Application of He's parameter-expansion method to a coupled van der Pol oscillators with two kinds of time-delay coupling

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Abstract—In this paper, the dynamics of a system of two van der Pol oscillators with delayed position and velocity is studied. We provide an approximate solution for this system using parameter-expansion method. Also, we obtain approximate values for frequencies of the system. The parameter-expansion method is more efficient than the perturbation method for this system because the method is independent of perturbation parameter assumption.

Keywords—Parameter-expansion method, Coupled van der Pol oscillator, Time-delay system.

I. INTRODUCTION

THE van der Pol oscillator was originally discovered by the Dutch electrical engineer and physicist Balthasar Van der Pol. Van der Pol found stable oscillations, now known as limit cycles, in electrical circuits employing vacuum tubes. To solve nonlinear oscillators many effective methods have been introduced, such as the

the variational iteration method [1], [2], [3], the Adomian decomposition method [4], [5], the homotopy perturbation method [6], [7], [8], [9], [10], spectral collocation method [11], [12], [13], [14], [15], homotopy analysis method [16], [17], [18], [19], [20], [21], [22], and the Exp-function method [23], [24], [25], [26], [27], [28]. Consider the following mathematical model of a coupled van der Pol oscillators

$$\begin{cases} \ddot{x} + \omega_1^2 x - \varepsilon(1 - x^2)\dot{x} = \varepsilon\alpha_2[y(t - \tau) + \dot{y}(t - \tau)], \\ \ddot{y} + \omega_2^2 y - \varepsilon(1 - y^2)\dot{y} = \varepsilon\alpha_1[x(t - \tau) + \dot{x}(t - \tau)], \end{cases} \quad (1)$$

where ε is a small positive parameter and ω_1 and ω_2 are linear undamped natural frequencies of x and y, respectively. Also dot shows differentiation with respect to time, t. In (1) the time delay τ is a positive constant, and two oscillators are coupled with two kinds of time-delay coupling. Li et al. [29] studied the dynamics of (1) by the method of averaging together with truncation of Taylor expansions. They determined the condition necessary for in-phase and out-phase modes, which is the condition necessary for saddle-node and Hopf bifurcation for symmetric modes.

Nonlinear oscillators with delay terms are very difficult to be solved, even more difficult for coupled systems. This paper suggests a universal method to the problem using a new technology called the parameter-expansion method, which

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is very effective to the problem. We apply the parameter-expansion method to obtain approximate solution of system (1), also we provide numerical approximations for frequencies of x and y.

II. PARAMETER EXPANSION METHOD

Parameter-expansion method is an easy and straightforward approach to nonlinear oscillators. Anyone can apply the method to find approximation of amplitude-frequency relationship of a nonlinear oscillator only with basic knowledge of advance calculus. The basic idea of the parameter-expansion method was provided in [30] and one may find several applications of the method over various areas in [31], [32], [33], [34], [35], [36], [37], [38], [39], [40].

To apply parameter-expansion method on (1) we rewrite the system as

$$\begin{aligned} x'' + \omega_1^2 x &= \varepsilon x'(1 - x^2) + \varepsilon\alpha_2(y + y')(t - \tau), \\ y'' + \omega_2^2 y &= \varepsilon y'(1 - y^2) + \varepsilon\alpha_1(x + x')(t - \tau). \end{aligned} \quad (2)$$

In this section we consider case of $\omega_1 \neq \omega_2$, the case of $\omega_1 = \omega_2$ is considered separately in the next section. According to the parameter-expansion method, all variables x and y can be expanded into a series of an artificial parameter p such as

$$\begin{aligned} x &= x_0 + px_1 + p^2 x_2 + \cdots \\ y &= y_0 + py_1 + p^2 y_2 + \cdots \end{aligned} \quad (3)$$

where p is called a bookkeeping parameter [30]. We also expand all coefficients of the system (1) into a series of p in a similar way

$$\begin{aligned} \omega_1^2 &= \mu^2 + p\mu_1 + p^2\mu_2 + \cdots \\ \omega_2^2 &= \nu^2 + p\nu_1 + p^2\nu_2 + \cdots \\ \varepsilon\alpha_1 &= p\delta_{11} + p^2\delta12 + \cdots \\ \varepsilon\alpha_2 &= p\delta_{21} + p^2\delta22 + \cdots \\ \varepsilon &= p\varepsilon_1 + p^2\varepsilon_2 + \cdots. \end{aligned} \quad (4)$$

