5} \]
where
\[ x^*_i = \frac{1}{r_i} \exp[r_i u^*_i - 1], \quad u^*_i = \frac{g_i^{x^*_i}}{f_i^u}, \quad u^*_i = \frac{g_i^{x^*_i}}{f_i^u}. \]

**Proof:** We first prove that
\[ \lim_{k \rightarrow +\infty} \sup_{x(k)} x(k) \leq x^*_i, \quad i = 1, 2. \]

By the first equation of system (2), we have
\[ x_1(k+1) = x_1(k) \exp[r_1(k)[1 - c_1(k)x_1(k)]] \]
\[ = x_1(k) \exp[r_1(k) - r_1(k)c_1(k)x_1(k)]. \]

By applying Lemma 4, we have
\[ \lim_{k \rightarrow +\infty} \sup_{x(k)} x_1(k) \leq \frac{1}{r_1^*} \exp[r_1^* - 1] \equiv x^*_1. \]
By using the second equation of system (2), similar to the above analysis, we can obtain
\[
\lim_{k \to +\infty} \sup x_2(k) \leq \frac{1}{r_2^*} \exp \{r_2^* - 1\} = x_2^*.
\] (7)

Therefore, for each \( \varepsilon > 0 \), there exists a large enough integer \( k_0 \) such that
\[
x_i(k) \leq x_i^* + \varepsilon, \quad i = 1, 2, \text{ whenever } k \geq k_0.
\]

Now we prove that
\[
\lim_{k \to +\infty} \sup u_i(k) \leq u_i^*, \quad i = 1, 2.
\]

By the third equation of system (2), we can get that
\[
u_i(k) = \prod_{i=0}^{k-1} (1 - f_i(k)) \left[ u_i(0) + \sum_{i=0}^{k-1} g_i(i) x_i(i) \right] \\
\leq (1 - f_i^1)^k (u_i(0) + v_i) + g_i^1 (x_i^* + \varepsilon) \prod_{i=k_0}^{k-1} (1 - f_i(j)) \\
\leq (1 - f_i^1)^k (u_i(0) + v_i) + g_i^1 (x_i^* + \varepsilon) \prod_{i=k_0}^{k-1} (1 - f_i^1)^{k-i-1},
\]
where \( v_i = \sum_{i=0}^{k-1} g_i(i) x_i(i) \). Since \( 0 < f_i^1 < 1 \), we can find a positive number \( d \) such that \( 1 - f_i^1 = e^{-d} \), then, by Stolz's theorem, we have
\[
\sum_{i=k_0}^{k-1} e^{d(i+1)} = e^{dk} \to \frac{1}{1 - e^{-d}} = f_i^1, \quad k \to \infty.
\]

Thus
\[
\lim_{k \to +\infty} \sup u_i(k) \leq \frac{g_i^1 x_i^*}{f_i^1} \triangleq u_i^*.
\] (8)

In the similar way, we can prove that
\[
\lim_{k \to +\infty} \sup u_2(k) \leq \frac{g_2^1 x_2^*}{f_2^1} \triangleq u_2^*.
\] (9)

Therefore, for each \( \varepsilon > 0 \), there exists \( k_0 \in N \) such that
\[
u_i(k) \leq u_i^* + \varepsilon, \quad i = 1, 2.
\]

Next, we prove that
\[
\lim_{k \to +\infty} \inf x_i(k) \geq x_i^* + \varepsilon, \quad i = 1, 2.
\]

By the first equation of system (2), we have
\[
x_1(k+1) \geq x_1(k) \exp \left\{ r_1(k) \left[ 1 - x_1(k) - c_1^1 x_1(k) \right] - d_1^u (u^*_1 + \varepsilon) \right\}
\]
\[
\geq x_1(k) \exp \left\{ r_1^* - d_1^u (u^*_1 + \varepsilon) \right\}
\]
\[
- \frac{r_1^*}{d_1^u} + r_1^* c_1^1 x_1(k).
\]

