

Statistical Reliability Based Modeling of Series and Parallel Operating Systems using Extreme Value Theory

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Abstract—This paper tries to represent a new method for computing the reliability of a system which is arranged in series or parallel model. In this method we estimate life distribution function of whole structure using the asymptotic Extreme Value (EV) distribution of Type I, or Gumbel theory. We use EV distribution in minimal mode, for estimate the life distribution function of series structure and maximal mode for parallel system. All parameters also are estimated by Moments method. Reliability function and failure (hazard) rate and p-th percentile point of each function are determined. Other important indexes such as Mean Time to Failure (MTTF), Mean Time to repair (MTTR), for non-repairable and renewal systems in both of series and parallel structure will be computed.

Keywords—Reliability, extreme value, parallel, series, life distribution

I. INTRODUCTION

THERE are some different methods to computing the reliability of a composite structure. We can use two general ways for these computations:

- 1) By using the reliability of components.
- 2) By using the life distribution function of system.

In the large systems which have a lot of different components with unknown reliability (complex structures), it will be very important to find the life distribution function of system. In this paper we're going to represent a new method for estimate the life distribution of these systems, when their components become a lot and different.

We try to use desire properties of Extreme Value (EV) theorem which has been offered by Gumbel in 1954 [1]. He demonstrated asymptotic probability distributions for minimal and maximal value in a random sample, under the special conditions.

Many fields of modern science and engineering have to deal with events which are rare but have significant consequences. EV theory provides a firm theoretical foundation on which we can build statistical models

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describing extreme events.

In many fields of modern science, engineering and insurance, Extreme Value theory is well established (see e.g. Embrechts et al [2], Reiss and Thomas [3]).

Recently, more and more research has been undertaken to analyze the extreme variations that financial markets are subject to, mostly because of currency crises, stock market crashes and large credit defaults. The tail behaviour of financial series has, among others, been discussed in Koedijk et al. [4], Dacorogna et al. [5], Loretan and Phillips [6], Neftci [7] and McNeil and Frey [8]. An interesting discussion about the potential of extreme value theory in risk management is given in Diebold et al [9].

The EV theory which is related to statistical behavior of maximum and minimum value in a random sample was suggested by Fisher and Tippett [10] initially, and extended by Gumbel [11] through determining the approximate distribution function of $X(1)$ and $X(n)$ as the extreme values (minimum and maximum) of a statistical random sample, X_1, X_2, \dots, X_n .

The intricate models like EV type II and type III, were represented and adapted by Ang and Tang as scientific and applied extension of this theory [12]. Kotz and Nagarajah listed over 50 application ranging from accelerated life testing through to earthquakes, foods horse racing, rainfall, queues in systems, sea currents, wind speeds, and track race records [13]. Developing the EV theory in reliability applications, studied through kinds of mathematical modeling by Alkallut and Lye et al [14, 15] and various dynamic evaluating of systems reliability [16, 17, 18, 19, 20, 21].

We find these PDFs to computing the reliability of series and parallel based systems by estimating the life distribution of each system.

According the obtained functions all important indexes of reliability and dependability of a system with any structure will be determined for renewal and nonrenewable mode. The Mean Time to Failure (MTTF), Mean Time to Replacement (MTTR), Hazard Function ($h(t)$), are some of these parameters. We acquire another useful relation for reliability calculations and analysis system.

II. EXTREME VALUE'S DISTRIBUTIONS OF A SAMPLE

In this section two types of extreme values and their PDFs with probability properties are introduced.

A. Maximal value distribution function in a random sample

Suppose the random sample x_1, x_2, \dots, x_n and define $X(1)$ and $X(n)$ as minimum and maximum of x_i in sample and call them the Extreme Values:

$$\begin{aligned} X(1) &= \text{Min}\{x_1, x_2, \dots, x_n\} \\ X(n) &= \text{Max}\{x_1, x_2, \dots, x_n\} \end{aligned} \quad (1)$$

In the specific conditions these will have the special behavior independent the initial distribution.

