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Design of Nonlinear Observer by Using Augmented Linear System based on Formal Linearization of Polynomial Type

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Abstract—The objective of this study is to propose an observer design for nonlinear systems by using an augmented linear system derived by application of a formal linearization method. A given nonlinear differential equation is linearized by the formal linearization method which is based on Taylor expansion considering up to the higher order terms, and a measurement equation is transformed into an augmented linear one. To this augmented dimensional linear system, a linear estimation theory is applied and a nonlinear observer is derived. As an application of this method, an estimation problem of transient state of electric power systems is studied, and its numerical experiments indicate that this observer design shows remarkable performances for nonlinear systems.

Keywords—nonlinear system, augmented linear system, nonlinear observer, formal linearization, electric power system.

I. Introduction

ESTIMATION problems of nonlinear system have been well studied and the design technique via linearization approaches is attractive ways by reason of making use of the linear theories [1]–[6]. Formal linearization [7], [8] is one of interesting approaches to solve nonlinear problems.

In this paper, we consider an observer design for nonlinear systems to be applied the formal linearization method [8]. A given nonlinear system is transformed into an augmented linear system by using the higher polynomials of both the state and measurement equations.

We introduce a linearization function which consists of polynomials of the state variables, and an augmented measurement equation which consists of polynomials of the measurement variables. The nonlinear state differential equation and the augmented measurement equation are linearized by the formal linearization method which is based on Taylor expansion considering up to the higher order. As a result, an augmented linear system is obtained from them. By this linear system, we can apply linear system theories and derive a nonlinear observer. Inversion is simple because of the original state variable involved in the linearization function. As an application of this method, a nonlinear observer for an estimation of the transient sate of electric power systems is synthesized. Numerical experiments indicate that this observer design shows remarkable performances for the nonlinear system.

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II. Statement of Problem

Consider a nonlinear system described by a state differential equation

$$\Sigma_1 : \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \subset R^n \tag{1}$$

where t denotes time, $\cdot = d/dt$, $\boldsymbol{x}$ is an $n \times 1$ state vector, and $\boldsymbol{f}$ is a sufficiently smooth nonlinear function. Assumed that a measurement equation is given by

$$\boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t)) \subset R^m \tag{2}$$

where $\boldsymbol{y}$ is an $m \times 1$ vector of independent outputs with $m < n$, and $\boldsymbol{h}(\boldsymbol{x})$ is a sufficiently smooth nonlinear function.

Our goal is to estimate the state vector from measurement data by using the formal linearization method.

III. Nonlinear Observer

The objective of this study is to develop a nonlinear observer by the formal linearization method using Taylor expansion [8]. We quote the previous method to linearize a state differential equation, present a new linearization method for a measurement equation, and then synthesize a nonlinear observer.

A. Fromal Linearization for Nonlinear State differential Equation

In order to linearize the given nonlinear state differential equation (Eq. (1)), an N-th order linearization function $\boldsymbol{\phi}(\cdot) = \boldsymbol{\phi}(\boldsymbol{x}(\cdot))$ which consists of polynomials is defined by

$$\begin{aligned}\boldsymbol{\phi} &= [\phi_1, \phi_2, \cdots, \phi_i, \cdots, \phi_{\alpha_{max}}]^T \\ &= [T_{(10\cdots0)}(\boldsymbol{x}), T_{(01\cdots0)}(\boldsymbol{x}), \cdots, T_{(0\cdots01)}(\boldsymbol{x}), \\ &\quad T_{(11\cdots0)}(\boldsymbol{x}), T_{(101\cdots0)}(\boldsymbol{x}), \cdots, T_{(10\cdots1)}(\boldsymbol{x}), \\ &\quad T_{(20\cdots0)}(\boldsymbol{x}), T_{(02\cdots0)}(\boldsymbol{x}), \cdots, T_{(r_1\cdots r_n)}(\boldsymbol{x})]^T \end{aligned} \tag{3}$$

where

$$T_{(r_1\cdots r_n)}(\boldsymbol{x}) = \prod_{i=1}^{n} x_i^{r_i} \ ,$$

$$1 \le r_1 + r_2 + \cdots + r_n \le N,$$

$$\alpha_{max} : \text{the number of combination of } \{r_1, \cdots, r_n\}.$$

Deriving the derivative of each element of $\boldsymbol{\phi}$ along with the solution of the given nonlinear system (Eq. (1)) becomes