By Lemma 5, we have
\[
\lim_{k \to +\infty} \inf x_1(k) \geq \frac{r_1^* - d_1^u (u^*_1 + \varepsilon)}{\frac{r_1^*}{d_1^u} + r_1^* c_1^1} x_1^* \triangleq x_{1*}.
\] (10)

In the similar way, we can prove that
\[
\lim_{k \to +\infty} \inf x_2(k) \geq \frac{r_2^* - d_2^u (u^*_2 + \varepsilon)}{\frac{r_2^*}{d_2^u} + r_2^* c_2^u} x_2^* \triangleq x_{2*}.
\] (11)

Therefore, for each \( \varepsilon > 0 \), there exists a large enough integer \( k_0 \) such that
\[
x_i(k) > x_i^* - \varepsilon, \quad i = 1, 2.
\]

Finally, we prove that
\[
\lim_{k \to +\infty} \inf u_i(k) \geq u_i^*, \quad i = 1, 2.
\]

By the third equation of system (2), we can get that
\[
u_1(k) = \prod_{i=0}^{k-1} (1 - f_i(k)) \left[ u_1(0) + \sum_{i=0}^{k-1} g_i(i) x_i(i) \right] \\
\geq (1 - f_i^u)^k (u_1(0) + v_1) + g_1^u (x_1^* - \varepsilon) \prod_{i=k_0}^{k-1} (1 - f_i(j)) \\
\geq (1 - f_i^u)^k (u_1(0) + v_1) + g_1^u (x_1^* - \varepsilon) \prod_{i=k_0}^{k-1} (1 - f_i^u)^{k-i-1},
\]
where \( v_1 = \sum_{i=0}^{k-1} g_i(i) x_i(i) \). Since \( 0 < f_i^u < 1 \), we can find a positive number \( d \) such that \( 1 - f_i^u = e^{-d} \), then, by Stolz’s theorem, we have
\[
\sum_{i=k_0}^{k-1} e^{d(i+1)} = e^{dk} \to \frac{1}{1 - e^{-d}} = f_i^u, \quad k \to \infty.
\]
Thus
\[
\lim_{k \to +\infty} \inf u_1(k) \geq \frac{g_1 x_1^*}{f_1^*} \triangleq u_1^*.
\]
(12)

In the similar way, we can prove that
\[
\lim_{k \to +\infty} \inf u_2(k) \geq \frac{g_2 x_2^*}{f_2^*} \triangleq u_2^*.
\]
(13)

Hence, from (5)-(13), we can get that
\[
x_{i*} \leq \lim_{k \to +\infty} \inf x(k) \leq \lim_{k \to +\infty} \sup x(k) \leq x_i^*,
\]
\[u_{i*} \leq \lim_{k \to +\infty} \inf u(k) \leq \lim_{k \to +\infty} \sup u(k) \leq u_i^*,
\]
where \(i = 1, 2\). This completes the proof of Theorem 1.

IV. MAIN RESULTS

Consider the following almost periodic difference system
\[
x(n+1) = f(n, x(n)), n \in Z^+,
\]
(14)
where \(f : Z \times E_B \to R^k, S_B = \{x \in R^k : \|x\| < B\}\), and \(f(n, x)\) is almost periodic in \(x\) uniformly for \(x \in S_B\) and is continuous in \(x\). The product system of (14) is as follows:
\[
x(n+1) = f(n, x(n)), y(n+1) = g(n, y(n)).
\]
(15)

Lemma 6. [11] Suppose that there exists a Lyapunov functional \(V(n, x, y)\) defined for \(n \in N, \|x\| < B, \|y\| < B\) satisfying the following conditions:

(i) \(a(\|x - y\|) \leq V(n, x, y) \leq b(\|x - y\|)\), where \(a, b \in K\), with \(K = \{a \in C(R^+, R^+) : a(0) = 0\}\) and \(a\) is increasing;

(ii) \(\|V(n, x_1, y_1) - V(n, x_2, y_2)\| \leq L(\|x_1 - x_2\| + \|y_1 - y_2\|)\), where \(L > 0\) is a constant;