At the 1954, Gumbel could present the $X(n)$, when the n closest infinitive, has a fix and independent probability model with CDF as (2), where γ and δ are Location and Scale parameters.

$$\begin{aligned} F_{EV}(x, \gamma, \delta) &= P(X \leq x) = \exp\left(-\exp\left(-\frac{x-\gamma}{\delta}\right)\right) \\ -\infty < x < +\infty, \quad x > \gamma \end{aligned} \quad (2)$$

This function is called Gumbel or EV distribution or EV Type I [22]. After him, Benjamin in 1970 and Ang in 1984 developed his concept on EV type II and III [1]. So in this paper, we just discuss about Type I. The PDF of Gumbel distribution that is denoted by $f_{EV}^{x(n)}$ can be found after differentiating as (3).

$$\begin{aligned} f_{EV}^{x(n)}(x; \gamma, \delta) &= \frac{d}{dx} F_{EV} \\ &= \frac{1}{\delta} \exp\left(-\left(\frac{x-\gamma}{\delta}\right) - \exp\left(-\frac{x-\gamma}{\delta}\right)\right) \\ -\infty < x < +\infty, \quad \delta > 0 \end{aligned} \quad (3)$$

If $\gamma = 0, \delta = 1$, we have the Standard Gumbel distribution with the formula same as (4) and a curve same as Fig. 1

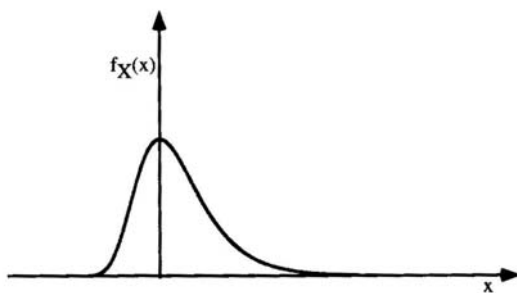


Fig. 1 A general curve of EV distribution for maximal

$$f(x) = e^{-x} e^{-e^{-x}} \quad (4)$$

Other statistical attributes of this function, are as below. where μ is expected value, σ is standard deviation, Me is median, t_p is p-th percentile point, $S.K$ is skewness, and K is kurtosis of this distribution function [23].

$$\mu = \gamma + 0.5772 \delta \quad (5)$$

$$\sigma = \frac{\pi}{\sqrt{6}} \delta = 1.283 \delta \quad (6)$$

$$Me = \gamma - \delta \ln(\ln(2)) = \gamma - 0.3665 \delta \quad (7)$$

$$t_p = \gamma - \delta \ln\left(\ln\left(\frac{1}{p}\right)\right) \quad (8)$$

$$S.K = 5.4 \quad (9)$$

$$K = 1.14 \quad (10)$$

B. Minimal value distribution function in a random sample

Second form of EV distribution relates to minimal value in a random sample. In the other hand, can demonstrate when the sample size, n , increase enough, the $X(1)$, has an independent probability model with the following asymptotic CDF as (11).

$$\begin{aligned} F_{EV}^{x(1)}(x; \lambda, \delta) &= P(X \leq x) = 1 - \exp\left(-\exp\left(\frac{x-\gamma}{\delta}\right)\right) \\ -\infty < x < +\infty, \quad x > \gamma \end{aligned} \quad (11)$$

By differentiating of formula (11), the PDF of $X(1)$ which is denoted by $f_{EV}^{x(1)}$ define as (12).

$$\begin{aligned} f_{EV}^{x(1)} &= \frac{1}{\delta} \exp\left(\frac{x-\gamma}{\delta} - \exp\left(\frac{x-\gamma}{\delta}\right)\right) \\ -\infty < x < +\infty, \quad x > \gamma \end{aligned} \quad (12)$$

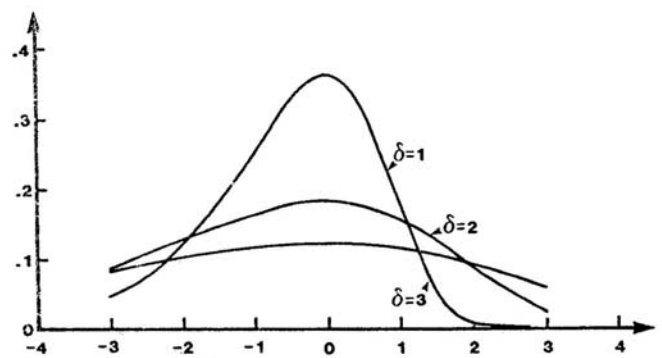


Fig. 2 A general curve of EV distribution for minimal

If $\gamma = 0, \delta = 1$, we have the Standard form of EV distribution (Min), with the formula same as (13) and the curve same as Fig. 2.