$$\dot{\phi}_\alpha(\boldsymbol{x}) = \dot{T}_{(r_1\cdots r_n)}(\boldsymbol{x})$$

$$= (\frac{d}{dt}x_1^{r_1})x_2^{r_2}\cdots x_{n-1}^{r_n-1}x_n^{r_n} + \cdots + x_1^{r_1}x_2^{r_2}\cdots x_{n-1}^{r_n-1}(\frac{d}{dt}x_n^{r_n})$$

$$= \sum_{k=1}^{n} r_k f_k(\boldsymbol{x}) \frac{T_{(r_1\cdots r_n)}(\boldsymbol{x})}{x_k}\ , \quad \alpha = \alpha(r_1, \cdots, r_n). \quad (4)$$

Note that Taylor expansion up to the N-th order derives

$$f_k(\boldsymbol{x}(t)) = [p_{k\ 1}, p_{k\ 2}, \cdots, p_{k\ \alpha_{max}}]\boldsymbol{\phi}(\boldsymbol{x}) + p_{k\ 0} + \text{higher order} \quad (5)$$

where

$$p_{k\ j} = \frac{\partial^{(r_1+r_2+\cdots+r_n)}}{\partial x_1^{r_1}\partial x_2^{r_2}\cdots\partial x_n^{r_n}} f_k(\boldsymbol{x}) \big|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_0},$$

$$p_{k\ 0} = f_k(\boldsymbol{x}) \big|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_0},\ j = j(r_1, \cdots, r_n),$$

and $\hat{\boldsymbol{x}}_0$ is an operating point of the nonlinear system. From Eqs. (4) and (5), it follows that

$$\begin{aligned}\dot{\phi}_\alpha(\boldsymbol{x}) &= r_1\Big([p_{11},\ \ p_{12}, \cdots, p_{1\ \alpha_{max}}]\boldsymbol{\phi}(\boldsymbol{x}) + p_{10}\Big) \\ &\times x_1^{r_1-1}x_2^{r_2}\cdots x_n^{r_n} + r_2\Big([p_{21},\ \ p_{22}, \\ &\cdots, p_{2\ \alpha_{max}}]\boldsymbol{\phi}(\boldsymbol{x}) + p_{20}\Big) \times x_1^{r_1}x_2^{r_2-1}\cdots x_n^{r_n} + \\ &\cdots\ + r_n\Big([p_{n1},\ \ p_{n2}, \cdots, p_{n\ \alpha_{max}}]\boldsymbol{\phi}(\boldsymbol{x}) + p_{n0})\Big) \\ &\times x_1^{r_1}x_2^{r_2}\cdots x_n^{r_n-1} + \text{higher order} \\ &= [G_{\alpha\ 1}, \cdots, G_{\alpha\ \beta}, \cdots, G_{\alpha\ \alpha_{max}}]\boldsymbol{\phi}(\boldsymbol{x}) + G_{\alpha\ 0} + \epsilon_\alpha(\boldsymbol{x}) \quad (6)\end{aligned}$$

where

$$G_{\alpha\ \beta} = \sum_{k=1}^{n} r_k p_{k\ \beta(s_1-r_1, s_2-r_2, \cdots, s_k-r_k+1, \cdots, s_n-r_n)},$$

$$\beta = \beta(s_1, \cdots, s_n)\ ,$$

$$p_{k\ \beta} = \begin{cases} p_{k\ j} & (\beta = j) \\ 0 & (\beta \neq j) \end{cases}\ .$$

The $\epsilon_\alpha(\boldsymbol{x})$ is the error term whose order includes higher than N. Using the above functions (Eq. (5)), $\dot{\boldsymbol{\phi}}$ of Eq. (4) is approximated by

$$\dot{\boldsymbol{\phi}}(\boldsymbol{x}) \approx A\boldsymbol{\phi}(\boldsymbol{x}) + b \quad (7)$$

where

$$A = [G_{i\ j}] \subset R^{\alpha_{max}\times\alpha_{max}}\ ,$$

$$b = [G_{i\ 0}] \subset R^{\alpha_{max}}\ .$$

Thus a formal linear state differential equation is derived by

$$\Sigma_2 : \dot{\boldsymbol{z}}(t) = A\boldsymbol{z}(t) + b\ , \quad (8)$$

$$\boldsymbol{z}(0) = \boldsymbol{\phi}(\boldsymbol{x}_0)\ .$$

B. Fromal Linearization for Measurement Equation

With the measurement equation (Eq. (2)), measurement data are exploited in order to develop the approximation error of the linearization.