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Vol:5, No:2, 2011

By substituting expansions (3) and (4) into system (2), we have

$$(x_0'' + p x_1'' + p^2 x_2'' + \cdots) + (\mu^2 + p\mu_1 + p^2\mu_2 + \cdots) \times$$
$$(x_0 + p x_1 + p^2 x_2 + \cdots) = (p\varepsilon_1 + p^2\varepsilon_2 + \cdots) \times$$
$$(x_0' + p x_1' + p^2 x_2' + \cdots)[1 - (x_0 + p x_1 + p^2 x_2 + \cdots)^2] +$$
$$(p\delta_{21} + p^2\delta_{22} + \cdots)[(y_0 + p y_1 + p^2 y_2 + \cdots) +$$
$$(y_0' + p y_1' + p^2 y_2' + \cdots)](t - \tau),$$

$$(y_0'' + p y_1'' + p^2 y_2'' + \cdots) + (\nu^2 + p\nu_1 + p^2\nu_2 + \cdots) \times$$
$$(y_0 + p y_1 + p^2 y_2 + \cdots) = (p\varepsilon_1 + p^2\varepsilon_2 + \cdots) \times$$
$$(y_0' + p y_1' + p^2 y_2' + \cdots)[1 - (y_0 + p y_1 + p^2 y_2 + \cdots)^2] +$$
$$(p\delta_{11} + p^2\delta_{12} + \cdots)[(x_0 + p x_1 + p^2 x_2 + \cdots) +$$
$$(x_0' + p x_1' + p^2 x_2' + \cdots)](t - \tau).$$

$$(5)$$

Equating in powers of p, yields

$$p^0 : \begin{cases} x_0'' + \mu^2 x_0 = 0 \\ y_0'' + \nu^2 y_0 = 0 \end{cases} \qquad (6)$$

and

$$p^1 : \begin{cases} x_1'' + \mu^2 x_1 + \mu_1 x_0 = \\ \quad \varepsilon_1 x_0'(1 - x_0^2) + \delta_{21}(y_0 + y_0')(t - \tau), \\ y_1'' + \nu^2 y_1 + \nu_1 y_0 = \\ \quad \varepsilon_1 y_0'(1 - y_0^2) + \delta_{11}(x_0 + x_0')(t - \tau). \end{cases} \qquad (7)$$

Solving Eq. (6), we obtain

$$\begin{cases} x_0 = A_1 \cos(\mu t) + A_2 \sin(\mu t) \\ y_0 = B_1 \cos(\nu t) + B_2 \sin(\nu t), \end{cases} \qquad (8)$$

where A_1, A_2, B_1 and B_2 are arbitrary constants. Substituting (8) into (7), we obtain

$$\begin{cases} x_1'' + \mu^2 x_1 = \\ \quad \sin(\mu t)[-\mu_1 A_2 - \mu\varepsilon_1 A_1 + \frac{1}{4}\mu\varepsilon_1 A_1 A_2^2 + \frac{1}{4}\mu\varepsilon_1 A_1^3] \\ + \cos(\mu t)[-\mu_1 A_1 + \mu\varepsilon_1 A_2 - \frac{1}{4}\mu\varepsilon_1 A_2^3 - \frac{1}{4}\mu\varepsilon_1 A_1^2 A_2] \\ + \sin(3\mu t)[-\frac{3}{4}\mu\varepsilon_1 A_1 A_2^2 + \frac{1}{4}\mu\varepsilon_1 A_1^3] \\ + \cos(3\mu t)[-\frac{3}{4}\mu\varepsilon_1 A_1^2 A_2 + \frac{1}{4}\mu\varepsilon_1 A_2^3] \\ + \sin(\nu t)[-\delta_{21} B_2 \tau + \delta_{21}\nu B_1 \tau] \\ + \cos(\nu t)[-\delta_{21} B_1 \tau - \delta_{21}\nu B_2 \tau] \\ + t[(\delta_{21} B_1 + \delta_{21}\nu B_2)\cos(\nu t) + \\ \quad (\delta_{21} B_2 - \delta_{21}\nu B_1)\sin(\nu t)] \\[6pt] y_1'' + \nu^2 y_1 = \\ \quad \sin(\nu t)[-\nu_1 B_2 - \nu\varepsilon_1 B_1 + \frac{1}{4}\nu\varepsilon_1 B_1 B_2^2 + \frac{1}{4}\nu\varepsilon_1 B_1^3] \\ + \cos(\nu t)[-\nu_1 B_1 + \nu\varepsilon_1 B_2 - \frac{1}{4}\nu\varepsilon_1 B_2^3 - \frac{1}{4}\nu\varepsilon_1 B_1^2 B_2] \\ + \sin(3\nu t)[-\frac{3}{4}\nu\varepsilon_1 B_1 B_2^2 + \frac{1}{4}\nu\varepsilon_1 B_1^3] \\ + \cos(3\nu t)[-\frac{3}{4}\nu\varepsilon_1 B_1^2 B_2 + \frac{1}{4}\nu\varepsilon_1 B_2^3] \\ + \sin(\mu t)[-\delta_{11} A_2 \tau + \delta_{11}\mu A_1 \tau] \\ + \cos(\mu t)[-\delta_{11} A_1 \tau - \delta_{11}\mu A_2 \tau] \\ + t[(\delta_{11} A_1 + \delta_{11}\mu A_2)\cos(\mu t) + \\ \quad (\delta_{11} A_2 - \delta_{11}\mu A_1)\sin(\mu t)]. \end{cases}$$

$$(9)$$

If the first-order approximation be enough, then setting $p = 1$ in equations (3) and (4), we have