$$f(x) = e^x e^{-e^x} \quad (13)$$

Other statistical attributes of this function, are:

$$\mu = \gamma - 0.5772 \delta \quad (14)$$

$$\sigma = \frac{\pi}{\sqrt{6}} \delta = 1.283 \delta \quad (15)$$

$$Me = \gamma + \delta \ln(\ln(2)) = \gamma + 0.3665 \delta \quad (16)$$

$$t_p = \gamma + \delta \ln\left(\ln\left(\frac{1}{1-p}\right)\right) \quad (17)$$

$$S.K = -5.4 \quad (18)$$

$$K = 1.14 \quad (19)$$

According to the relations of the above, Gumbel distribution (EV distribution) curves is skewed to right in maximal mode, and skewed to left in minimal mode. Both of these forms are more kurtosis than Normal distribution.

III. RELIABILITY OF COMPOSITE STRUCTURES

Two single combinations of systems are series and parallel. Fig. 3 and Fig. 4 show them. Suppose C_i is the symbol of i -th Component. As we know the reliability of a series structure is:

$$R_{sys}(1) = \prod_{i=1}^n R_i \quad (20)$$

where R_i is reliability of i -th component. This relation for a parallel system is same to (21).

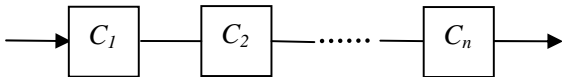


Fig. 3 A simple view of series structure

$$R_{sys}(2) = 1 - \prod_{i=1}^n (1 - R_i) \quad (21)$$

When we don't have any information on the reliability of subsystems, or components, R_i , we can not use formulas (20) and (21). So we must determine the life distribution function of whole system [24].

Suppose $C(i)$ is the i -th part of system that fails. Thus in series system, duration life of whole system equals to the first part which fails, as $C(1)$. So life duration of a parallel system equals to last failed part, $C(n)$. Thus by given PDF of $C(1)$ and $C(n)$ we can determine the life distribution of series and parallel system. We want to apply the results of section II.A and II.B for estimation the PDF of $C(1)$ and $C(n)$.

Consider the series structure in Fig. 3. Suppose the quantity of subsystems or components, n , is large. The life duration of

this system is equal to life duration of $C(1)$ and meet the PDF same EV dis. in minimum mode, $f_{EV}^{x(1)}$. At the same way, the structure in Fig. 4, has life duration same the $C(n)$ and meet the PDF of EV dis. in maximum mode, $f_{EV}^{x(n)}$.

According to these concepts we're going to determine the reliability indexes of both of the series and parallel systems.

IV. COMPUTING THE RELIABILITY OF COMPLEX STRUCTURE USING THE EV DISTRIBUTION

A. Series structure

Consider the given system in Fig. 4, again. According to results of previous section, the reliability of this system can be computed by using (11), $f_{EV}^{x(1)}$.

$$\begin{aligned} R(t) &= P(T \geq t) = \int_t^{\infty} f(x) dx \\ &= \int_t^{\infty} f_{EV}^{x(1)}(x; \gamma, \delta) dx \\ &= 1 - F_{EV}^{x(1)}(t; \gamma, \delta) \\ &= \exp\left(-\exp\left(\frac{t-\gamma}{\delta}\right)\right) \end{aligned} \quad (22)$$

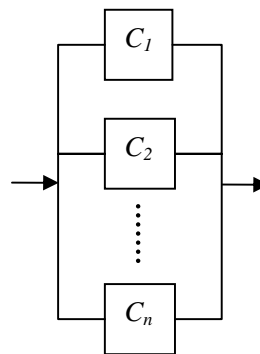


Fig. 4 A simple view of parallel structure

For determining the failure rate or hazard rate, we should try for $h(t)$:

$$\begin{aligned} H(t) &= -\ln(1 - F_{EV}^{x(1)}(t)) \\ &= -\ln\left(\exp\left(-\exp\left(\frac{t-\gamma}{\delta}\right)\right)\right) \\ &= \exp\left(\frac{t-\gamma}{\delta}\right) \end{aligned} \quad (23)$$

$$\begin{aligned} h(t) &= \frac{d}{dt} H(t) \\ &= \frac{1}{\delta} \exp\left(\frac{t-\gamma}{\delta}\right) \end{aligned} \quad (24)$$

In the Preventive Maintenance (PM) for non-repairable parts which should be replaced with any failure, we need to predict the time of the first failure. So determining the Mean Time to Failure (MTTF) will be necessary. By using this Index determine the scheduled inspections for on time replacements that can always keep the system in an available condition.