Let us define a new N-th order measurement vector $\boldsymbol{Y}(\cdot) = \boldsymbol{Y}(\boldsymbol{y}(\cdot))$ which consists of polynomials by

$$\begin{aligned}\boldsymbol{Y} &= [Y_1(\boldsymbol{y}), Y_2(\boldsymbol{y}), \cdots, Y_i(\boldsymbol{y}), \cdots, Y_{\alpha'_{max}}(\boldsymbol{y})]^T \\ &= [T_{(10\cdots0)}(\boldsymbol{y}), T_{(01\cdots0)}(\boldsymbol{y}), \cdots, T_{(0\cdots01)}(\boldsymbol{y}), \\ &\quad T_{(11\cdots0)}(\boldsymbol{y}), T_{(101\cdots0)}(\boldsymbol{y}), \cdots, T_{(10\cdots1)}(\boldsymbol{y}), \\ &\quad T_{(20\cdots0)}(\boldsymbol{y}), T_{(02\cdots0)}(\boldsymbol{y}), \cdots, T_{(r_1\cdots r_m)}(\boldsymbol{y})]^T \quad (9)\end{aligned}$$

where

$$T_{(r_1\cdots r_m)}(\boldsymbol{y}) = \prod_{i=1}^{m} y_i^{r_i}\ ,$$

$$1 \le r_1 + r_2 + \cdots + r_m \le N,$$

$$\alpha'_{max} : \text{the number of combination of } \{r_1, \cdots, r_m\}.$$

From this augmented measurement vector, the measurement equation (Eq. (2)) is reconstructed as

$$Y_{\alpha'}(\boldsymbol{y}) = T_{(r_1\cdots r_m)}(\boldsymbol{y}) = \prod_{i=1}^{m} y_i^{r_i}$$

$$= h_1^{r_1}(\boldsymbol{x})h_2^{r_2}(\boldsymbol{x})\cdots h_m^{r_m}(\boldsymbol{x}), \quad \alpha' = \alpha'(r_1, \cdots, r_m). \quad (10)$$

Applying Taylor expansion up to the N-th order derives

$$Y_{\alpha'} = [q_{\alpha'\ 1}, \cdots, q_{\alpha'\ \beta}, \cdots, q_{\alpha'\ \alpha_{max}}]\boldsymbol{\phi}(\boldsymbol{x}) + q_{\alpha'\ 0} + \text{higher order} \quad (11)$$

where

$$q_{\alpha'\ \beta} = \frac{\partial^{(s_1+r_2+\cdots+s_n)}}{\partial x_1^{s_1}\partial x_2^{s_2}\cdots\partial x_n^{s_n}} h_1^{r_1}(\boldsymbol{x})h_2^{r_2}(\boldsymbol{x})\cdots h_m^{r_m}(\boldsymbol{x}) \big|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_0},$$

$$q_{\alpha'\ 0} = h_1^{r_1}(\boldsymbol{x})h_2^{r_2}(\boldsymbol{x})\cdots h_m^{r_m}(\boldsymbol{x}) \big|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_0}\ .$$

From Eqs. (9) and (11), the augmented measurement equation becomes

$$\boldsymbol{Y} \approx C\boldsymbol{\phi}(\boldsymbol{x}) + d \quad (12)$$

where

$$C = [q_{i\ j}] \subset R^{\alpha'_{max}\times\alpha_{max}}\ ,$$

$$d = [q_{i\ 0}] \subset R^{\alpha'_{max}}\ .$$

Thus a formal linear measurement equation is derived by

$$\boldsymbol{Y}(t) = C\boldsymbol{z}(t) + d\ . \quad (13)$$

C. Design of Nonlinear Observer

To the linearized system (Eqs. (8) and (13)), a linear estimation theory is applied so that the identity observer [9] is synthesized as

$$\dot{\hat{\boldsymbol{z}}}(t) = A\hat{\boldsymbol{z}}(t) + b + K(t)\big(\boldsymbol{Y}(t) - \hat{\boldsymbol{Y}}(t)\big), \tag{14}$$
$$\hat{\boldsymbol{Y}}(t) = C\hat{\boldsymbol{z}}(t) + d,$$
$$\hat{\boldsymbol{z}}(0) = \boldsymbol{\phi}(\hat{\hat{\boldsymbol{x}}}(0))$$