$$x = x_0 + x_1, \quad \omega_1^2 = \mu^2 + \mu_1, \quad \varepsilon\alpha_1 = \delta_{11}, \quad \varepsilon = \varepsilon_1$$
$$y = y_0 + y_1, \quad \omega_2^2 = \nu^2 + \nu_1, \quad \varepsilon\alpha_2 = \delta_{21}.$$

$$(10)$$

Now substituting (10) into (9) yields:

$$\begin{cases} x_1'' + \mu^2 x_1 = \\ \quad \sin(\mu t)[-\mu_1 A_2 - \mu\varepsilon A_1 + \frac{1}{4}\mu\varepsilon A_1 A_2^2 + \frac{1}{4}\mu\varepsilon A_1^3] \\ + \cos(\mu t)[-\mu_1 A_1 + \mu\varepsilon A_2 - \frac{1}{4}\mu\varepsilon A_2^3 - \frac{1}{4}\mu\varepsilon A_1^2 A_2] \\ + \sin(3\mu t)[-\frac{3}{4}\mu\varepsilon A_1 A_2^2 + \frac{1}{4}\mu\varepsilon A_1^3] \\ + \cos(3\mu t)[-\frac{3}{4}\mu\varepsilon A_1^2 A_2 + \frac{1}{4}\mu\varepsilon A_2^3] \\ + \sin(\nu t)[-\varepsilon\alpha_2 B_2 \tau + \varepsilon\alpha_2\nu B_1 \tau] \\ + \cos(\nu t)[-\varepsilon\alpha_2 B_1 \tau - \varepsilon\alpha_2\nu B_2 \tau] \\ + t[(\varepsilon\alpha_2 B_1 + \varepsilon\alpha_2\nu B_2)\cos(\nu t) \\ + (\varepsilon\alpha_2 B_2 - \varepsilon\alpha_2\nu B_1)\sin(\nu t)] \\[6pt] y_1'' + \nu^2 y_1 = \\ \quad \sin(\nu t)[-\nu_1 B_2 - \nu\varepsilon B_1 + \frac{1}{4}\nu\varepsilon B_1 B_2^2 + \frac{1}{4}\nu\varepsilon B_1^3] \\ + \cos(\nu t)[-\nu_1 B_1 + \nu\varepsilon B_2 - \frac{1}{4}\nu\varepsilon B_2^3 - \frac{1}{4}\nu\varepsilon B_1^2 B_2] \\ + \sin(3\nu t)[-\frac{3}{4}\nu\varepsilon B_1 B_2^2 + \frac{1}{4}\nu\varepsilon B_1^3] \\ + \cos(3\nu t)[-\frac{3}{4}\nu\varepsilon B_1^2 B_2 + \frac{1}{4}\nu\varepsilon B_2^3] \\ + \sin(\mu t)[-\varepsilon\alpha_1 A_2 \tau + \varepsilon\alpha_1\mu A_1 \tau] \\ + \cos(\mu t)[-\varepsilon\alpha_1 A_1 \tau - \varepsilon\alpha_1\mu A_2 \tau] \\ + t[(\varepsilon\alpha_1 A_1 + \varepsilon\alpha_1\mu A_2)\cos(\mu t) \\ + (\varepsilon\alpha_1 A_2 - \varepsilon\alpha_1\mu A_1)\sin(\mu t)]. \end{cases}$$

$$(11)$$

No secular term in x_1 and y_1 requires that

$$\begin{cases} -\mu_1 A_2 - \mu\varepsilon A_1 + \frac{1}{4}\mu\varepsilon A_1 A_2^2 + \frac{1}{4}\mu\varepsilon A_1^3 = 0, \\ -\mu_1 A_1 + \mu\varepsilon A_2 - \frac{1}{4}\mu\varepsilon A_2^3 - \frac{1}{4}\mu\varepsilon A_1^2 A_2 = 0, \end{cases} \qquad (12)$$

and

$$\begin{cases} -\nu_1 B_2 - \nu\varepsilon B_1 + \frac{1}{4}\nu\varepsilon B_1 B_2^2 + \frac{1}{4}\nu\varepsilon B_1^3 = 0, \\ -\nu_1 B_1 + \nu\varepsilon B_2 - \frac{1}{4}\nu\varepsilon B_2^3 - \frac{1}{4}\nu\varepsilon B_1^2 B_2 = 0, \end{cases} \qquad (13)$$

using (12), (13) and (10), we have

$$\begin{cases} \mu = \omega_1 \\ \nu = \omega_2. \end{cases} \qquad (14)$$

The frequencies of equations are

$$\begin{cases} T_x = \frac{2\pi}{\omega_1} \\ T_y = \frac{2\pi}{\omega_2}. \end{cases} \qquad (15)$$