(iii) \(\Delta V_{ij}(n, x, y) \leq -a V(n, x, y), \) where \(0 \leq a < 1\) is a constant

\[\Delta V_{ij}(n, x, y) = V(n+1, f(n, x), g(n, y)) - V(n, x, y).
\]

Moreover, if there exists a solution \(\varphi(n)\) of (14) such that \(\|\varphi(n)\| \leq B^* < B\), for \(n \in Z^+\), then there exists a unique uniformly asymptotically stable almost periodic solution \(p(n)\) of system (14) which is bounded by \(B^*\). In particular, if \(f(n, x)\) is periodic of period \(\omega\), then there exists a unique uniformly asymptotically stable periodic solution of (14) of period \(\omega\).

According to Lemma 6, we first prove that there exists a bounded solution of (2), then construct an adaptive Lyapunov functional for (2).

We denote by \(\Omega\) the set of all solutions \(X(k) = (x_1(k), x_2(k), u_1(k), u_2(k))\) of system (2) satisfying \(x_{i*} \leq x_i(k) \leq x_i^*, u_{i*} \leq u_i(k) \leq u_i^*, i = 1, 2\) for all \(k \in Z^+\).

Lemma 7. Assume that (H) and the conditions of Theorem 1 hold, then \(\Omega \neq \emptyset\).

Proof: It is now possible to show by an inductive argument that the system (2) leads to
\[
x_1(k) = x_1(0) \exp \sum_{l=0}^{k-1} \left\{ r_1(l) \left[ 1 - \frac{x_1(l)}{a_1(l) + b_1(l)x_2(l)} \right] - c_1(l)x_1(l) - d_1(l)u_1(l) \right\},
\]
\[
x_2(k) = x_2(0) \exp \sum_{l=0}^{k-1} \left\{ r_2(l) \left[ 1 - \frac{x_2(l)}{a_2(l) + b_2(l)x_1(l)} \right] - c_2(l)x_2(l) - d_2(l)u_2(l) \right\},
\]
\[
u_1(k) = u_1(0) - \sum_{l=0}^{k-1} f_1(l)u_1(l) - g_1(l)x_1(l),
\]
\[
u_2(k) = u_2(0) - \sum_{l=0}^{k-1} f_2(l)u_2(l) - g_2(l)x_2(l).
\]

From Theorem 1, for any solution \(X(k) = x_1(k), x_2(k), u_1(k), u_2(k)\) of system (2) with initial condition (4) satisfy (5). Hence, for any \(\epsilon > 0\), there exist \(k_0\), if \(k_0\) is sufficiently large, we have
\[
x_{i*} - \epsilon \leq x_i(k) \leq x_i^* + \epsilon, u_{i*} - \epsilon \leq u_i(k) \leq u_i^* + \epsilon, i = 1, 2.
\]

Let \(\{\tau_0\} \) be any integer valued sequence such that \(\tau_0 \to \infty\) as \(\alpha \to \infty\), we claim that there exists a subsequence of \(\{\tau_0\}\), we still denote it by \(\{\tau_0\}\), such that
\[
x_i(k + \tau_0) \to x_i^*(k)
\]
uniformly in \(n\) on any finite subset \(B\) of \(Z\) as \(\alpha \to \infty\), where \(B = \{a_1, a_2, \ldots, a_m\}, a_h \in Z, h = (1, 2, \ldots, m)\) and \(m\) is a finite number.

In fact, for any finite subset \(B \subset Z\), when \(\alpha\) is large enough, \(\tau_0 + a_h > k_0, h = 1, 2, \ldots, m\). So
\[
x_{i*} - \epsilon \leq x_i(k + \tau_0) \leq x_i^* + \epsilon, i = 1, 2,
\]
\[
u_{i*} - \epsilon \leq u_i(k + \tau_0) \leq u_i^* + \epsilon, i = 1, 2.
\]
That is, \(\{x_i(k + \tau_0), u_i(k + \tau_0)\}\) are uniformly bounded for large enough \(\alpha\).