However this index is very useful for non-repairable systems and we try to compute for described systems in the condition of this paper given in Fig. 4. Equation (25) presents the result of the computations.

$$\begin{aligned}
 MTTF &= E(T) \\
 &= \int_0^{\infty} tf(t)dt \\
 &= \int_0^{\infty} R(t)dt \\
 &= \int_0^{\infty} \exp\left(-\exp\left(\frac{t-\gamma}{\delta}\right)\right)dt \\
 &= -\delta \exp\left(-\frac{t-\gamma}{\delta} - \exp\left(\frac{t-\gamma}{\delta}\right)\right) \Big|_0^{\infty} \\
 &= \delta \exp\left(\frac{\gamma}{\delta} - \exp\left(-\frac{\gamma}{\delta}\right)\right) \quad (25)
 \end{aligned}$$

For renewal systems we need to predict during times between failures for periodic service before breaking down. So it was necessary to determine the Mean Time Between Failure (MTBF) which can be computed by statistics data of performance during time in the normal or intensive condition [10], [24].

B. Parallel systems

Consider given structure in Fig. 4 and suppose the same condition in section IV.A. According the results we discussed in section II.A and III, life distribution of this system is equal to $C(n)$ PDF. So we can determine reliability as bellow:

$$\begin{aligned}
 R(t) &= P(T \geq t) = \int_0^{\infty} f(x) dx \\
 &= 1 - F_{EV}^{x(n)}(t; \gamma, \delta) \\
 &= 1 - \exp\left(-\exp\left(-\frac{t-\gamma}{\delta}\right)\right) \quad (26)
 \end{aligned}$$

Other indexes of reliability can be computed same the series system [11]. For example the Failure rate or hazard function is:

$$\begin{aligned}
 H(t) &= -\ln(1 - F_{EV}^{x(n)}(t)) \\
 &= -\ln\left(1 - \exp\left(-\exp\left(-\frac{t-\gamma}{\delta}\right)\right)\right) \\
 h(t) &= \frac{d}{dt} H(t) \\
 &= \frac{\frac{1}{\delta} \exp\left(-\frac{t-\gamma}{\delta} - \exp\left(-\frac{t-\gamma}{\delta}\right)\right)}{1 - \exp\left(-\exp\left(-\frac{t-\gamma}{\delta}\right)\right)} \quad (27)
 \end{aligned}$$

V. ESTIMATING PARAMETERS

In this step we must present the statistical estimators for each parameter in determined formulas. The main parameters of Gumbel distribution are δ (scale parameter), and γ (location parameter).

Using the Moments method for estimate the scale parameter of the above we obtain:

$$\hat{\delta} = \frac{\sqrt{6}}{\pi} S \quad (28)$$

where S is the standard deviation of sample.

The below formulas will be obtained for location parameters:

$$\hat{\gamma}_{C(1)} = \bar{x} + 0.5772 \hat{\delta} \quad (29)$$

$$\hat{\gamma}_{C(n)} = \bar{x} - 0.5772 \hat{\delta} \quad (30)$$

where \bar{x} is mean of the random sample and $\hat{\gamma}_{C(i)}$ is momentum estimator of location parameter for minimum ($C(1)$) or maximum ($C(n)$) mode of EV distribution. Formula (29) is for minimal mode and (30) for maximal mode.

VI. CONCLUSION

During this paper we tried to present a new method for determining the reliability of composite systems with components with unknown reliabilities. In this letter some useful relations and formulas were exhibited for reliability indexes computations. This new method is recommended strongly for large structures with various components.

Although in this paper we discussed about single series or parallel systems, but it can be developed for complex systems. However we offer to apply this presented theorem for large complex and applicable systems.

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