Furthermore, (11) can be simplified as

$$\begin{cases} x_1'' + \mu^2 x_1 = \\ \quad \sin(3\mu t)[-\frac{3}{4}\mu\varepsilon A_1 A_2^2 + \frac{1}{4}\mu\varepsilon A_1^3] \\ + \cos(3\mu t)[-\frac{3}{4}\mu\varepsilon A_1^2 A_2 + \frac{1}{4}\mu\varepsilon A_2^3] \\ + \sin(\nu t)[-\varepsilon\alpha_2 B_2 \tau + \varepsilon\alpha_2\nu B_1 \tau] \\ + \cos(\nu t)[-\varepsilon\alpha_2 B_1 \tau - \varepsilon\alpha_2\nu B_2 \tau] \\ + t[(\varepsilon\alpha_2 B_1 + \varepsilon\alpha_2\nu B_2)\cos(\nu t) \\ + (\varepsilon\alpha_2 B_2 - \varepsilon\alpha_2\nu B_1)\sin(\nu t)] \\[6pt] y_1'' + \nu^2 y_1 = \\ \quad \sin(3\nu t)[-\frac{3}{4}\nu\varepsilon B_1 B_2^2 + \frac{1}{4}\nu\varepsilon B_1^3] \\ + \cos(3\nu t)[-\frac{3}{4}\nu\varepsilon B_1^2 B_2 + \frac{1}{4}\nu\varepsilon B_2^3] \\ + \sin(\mu t)[-\varepsilon\alpha_1 A_2 \tau + \varepsilon\alpha_1\mu A_1 \tau] \\ + \cos(\mu t)[-\varepsilon\alpha_1 A_1 \tau - \varepsilon\alpha_1\mu A_2 \tau] \\ + t[(\varepsilon\alpha_1 A_1 + \varepsilon\alpha_1\mu A_2)\cos(\mu t) \\ + (\varepsilon\alpha_1 A_2 - \varepsilon\alpha_1\mu A_1)\sin(\mu t)]. \end{cases}$$

$$(16)$$

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Solving (16) yields

$$
\left\{
\begin{array}{l}
x_1 = \\
\frac{\varepsilon A_1}{32\mu}(3A_2^2 - A_1^2)\sin(3\mu t) + \frac{\varepsilon A_2}{32\mu}(3A_1^2 - A_2^2)\cos(3\mu t) \\
+\big[\frac{\varepsilon \alpha_2(B_2 - \nu B_1)}{\mu^2 - \nu^2}t+ \\
\quad \frac{\varepsilon \alpha_2\tau(\mu^2-\nu^2)(\nu B_1 - B_2)+2\varepsilon\alpha_2\nu(\nu B_2+B_1)}{(\mu^2-\nu^2)^2}\big]\sin(\nu t) \\
+\big[\frac{\varepsilon \alpha_2(B_1 + \nu B_2)}{\mu^2 - \nu^2}t+ \\
\quad \frac{-\varepsilon \alpha_2\tau(\mu^2-\nu^2)(\nu B_2 + B_1)+2\varepsilon\alpha_2\nu(\nu B_1-B_2)}{(\mu^2-\nu^2)^2}\big]\cos(\nu t), \\
\\
y_1 = \\
\frac{\varepsilon B_1}{32\nu}(3B_2^2 - B_1^2)\sin(3\nu t) + \frac{\varepsilon B_2}{32\nu}(3B_1^2 - B_2^2)\cos(3\nu t) \\
+\big[\frac{\varepsilon \alpha_1(A_2 - \mu A_1)}{\nu^2 - \mu^2}t+ \\
\quad \frac{\varepsilon \alpha_1\tau(\nu^2-\mu^2)(\mu A_1 - A_2)+2\varepsilon\alpha_1\mu(\mu A_2+A_1)}{(\nu^2-\mu^2)^2}\big]\sin(\mu t) \\
+\big[\frac{\varepsilon \alpha_1(A_1 + \mu A_2)}{\nu^2 - \mu^2}t+ \\
\quad \frac{-\varepsilon \alpha_1\tau(\nu^2-\mu^2)(\mu A_2 + A_1)+2\varepsilon\alpha_1\mu(\mu A_1-A_2)}{(\nu^2-\mu^2)^2}\big]\cos(\mu t).
\end{array}
\right.
\tag{17}
$$