Now, for \(a_1 \in B\), we can choose a subsequence \(\{\tau_{a_1}^{(1)}\}\) of \(\{\tau_0\}\) such that \(\{x_i(a_1 + \tau_{a_1}^{(1)}), u_i(a_1 + \tau_{a_1}^{(1)})\}\) uniformly converges on \(Z^+\) for \(\alpha\) large enough.

Similarly, for \(a_2 \in B\), we can choose a subsequence \(\{\tau_{a_2}^{(2)}\}\) of \(\{\tau_{a_1}^{(1)}\}\) such that \(\{x_i(a_2 + \tau_{a_2}^{(2)}), u_i(a_2 + \tau_{a_2}^{(2)})\}\) uniformly converges on \(Z^+\) for \(\alpha\) large enough.

Repeating this procedure, for \(a_m \in B\), we can choose a subsequence \(\{\tau_{a_m}^{(m-1)}\}\) of \(\{\tau_{a_{m-1}}^{(m-2)}\}\) such that \(\{x_i(a_m + \tau_{a_m}^{(m-1)}), u_i(a_m + \tau_{a_m}^{(m-1)})\}\) uniformly converges on \(Z^+\) for \(\alpha\) large enough.

Now pick the sequence \(\{\tau_{a_m}^{(m)}\}\) which is a subsequence of \(\{\tau_0\}\), we still denote it as \(\{\tau_0\}\) then for all \(k \in B\), we have
\[
x_i(k + \tau_0) \to x_i^*, u_i(k + \tau_0) \to u_i^*
\]
uniformly in \(k \in B\) as \(\alpha \to \infty\).

By the arbitrary of \(B\), the conclusion is valid.

Since \(\{\tau_0\}, \{a_1(k), b_1(k), c_1(k), d_1(k), f_1(k)\}\) and \(\{g_i(k)\}\) are almost periodic sequence, for above sequence
Suppose the conditions of Lemma 7 are satisfied, so let

as \( \alpha \to \infty \) uniformly on \( Z^+ \).

For any \( \delta \in Z \), we can assume that \( \tau_{n0} + \delta \geq k_0 \) for \( \delta \) large enough. Let \( k \geq 0 \) and \( k \leq 0 \), by an inductive argument of (2) from \( \tau_n + \delta \) to \( k + \tau_n + \delta \) leads to

Then, for \( i, j = 1, 2, i \neq j \), we have

where

\( p_i(k) = \ln x_i(k) \), from system (2), we have

\( p_i(k + 1) = p_i(k) + r_i(k) \left[ 1 - \frac{\exp\{p_i(k)\}}{a_i(k) + b_i(k) \exp\{p_i(k)\}} \right] \)

Hence, \( |p_i(k)| \leq A_i, |u_i(k)| \leq B_i, A_i = \max\{|\ln x_i|, |\ln x_i^*|\}, B_i = \max\{|u_i|, |u_i^*|\}, i = 1, 2, k \leq 0 \).

For \( (X, U) \in R^{2^3+2^3} \), we define the norm \( \|\|X, U\|\| = \sum_{i=1}^{2^3} |x_i| + \sum_{i=1}^{2^3} |u_i| \).

Consider the product system of system (16)

Suppose that \( Z = (p_1(k), p_2(k), u_1(k), u_2(k)) \) are any two solutions of system (16) defined on \( Z^+ \times S^* \times S^* \), then \( \|Z]\) \leq B, \( \|W]\) \leq B, where \( B = \sum_{i=1}^{2^3} |A_i + B_i|, S^* = \sum_{i=1}^{2^3} |A_i + B_i| \). \( S^* \)
Consider a Lyapunov function defined on $\mathbb{Z}^+ \times S^* \times S^*$ as follows

$$V(k, Z, W) = \sum_{i=1}^{2} \left\{ (p_i(k) - q_i(k))^2 + (u_i(k) - \omega_i(k))^2 \right\},$$