where $\hat{\hat{\boldsymbol{x}}}(0)$ is an initial value of the observer, $K(t)$ is an observer gain as

$$K(t) = \frac{1}{2}P(t)CL(t).$$

$P(t)$ satisfies the matrix Riccati differential equation as

$$\dot{P}(t) = AP(t) + P(t)A^T + Q(t) - P(t)C^T L(t)CP(t) \tag{15}$$

where $Q(t), L(t)$ and $P(0)$ are chosen to be arbitrary real, symmetric, and positive definite. With the reference to the exponential estimator [9], the error in the state estimate

$$\boldsymbol{e} = \boldsymbol{z} - \hat{\boldsymbol{z}}$$

is uniformly asymptotically stable in the sense of Lyapunov.

From Eq.(3), the estimate of the nonlinear observer $\hat{\boldsymbol{x}}(t)$ becomes

$$\hat{\boldsymbol{x}}(t) = [I\ 0\ \cdots\ 0]\boldsymbol{\phi}(\hat{\boldsymbol{x}}(t)) = [I\ 0\ \cdots\ 0]\hat{\boldsymbol{z}}(t) \tag{16}$$

where I is an $n \times n$ unit matrix.

IV. Numerical experiments

We carry out numerical experiments of an estimation of the transient state of an electric power system [7]. The dynamic equation of a synchronous machine under certain assumptions is described by

$$M\ddot{\delta} + D\dot{\delta} + \frac{e_s e_f'}{x_d}\sin\delta = P_{in} \tag{17}$$

where δ is a load angle, M is a moment of inertia, D is a damping coefficient, e_s is an infinite bus voltage, e_f' is an excitation voltage, x_d is a synchronous reactance, and P_{in} is a mechanical input power.

If the reactive power is measured, the measurement equation is

$$y = \frac{e_s}{x_d}(e_s - e_f'\cos\delta). \tag{18}$$

Putting $x_1 = \delta$ and $x_2 = \dot{\delta}$, the electric power system is described by

$$\begin{cases} \dot{x}_1 = x_2 = f_1(\boldsymbol{x}) \\ \dot{x}_2 = -\dfrac{e_s e_f'}{M x_d}\sin x_1 - \dfrac{D}{M}x_2 + \dfrac{P_{in}}{M} = f_2(\boldsymbol{x}) \end{cases}, \tag{19}$$

$$y = \frac{e_s^2}{x_d} - \frac{e_s e_f'}{x_d}\cos x_1 = h(\boldsymbol{x}). \tag{20}$$

Applying the above formal linearization in Sec. III, a formal linear system is obtained by

$$\dot{\boldsymbol{z}}(t) = A\boldsymbol{z}(t) + b\ , \tag{21}$$

$$\boldsymbol{Y}(t) = C\boldsymbol{z}(t) + d\ , \tag{22}$$

where an operating point $\hat{\boldsymbol{x}}_0$ is set to satisfy $\boldsymbol{f}(\hat{\boldsymbol{x}}_0) = 0$ so that

$$\hat{\boldsymbol{x}}_0 = \begin{pmatrix} \sin^{-1} P_{in} \\ 0 \end{pmatrix}$$

and the observer of this power system is

$$\dot{\hat{\boldsymbol{z}}}(t) = A\hat{\boldsymbol{z}}(t) + b + K(t)\big(\boldsymbol{Y}(t) - C\hat{\boldsymbol{z}}(t) - d\big), \tag{23}$$
$$\hat{\boldsymbol{z}}(0) = \boldsymbol{\phi}(\hat{\hat{\boldsymbol{x}}}(0)).$$

Throughout this experiments, the system parameters are set as

$$M = 0.0265,\ D = 0.005,\ P_{in} = 0.8,\ e_s = e_f' = x_d = 1.0.$$

The initial values of the power system $\boldsymbol{x}(0)$ and of the observer $\hat{\hat{\boldsymbol{x}}}(0)$ are

$$\boldsymbol{x}(0) = [2,\ 0.8]^T,\ \hat{\hat{\boldsymbol{x}}}(0) = [0,\ 0]^T.$$

Figs. 1 and 2 show the true value $x_i(t)$ and the estimated values $\hat{x}_i(t)$ for $i = 1$ and 2, respectively. $\hat{x}_i$(new) is the result through this method when the order of the linearization function is $N = 3$, and the parameters of the observer (Eq. (15)) are

$$Q(t) = P(0) = I \subset R^{9\times 9},\ L(t) = \begin{pmatrix} 60 & 0 & 0 \\ 0 & 22 & 0 \\ 0 & 0 & 0.05 \end{pmatrix}.$$