Now using (17), (8) and (10) we obtain the first order solution for x and y

$$
\left\{
\begin{array}{l}
x = x_0 + x_1 = \\
A_1\cos(\mu t) + A_2\sin(\mu t) \\
+\frac{\varepsilon A_1}{32\mu}(3A_2^2 - A_1^2)\sin(3\mu t) + \frac{\varepsilon A_2}{32\mu}(3A_1^2 - A_2^2)\cos(3\mu t) \\
+\big[\frac{\varepsilon \alpha_2(B_2 - \nu B_1)}{\mu^2 - \nu^2}t+ \\
\quad \frac{\varepsilon \alpha_2\tau(\mu^2-\nu^2)(\nu B_1 - B_2)+2\varepsilon\alpha_2\nu(\nu B_2+B_1)}{(\mu^2-\nu^2)^2}\big]\sin(\nu t) \\
+\big[\frac{\varepsilon \alpha_2(B_1 + \nu B_2)}{\mu^2 - \nu^2}t+ \\
\quad \frac{-\varepsilon \alpha_2\tau(\mu^2-\nu^2)(\nu B_2 + B_1)+2\varepsilon\alpha_2\nu(\nu B_1-B_2)}{(\mu^2-\nu^2)^2}\big]\cos(\nu t), \\
\\
y = y_0 + y_1 = \\
B_1\cos(\nu t) + B_2\sin(\nu t) \\
+\frac{\varepsilon B_1}{32\nu}(3B_2^2 - B_1^2)\sin(3\nu t) + \frac{\varepsilon B_2}{32\nu}(3B_1^2 - B_2^2)\cos(3\nu t) \\
+\big[\frac{\varepsilon \alpha_1(A_2 - \mu A_1)}{\nu^2 - \mu^2}t+ \\
\quad \frac{\varepsilon \alpha_1\tau(\nu^2-\mu^2)(\mu A_1 - A_2)+2\varepsilon\alpha_1\mu(\mu A_2+A_1)}{(\nu^2-\mu^2)^2}\big]\sin(\mu t) \\
+\big[\frac{\varepsilon \alpha_1(A_1 + \mu A_2)}{\nu^2 - \mu^2}t+ \\
\quad \frac{-\varepsilon \alpha_1\tau(\nu^2-\mu^2)(\mu A_2 + A_1)+2\varepsilon\alpha_1\mu(\mu A_1-A_2)}{(\nu^2-\mu^2)^2}\big]\cos(\mu t).
\end{array}
\right.
\tag{18}
$$

III. CASE $\omega_1 = \omega_2$

In this section we suppose that $\omega_1 = \omega_2 = \omega$ and provide approximate solution of system (1) using parameter-expansion method. In this case system (1) is rewritten as

$$
\begin{array}{l}
x'' + \omega^2 x = \varepsilon x'(1 - x^2) + \varepsilon \alpha_2(y + y')(t - \tau), \\
y'' + \omega^2 y = \varepsilon y'(1 - y^2) + \varepsilon \alpha_1(x + x')(t - \tau).
\end{array}
\tag{19}
$$

As usual we expand all variables and parameters into a series of the artificial parameter p as

$$
\begin{array}{l}
x = x_0 + px_1 + p^2 x_2 + \cdots \\
y = y_0 + py_1 + p^2 y_2 + \cdots \\
\omega^2 = \mu^2 + p\mu_1 + p^2\mu_2 + \cdots \\
\varepsilon\alpha_1 = p\delta_{11} + p^2\delta 12 + \cdots \\
\varepsilon\alpha_2 = p\delta_{21} + p^2\delta 22 + \cdots \\
\varepsilon = p\varepsilon_1 + p^2\varepsilon_2 + \cdots .
\end{array}
\tag{20}
$$

Substituting expansions (20) into system (19), and equating in powers of p, yields

$$
p^0 : \left\{
\begin{array}{l}
x_0'' + \mu^2 x_0 = 0 \\
y_0'' + \mu^2 y_0 = 0
\end{array}
\right.
\tag{21}
$$

and

$$
p^1 : \left\{
\begin{array}{l}
x_1'' + \mu^2 x_1 + \mu_1 x_0 = \\
\quad \varepsilon_1 x_0'(1 - x_0^2) + \delta_{21}(y_0 + y_0')(t - \tau), \\
y_1'' + \mu^2 y_1 + \mu_1 y_0 = \\
\quad \varepsilon_1 y_0'(1 - y_0^2) + \delta_{11}(x_0 + x_0')(t - \tau).
\end{array}
\right.
\tag{22}
$$

Solving (21), we obtain

$$
\left\{
\begin{array}{l}
x_0 = A_1\cos(\mu t) + A_2\sin(\mu t) \\
y_0 = B_1\cos(\mu t) + B_2\sin(\mu t),
\end{array}
\right.
\tag{23}
$$

where A_1, A_2, B_1 and B_2 are arbitrary constants. Substituting (23) into (22), yields