It is easy to see that the norm

$$\|Z - W\| = \sum_{i=1}^{2} \left\{ |p_i(k) - q_i(k)| + |u_i(k) - \omega_i(k)| \right\}$$

and the norm

$$\|Z - W\|_* = \left\{ \sum_{i=1}^{2} \left\{ (p_i(k) - q_i(k))^2 + (u_i(k) - \omega_i(k))^2 \right\} \right\}^{1/2}$$

are equivalent. That is, there exist two constants $C_1 > 0$, $C_2 > 0$, such that

$$C_1 \|Z - W\| \leq \|Z - W\|_* \leq C_2 \|Z - W\|,$$

then

$$(C_1 \|Z - W\|)^2 \leq V(k, Z, W) \leq (C_2 \|Z - W\|)^2.$$ 

Let $a \in C(R^+, R^+)$, $a(x) = C_2 x^2$, $b \in C(R^+, R^+)$, $b(x) = C_2^2 x^2$, thus the condition (i) in Lemma 6 is satisfied. In addition,

$$|V(n, Z, W) - V(n, Z, \tilde{W})|$$

$$= \sum_{i=1}^{2} \left\{ (p_i(n) - q_i(n))^2 + (u_i(n) - \omega_i(n))^2 \right\}$$

$$- \sum_{i=1}^{2} \left\{ (\tilde{p}_i(n) - \tilde{q}_i(n))^2 + (\tilde{u}_i(n) - \tilde{\omega}_i(n))^2 \right\}$$

$$\leq \sum_{i=1}^{2} \left\{ (p_i(n) - q_i(n))^2 - (\tilde{p}_i(n) - \tilde{q}_i(n))^2 \right\}$$

$$+ \sum_{i=1}^{2} \left\{ (u_i(n) - w_i(n))^2 - (\tilde{u}_i(n) - \tilde{w}_i(n))^2 \right\}$$

$$\leq \sum_{i=1}^{2} \left\{ (p_i(n) - q_i(n)) + (\tilde{p}_i(n) - \tilde{q}_i(n)) \right\}$$

$$\times \left\{ (p_i(n) - q_i(n)) - (\tilde{p}_i(n) - \tilde{q}_i(n)) \right\}$$

$$+ \sum_{i=1}^{2} \left\{ (u_i(n) - w_i(n)) + (\tilde{u}_i(n) - \tilde{w}_i(n)) \right\}$$

$$\times \left\{ (u_i(n) - w_i(n)) - (\tilde{u}_i(n) - \tilde{w}_i(n)) \right\}$$

$$\leq \sum_{i=1}^{2} \left\{ (|p_i(n)| + |q_i(n)| + |\tilde{p}_i(n)| + |\tilde{q}_i(n)|) \right\}$$

$$\times \left\{ (|p_i(n)| + |q_i(n)| - |\tilde{p}_i(n)| - |\tilde{q}_i(n)|) \right\}$$

$$+ \sum_{i=1}^{2} \left\{ (|u_i(n)| + |w_i(n)| + |\tilde{u}_i(n)| + |\tilde{w}_i(n)|) \right\}$$

$$\times \left\{ (|u_i(n)| + |w_i(n)| - |\tilde{u}_i(n)| - |\tilde{w}_i(n)|) \right\}.$$
\[
-2d_i(k)(p_i(k) - q_i(k))(u_i(k) - \omega_i(k))
+ \left[ \frac{2r_i(k)d_k}{a_i(k) + b_i(k)e^{p_i(k)}} + 2c_i(k)r_i(d_k) \right]
+ 2(1 - f_i(k))\left\{ (u_i(k) - \omega_i(k))(e^{p_i(k)} - e^{q_i(k)})
- \frac{2r_i(k)b_i}{a_i(k) + b_i(k)e^{p_i(k)}}r_i(d_k)
+ \frac{2r_i(k)}{(a_i(k) + b_i(k)e^{p_i(k)})(a_i(k) + b_i(k)e^{q_i(k)})}
\times (u_i(k) - \omega_i(k))(e^{p_i(k)} - e^{q_i(k)}) + (d_i^2(k) - 2f_i(k) + f_i^2(k))(u_i(k) - \omega_i(k))^2 \right\}.
\]