$\hat{x}_i$(old) is by the previous work [8] when the parameters of the linearization are the same as $N = 3$, $Q(t) = P(0) = I \subset R^{9\times 9}$, except for $L(t) = 30$. $\hat{x}_i$(Taylor) is the result by the conventional first order Taylor expansion [10] and the parameters of the identity observer are $Q(t) = P(0) = I \subset R^{2\times 2}, L(t) = 30$.

Fig. 3 shows the integral square errors of estimation

$$J(t) = \int_0^t \big(\boldsymbol{x}(\tau) - \hat{\boldsymbol{x}}(\tau)\big)^T \big(\boldsymbol{x}(\tau) - \hat{\boldsymbol{x}}(\tau)\big)d\tau$$

for the various order of the linearization function from $N = 1$ to $N = 3$. $N = 3$(old) is the error by the previous method [8]. When the order is $N = 1$, it's result is the same as those of the conventional first order Taylor expansion.

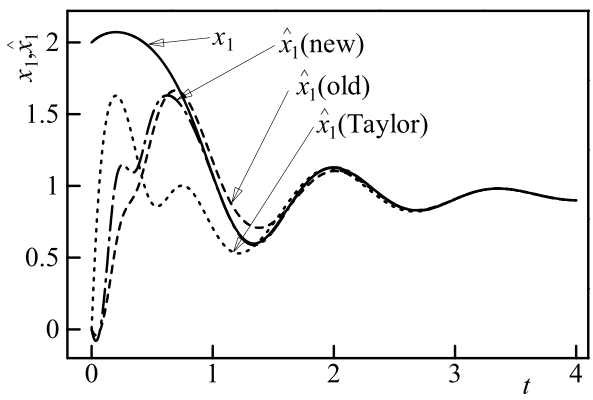

Fig. 1. Estimates $\hat{x}_1(t)$ of the power system by various methods

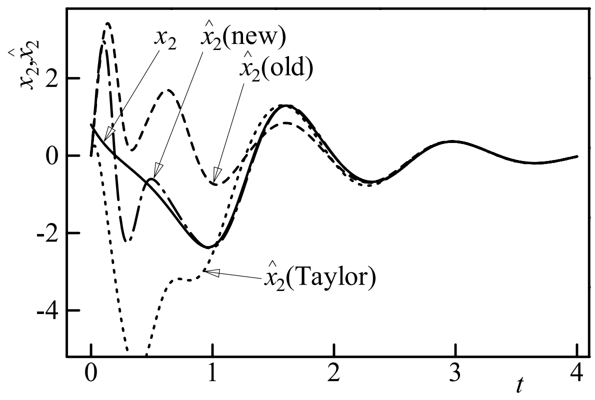

Fig. 2. Estimates $\hat{x}_2(t)$ of the power system by various methods

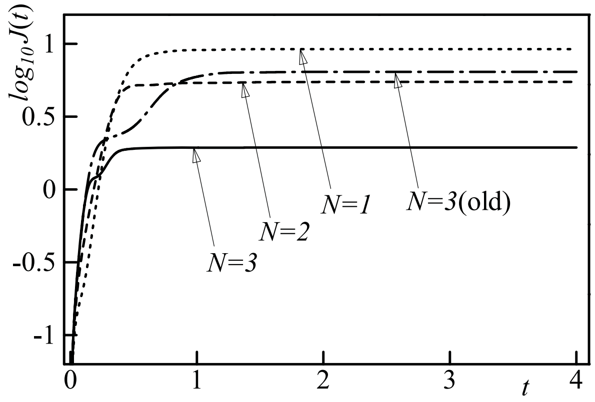

Fig. 3. Integral square errors of estimation by various N

V. Conclusions

We have developed an observer design for a nonlinear system by using a formal linearization method. Introducing an augmented measurement vector, the given nonlinear state and measurement equations make up an augmented linear system to be applied the linear estimation theory. Numerical experiments show that our method is better than the previous works and the accuracy is improved as the order of the linearization function increases.

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