$$
\left\{
\begin{array}{l}
x_1'' + \mu^2 x_1 = \\
\sin(\mu t)[-\mu_1 A_2 - \mu\varepsilon_1 A_1 + \frac{1}{4}\mu\varepsilon_1 A_1 A_2^2+ \\
\quad \frac{1}{4}\mu\varepsilon_1 A_1^3 - \delta_{21}B_2\tau + \delta_{21}\mu B_1\tau] \\
+\cos(\mu t)[-\mu_1 A_1 + \mu\varepsilon_1 A_2 - \frac{1}{4}\mu\varepsilon_1 A_2^3- \\
\quad \frac{1}{4}\mu\varepsilon_1 A_1^2 A_2 - \delta_{21}B_1\tau - \delta_{21}\mu B_2\tau] \\
+\sin(3\mu t)[-\frac{3}{4}\mu\varepsilon_1 A_1 A_2^2 + \frac{1}{4}\mu\varepsilon_1 A_1^3] \\
+\cos(3\mu t)[-\frac{3}{4}\mu\varepsilon_1 A_1^2 A_2 + \frac{1}{4}\mu\varepsilon_1 A_2^3] \\
+t[(\delta_{21}B_1 + \delta_{21}\mu B_2)\cos(\mu t)+ \\
\quad (\delta_{21}B_2 - \delta_{21}\mu B_1)\sin(\mu t)] \\
\\
y_1'' + \mu^2 y_1 = \\
\sin(\mu t)[-\mu_1 B_2 - \mu\varepsilon_1 B_1 + \frac{1}{4}\mu\varepsilon_1 B_1 B_2^2+ \\
\quad \frac{1}{4}\mu\varepsilon_1 B_1^3 - \delta_{11}A_2\tau + \delta_{11}\mu A_1\tau] \\
+\cos(\mu t)[-\mu_1 B_1 + \mu\varepsilon_1 B_2- \\
\quad \frac{1}{4}\mu\varepsilon_1 B_2^3 - \frac{1}{4}\mu\varepsilon_1 B_1^2 B_2 - \delta_{11}A_1\tau - \delta_{11}\mu A_2\tau] \\
+\sin(3\mu t)[-\frac{3}{4}\mu\varepsilon_1 B_1 B_2^2 + \frac{1}{4}\mu\varepsilon_1 B_1^3] \\
+\cos(3\mu t)[-\frac{3}{4}\mu\varepsilon_1 B_1^2 B_2 + \frac{1}{4}\mu\varepsilon_1 B_2^3] \\
+t[(\delta_{11}A_1 + \delta_{11}\mu A_2)\cos(\mu t)+ \\
\quad (\delta_{11}A_2 - \delta_{11}\mu A_1)\sin(\mu t)].
\end{array}
\right.
\tag{24}
$$

If the first-order approximation be enough, then setting $p = 1$ in equations (20), we have

$$
\begin{array}{ll}
x = x_0 + x_1, & \omega^2 = \mu^2 + \mu_1, \quad \varepsilon\alpha_1 = \delta_{11}, \\
y = y_0 + y_1, & \varepsilon = \varepsilon_1, \qquad \varepsilon\alpha_2 = \delta_{21}.
\end{array}
\tag{25}
$$

Now substituting (25) into (24) yields:

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
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$$\begin{cases} x_1'' + \mu^2 x_1 = \\ \sin(\mu t)[-\mu_1 A_2 - \mu\varepsilon A_1 + \frac{1}{4}\mu\varepsilon A_1 A_2^2 + \\ \qquad \frac{1}{4}\mu\varepsilon A_1^3 - \varepsilon\alpha_2 B_2 \tau + \varepsilon\alpha_2\mu B_1\tau] \\ + \cos(\mu t)[-\mu_1 A_1 + \mu\varepsilon A_2 - \frac{1}{4}\mu\varepsilon A_2^3 - \\ \qquad \frac{1}{4}\mu\varepsilon A_1^2 A_2 - \varepsilon\alpha_2 B_1\tau - \varepsilon\alpha_2\mu B_2\tau] \\ + \sin(3\mu t)[-\frac{3}{4}\mu\varepsilon A_1 A_2^2 + \frac{1}{4}\mu\varepsilon A_1^3] \\ + \cos(3\mu t)[-\frac{3}{4}\mu\varepsilon A_1^2 A_2 + \frac{1}{4}\mu\varepsilon A_2^3] \\ + t[(\varepsilon\alpha_2 B_1 + \varepsilon\alpha_2\mu B_2)\cos(\mu t) + \\ \qquad (\varepsilon\alpha_2 B_2 - \varepsilon\alpha_2\mu B_1)\sin(\mu t)] \\ \\ y_1'' + \mu^2 y_1 = \\ \sin(\mu t)[-\mu_1 B_2 - \mu\varepsilon B_1 + \frac{1}{4}\mu\varepsilon B_1 B_2^2 + \\ \qquad \frac{1}{4}\mu\varepsilon B_1^3 - \varepsilon\alpha_1 A_2\tau + \varepsilon\alpha_1\mu A_1\tau] \\ + \cos(\mu t)[-\mu_1 B_1 + \mu\varepsilon B_2 - \frac{1}{4}\mu\varepsilon B_2^3 - \\ \qquad \frac{1}{4}\mu\varepsilon B_1^2 B_2 - \varepsilon\alpha_1 A_1\tau - \varepsilon\alpha_1\mu A_2\tau] \\ + \sin(3\mu t)[-\frac{3}{4}\mu\varepsilon B_1 B_2^2 + \frac{1}{4}\mu\varepsilon B_1^3] \\ + \cos(3\mu t)[-\frac{3}{4}\mu\varepsilon B_1^2 B_2 + \frac{1}{4}\mu\varepsilon B_2^3] \\ + t[(\varepsilon\alpha_1 A_1 + \varepsilon\alpha_1\mu A_2)\cos(\mu t) + \\ \qquad (\varepsilon\alpha_1 A_2 - \varepsilon\alpha_1\mu A_1)\sin(\mu t)]. \end{cases} \quad (26)$$