Using the mean value theorem we get
\[
e^{p_i(k)} - e^{q_i(k)} = \xi_i(k)(p_i(k) - q_i(k)), \quad i = 1, 2,
\]
where \(\xi_i(k)\) lies between \(e^{p_i(k)}\) and \(e^{q_i(k)}, i = 1, 2\). From (18) and (19), we have
\[
\Delta V_{1(7)}(k)
= \sum_{i=1}^{2} \left\{ \frac{r_i^2(k)}{(a_i(k) + b_i(k)e^{p_i(k)})^2} + r_i^2(k)c_i^2(k)
+ 2r_i^2(k)c_i(k) + g_i^2(k) \right\} \xi_i^2(k)(p_i(k) - q_i(k))^2
+ \frac{b_i^2(k)(e^{p_i(k)})^2}{(a_i(k) + b_i(k)e^{p_i(k)})(a_i(k) + b_i(k)e^{q_i(k)})}
\times (u_i(k) - \omega_i(k))(e^{p_i(k)} - e^{q_i(k)})^2
- \frac{2r_i^2(k)}{(a_i(k) + b_i(k)e^{p_i(k)})(a_i(k) + b_i(k)e^{q_i(k)})}
\times \xi_i^2(k)(p_i(k) - q_i(k))
- \frac{r_i^2(k)}{(a_i(k) + b_i(k)e^{p_i(k)})(a_i(k) + b_i(k)e^{q_i(k)})}
\times \xi_i^2(k)(p_i(k) - q_i(k))
- \frac{r_i^2(k)}{(a_i(k) + b_i(k)e^{q_i(k)})}
\times (u_i(k) - \omega_i(k))^2 + \frac{2r_i(k)}{(a_i(k) + b_i(k)e^{p_i(k)})(a_i(k) + b_i(k)e^{q_i(k)})}
\times (u_i(k) - \omega_i(k))^2 + \frac{2r_i(k)d_k}{a_i(k) + b_i(k)e^{q_i(k)}}
\times (u_i(k) - \omega_i(k))(p_i(k) - q_i(k))
+ 2c_i(k)r_i(d_k)
+ 2(1 - f_i(k))\left\{ (u_i(k) - \omega_i(k))(p_i(k) - q_i(k))
- 2d_i(k)(p_i(k) - q_i(k))
+ \frac{2r_i(k)d_k}{a_i(k) + b_i(k)e^{q_i(k)}}
\times (u_i(k) - \omega_i(k))(p_i(k) - q_i(k))
+ (d_i^2(k) - 2f_i(k) + f_i^2(k))(u_i(k) - \omega_i(k))^2 \right\}
\leq \sum_{i=1}^{2} \left\{ \frac{r_i^2(k)}{(a_i(k) + b_i(k)e^{p_i(k)})^2} + r_i^2(k)c_i^2(k)\xi_i^2(k)
+ 2r_i^2(k)c_i(k)\xi_i^2(k) + g_i^2(k)\xi_i^2(k)
- \frac{2r_i(k)}{(a_i(k) + b_i(k)e^{p_i(k)})(a_i(k) + b_i(k)e^{q_i(k)})}
\times (u_i(k) - \omega_i(k))\xi_i^2(k)(p_i(k) - q_i(k))
+ (d_i^2(k) - 2f_i(k) + f_i^2(k))(u_i(k) - \omega_i(k))^2 \right\}
+ \frac{b_i^2(k)(e^{p_i(k)})^2}{(a_i(k) + b_i(k)e^{p_i(k)})(a_i(k) + b_i(k)e^{q_i(k)})}
\times (p_i(k) - q_i(k))^2
+ 2\left\{ \frac{-r_i^2(k)b_i}{(a_i(k) + b_i(k)e^{p_i(k)})(a_i(k) + b_i(k)e^{q_i(k)})}
\times \xi_i^2(k)(p_i(k) - q_i(k))
+ \frac{r_i^2(k)}{(a_i(k) + b_i(k)e^{p_i(k)})(a_i(k) + b_i(k)e^{q_i(k)})}
\times \xi_i^2(k)(p_i(k) - q_i(k))
+ (d_i^2(k) - 2f_i(k) + f_i^2(k))(u_i(k) - \omega_i(k))^2 \right\}.
\]