No secular term in x_1 requires that

$$\begin{cases} -\mu_1 A_2 - \mu\varepsilon A_1 + \frac{1}{4}\mu\varepsilon A_1 A_2^2 + \frac{1}{4}\mu\varepsilon A_1^3 - \\ \qquad \varepsilon\alpha_2 B_2\tau + \varepsilon\alpha_2\mu B_1\tau = 0, \\ -\mu_1 A_1 + \mu\varepsilon A_2 - \frac{1}{4}\mu\varepsilon A_2^3 - \frac{1}{4}\mu\varepsilon A_1^2 A_2 - \\ \qquad \varepsilon\alpha_2 B_1\tau - \varepsilon\alpha_2\mu B_2\tau = 0 \end{cases} \quad (27)$$

then

$$\mu_1 = \frac{\varepsilon\alpha_2\tau}{A_1^2 + A_2^2}\Big(\mu(A_2 B_1 - A_1 B_2) - (A_2 B_2 + A_1 B_1)\Big). \quad (28)$$

Now using $\omega^2 = \mu^2 + \mu_1$ into (28) yields:

$$\omega^2 = \mu^2 + \frac{\varepsilon\alpha_2\tau(A_2 B_1 - A_1 B_2)}{A_1^2 + A_2^2}\mu - \frac{\varepsilon\alpha_2\tau(A_2 B_2 + A_1 B_1)}{A_1^2 + A_2^2} \quad (29)$$

solving equation (29) with respect to μ yields

$$\mu = \frac{\varepsilon\alpha_2\tau(A_1 B_2 - A_2 B_1) \pm K}{2(A_1^2 + A_2^2)^2} \quad (30)$$

where $K = \sqrt{K_1 + K_2}$ for $K_1 = (\varepsilon\alpha_2\tau(A_2 B_1 - A_1 B_2))^2$ and $K_2 = 4\varepsilon\alpha_2\tau(A_1 B_1 + A_2 B_2)(A_1^2 + A_2^2)$. No secular term in y_1 requires that

$$\begin{cases} -\mu_1 B_2 - \mu\varepsilon B_1 + \frac{1}{4}\mu\varepsilon B_1 B_2^2 + \frac{1}{4}\mu\varepsilon B_1^3 - \\ \qquad \varepsilon\alpha_1 A_2\tau + \varepsilon\alpha_1\mu A_1\tau = 0, \\ -\mu_1 B_1 + \mu\varepsilon B_2 - \frac{1}{4}\mu\varepsilon B_2^3 - \frac{1}{4}\mu\varepsilon B_1^2 B_2 - \\ \qquad \varepsilon\alpha_1 A_1\tau - \varepsilon\alpha_1\mu A_2\tau = 0 \end{cases} \quad (31)$$

then

$$\mu_1 = \frac{\varepsilon\alpha_1\tau}{B_1^2 + B_2^2}\Big(\mu(B_2 A_1 - B_1 A_2) - (B_2 A_2 + B_1 A_1)\Big). \quad (32)$$

Now using $\omega^2 = \mu^2 + \mu_1$ into (32) yields

$$\omega^2 = \mu^2 + \frac{\varepsilon\alpha_1\tau(B_2 A_1 - B_1 A_2)}{B_1^2 + A_2^2}\mu - \frac{\varepsilon\alpha_1\tau(B_2 A_2 + B_1 A_1)}{B_1^2 + A_2^2} \quad (33)$$

solving equation (33) with respect to μ yields

$$\mu = \frac{E \pm F}{2(B_1^2 + B_2^2)^2} \quad (34)$$

where

$$E = \varepsilon\alpha_1\tau(B_1 A_2 - B_2 A_1)$$

and $F = \sqrt{G + H}$ for $G = (\varepsilon\alpha_1\tau(B_2 A_1 - B_1 A_2))^2$ and $H = 4\varepsilon\alpha_1\tau(B_1 A_1 + B_2 A_2)(B_1^2 + B_2^2)$. The frequencies of equations are

$$\begin{cases} T_x = \dfrac{4\pi(A_1^2 + A_2^2)^2}{X_1 \pm \sqrt{X_2}} \\ \\ T_y = \dfrac{4\pi(B_1^2 + B_2^2)^2}{Y_1 \pm \sqrt{Y_2}}. \end{cases} \quad (35)$$

where

$$\begin{aligned} X_1 &= \varepsilon\alpha_2\tau(A_1 B_2 - A_2 B_1), \\ X_2 &= (\varepsilon\alpha_2\tau(A_2 B_1 - A_1 B_2))^2 \\ &\quad + 4\varepsilon\alpha_2\tau(A_1 B_1 + A_2 B_2)(A_1^2 + A_2^2), \\ Y_1 &= \varepsilon\alpha_1\tau(B_1 A_2 - B_2 A_1) \end{aligned}$$

and

$$\begin{aligned} Y_2 &= (\varepsilon\alpha_1\tau(B_2 A_1 - B_1 A_2))^2 \\ &\quad + 4\varepsilon\alpha_1\tau(B_1 A_1 + B_2 A_2)(B_1^2 + B_2^2). \end{aligned}$$