We get for \(j = 1, 2\),
\[
\Delta V_{1(7)} = \sum_{i=1}^{2} \left\{ V_{1ij} + V_{2ij} + V_{3ij} + V_{4ij} + V_{5ij} + V_{6ij} \right\},
\]
where
\[
V_{1ij} = \left[ \frac{r_i^2(k)}{(a_i(k) + b_i(k)e^{p_i(k)})^2} + r_i^2(k)c_i^2(k)\xi_i^2(k)
+ 2r_i^2(k)c_i(k)\xi_i^2(k) + g_i^2(k)\xi_i^2(k)
- \frac{2r_i(k)}{(a_i(k) + b_i(k)e^{p_i(k)})(a_i(k) + b_i(k)e^{q_i(k)})}
\times (u_i(k) - \omega_i(k))\xi_i^2(k)(p_i(k) - q_i(k))
+ (d_i^2(k) - 2f_i(k) + f_i^2(k))(u_i(k) - \omega_i(k))^2 \right\}
\leq \frac{r_i^2(k)}{(a_i^2 + b_i^2x_i^2)^2} + r_i^2(k)c_i^2x_i^2 + 2r_i^2(k)c_i^2x_i^2
+ g_i^2(k)x_i^2
+ 2r_i(x_i^2 - \frac{2r_i}{a_i^2 + b_i^2x_i^2} - 2r_i^2c_i^2x_i^2)
\times (p_i(k) - q_i(k))^2
\]
\[
V_{2ij} = \left[ \frac{b_i^2(k)(e^{p_i(k)})^2}{(a_i(k) + b_i(k)e^{p_i(k)})(a_i(k) + b_i(k)e^{q_i(k)})}
\times (p_i(k) - q_i(k))^2
\leq \frac{b_i^2(k)(e^{p_i(k)})^2}{(a_i^2 + b_i^2x_i^2)^2} (p_i(k) - q_i(k))^2,
\]
\[
V_{3ij} = \left[ \frac{-r_i^2(k)b_i}{(a_i(k) + b_i(k)e^{p_i(k)})(a_i(k) + b_i(k)e^{q_i(k)})}
\times \xi_i^2(k)(p_i(k) - q_i(k))
+ \frac{r_i^2(k)}{(a_i(k) + b_i(k)e^{p_i(k)})(a_i(k) + b_i(k)e^{q_i(k)})}
\times \xi_i^2(k)(p_i(k) - q_i(k))
+ (d_i^2(k) - 2f_i(k) + f_i^2(k))(u_i(k) - \omega_i(k))^2 \right\}.
\]

\begin{align*}
\Delta V_{ij} & \leq \left| (p_i(k) - q_i(k))\left( p_j(k) - q_j(k) \right) \right| \\
& \leq \frac{r_i^u b_i^p c_i^p x_i^p x_i^p}{(a_i^l + b_i^l x_j^l)} + \frac{r_i^v b_i^p c_i^p x_i^p x_i^p}{(a_i^l + b_i^l x_j^l)^2} + \frac{r_i^w b_i^p x_i^w x_i^2}{(a_i^l + b_i^l x_j^l)^2} \\
& \times \left( (p_i(k) - q_i(k))^2 + p_j(k) - q_j(k)^2 \right), \\
V_{ij} & = \left[ r_i^u b_i^p c_i^p x_i^p + r_i^v b_i^p c_i^p x_i^p + (1 - f_i^x)x_i^+ - d_i \right] \\
& \times \left( p_i(k) - q_i(k)(u_i(k) - \omega_i(k))^2 \right) \\
& \leq \frac{r_i^u b_i^p d_i^p x_i^p x_i^2}{(a_i^l + b_i^l x_j^l)^2} \times \left( (p_i(k) - q_i(k))^2 + (u_i(k) - \omega_i(k))^2 \right) \\
V_{ij} & = \left[ r_i^u b_i^p d_i^p x_i^p x_i^2 + r_i^v b_i^p d_i^p x_i^p x_i^2 + (1 - f_i^x)x_i^+ - d_i \right] \\
& \times \left( (p_i(k) - q_i(k))^2 + (u_i(k) - \omega_i(k))^2 \right) \\
& \leq \frac{r_i^u b_i^p d_i^p x_i^p x_i^2}{(a_i^l + b_i^l x_j^l)^2} \\
& \times \left( (p_i(k) - q_i(k))^2 + (u_i(k) - \omega_i(k))^2 \right) \\
& \leq \frac{2r_i^u b_i^p d_i^p x_i^p x_i^2}{(a_i^l + b_i^l x_j^l)^2} \\
& \times \left( (p_i(k) - q_i(k))^2 + (u_i(k) - \omega_i(k))^2 \right)
\end{align*}

Hence,

\begin{align*}
\Delta V_{ij} \leq & \sum_{i=1}^{2} \left[ r_i^u b_i^p c_i^p x_i^p + \frac{r_i^v b_i^p c_i^p x_i^p}{(a_i^l + b_i^l x_j^l)^2} \right. \\
& + \frac{2r_i^u b_i^p c_i^p x_i^p}{(a_i^l + b_i^l x_j^l)^2} + \frac{2r_i^v b_i^p c_i^p x_i^p}{(a_i^l + b_i^l x_j^l)^2} - 2r_i^v b_i^p c_i^p x_i^p \\
& + \frac{r_i^u b_i^p c_i^p x_i^p}{(a_i^l + b_i^l x_j^l)^2} + \frac{r_i^v b_i^p c_i^p x_i^p}{(a_i^l + b_i^l x_j^l)^2} + \frac{r_i^w b_i^p c_i^p x_i^p}{(a_i^l + b_i^l x_j^l)^2} \\
& + \frac{r_i^u b_i^p c_i^p x_i^p}{(a_i^l + b_i^l x_j^l)^2} + \frac{r_i^v b_i^p c_i^p x_i^p}{(a_i^l + b_i^l x_j^l)^2} + \frac{r_i^w b_i^p c_i^p x_i^p}{(a_i^l + b_i^l x_j^l)^2} \\
& \times (p_i(k) - q_i(k))^2 + \frac{r_i^u b_i^p d_i^p x_i^p x_i^2}{(a_i^l + b_i^l x_j^l)^2} \\
& + \frac{r_i^v b_i^p d_i^p x_i^p x_i^2}{(a_i^l + b_i^l x_j^l)^2} + \frac{r_i^w b_i^p d_i^p x_i^p x_i^2}{(a_i^l + b_i^l x_j^l)^2} \\
& \times (p_i(k) - q_i(k))^2 + \frac{r_i^u b_i^p d_i^p x_i^p x_i^2}{(a_i^l + b_i^l x_j^l)^2} \\
& \left. + \frac{r_i^v b_i^p d_i^p x_i^p x_i^2}{(a_i^l + b_i^l x_j^l)^2} + \frac{r_i^w b_i^p d_i^p x_i^p x_i^2}{(a_i^l + b_i^l x_j^l)^2} \right] \\
& \times (p_i(k) - q_i(k))^2 \\
& \leq -\sum_{i=1}^{2} \left\{ r_i^u b_i^p d_i^p x_i^p x_i^2 \right\} \\
& \leq -\beta V(k)
\end{align*}

From $0 < \beta < 1$, the condition (iii) of Lemma 6 is satisfied. So, from Lemma 6, there exists a uniqueness uniformly asymptotically stable almost periodic solution $X(k) = (p_1(k), p_2(k), u_1(k), u_2(k))$ of (16) which is bounded by $s_s$ for all $k \in Z^+$, which means that there exists a uniqueness uniformly asymptotically stable almost periodic solution $X(k) = (x_1(k), x_2(k), u_1(k), u_2(k))$ of (2) which is bounded by $\Omega$ for all $k \in Z^+$. This completes the proof.

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**References**