Furthermore, (26) can be simplified as

$$\begin{cases} x_1'' + \mu^2 x_1 = \\ \sin(3\mu t)[-\frac{3}{4}\mu\varepsilon A_1 A_2^2 + \frac{1}{4}\mu\varepsilon A_1^3] \\ + \cos(3\mu t)[-\frac{3}{4}\mu\varepsilon A_1^2 A_2 + \frac{1}{4}\mu\varepsilon A_2^3] \\ + t[(\varepsilon\alpha_2 B_1 + \varepsilon\alpha_2\mu B_2)\cos(\mu t) + \\ \qquad (\varepsilon\alpha_2 B_2 - \varepsilon\alpha_2\mu B_1)\sin(\mu t)] \\ \\ y_1'' + \mu^2 y_1 = \\ \sin(3\mu t)[-\frac{3}{4}\mu\varepsilon B_1 B_2^2 + \frac{1}{4}\mu\varepsilon B_1^3] \\ + \cos(3\mu t)[-\frac{3}{4}\mu\varepsilon B_1^2 B_2 + \frac{1}{4}\mu\varepsilon B_2^3] \\ + t[(\varepsilon\alpha_1 A_1 + \varepsilon\alpha_1\mu A_2)\cos(\mu t) + \\ \qquad (\varepsilon\alpha_1 A_2 - \varepsilon\alpha_1\mu A_1)\sin(\mu t)]. \end{cases} \quad (36)$$

Solving (36) yields

$$\begin{cases} x_1 = \\ \frac{\varepsilon A_1}{32\mu}(3A_2^2 - A_1^2)\sin(3\mu t) + \frac{\varepsilon A_2}{32\mu}(3A_1^2 - A_2^2)\cos(3\mu t) \\ + \frac{\varepsilon\alpha_2}{4\mu}[(B_1 + \mu B_2)t^2 + (\frac{B_2}{\mu} - B_1)t]\sin(\mu t) \\ + \frac{\varepsilon\alpha_2}{4\mu}[(\mu B_1 - B_2)t^2 + (\frac{B_1}{\mu} + B_2)t]\cos(\mu t) \\ \\ y_1 = \\ \frac{\varepsilon B_1}{32\mu}(3B_2^2 - B_1^2)\sin(3\mu t) + \frac{\varepsilon B_2}{32\mu}(3B_1^2 - B_2^2)\cos(3\mu t) \\ + \frac{\varepsilon\alpha_1}{4\mu}[(A_1 + \mu A_2)t^2 + (\frac{A_2}{\mu} - A_1)t]\sin(\mu t) \\ + \frac{\varepsilon\alpha_1}{4\mu}[(\mu A_1 - A_2)t^2 + (\frac{A_1}{\mu} + A_2)t]\cos(\mu t). \end{cases} \quad (37)$$

Now using (37), (23) and (25) we obtain the first order solution of x and y

$$\begin{cases} x = x_0 + x_1 = A_1\cos(\mu t) + A_2\sin(\mu t) \\ + \frac{\varepsilon A_1}{32\mu}(3A_2^2 - A_1^2)\sin(3\mu t) + \frac{\varepsilon A_2}{32\mu}(3A_1^2 - A_2^2)\cos(3\mu t) \\ + \frac{\varepsilon\alpha_2}{4\mu}[(B_1 + \mu B_2)t^2 + (\frac{B_2}{\mu} - B_1)t]\sin(\mu t) \\ + \frac{\varepsilon\alpha_2}{4\mu}[(\mu B_1 - B_2)t^2 + (\frac{B_1}{\mu} + B_2)t]\cos(\mu t) \\ \\ y = y_0 + y_1 = B_1\cos(\mu t) + B_2\sin(\mu t) \\ + \frac{\varepsilon B_1}{32\mu}(3B_2^2 - B_1^2)\sin(3\mu t) + \frac{\varepsilon B_2}{32\mu}(3B_1^2 - B_2^2)\cos(3\mu t) \\ + \frac{\varepsilon\alpha_1}{4\mu}[(A_1 + \mu A_2)t^2 + (\frac{A_2}{\mu} - A_1)t]\sin(\mu t) \\ + \frac{\varepsilon\alpha_1}{4\mu}[(\mu A_1 - A_2)t^2 + (\frac{A_1}{\mu} + A_2)t]\cos(\mu t). \end{cases} \quad (38)$$
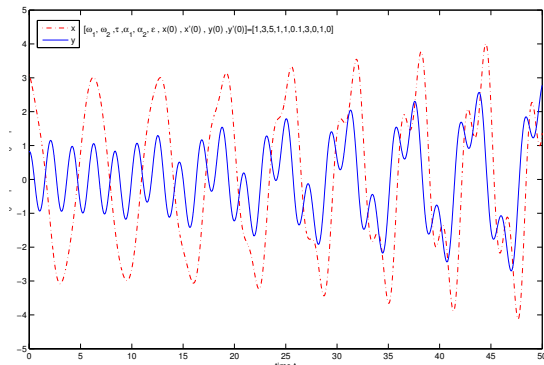
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Fig. 1.   Plots of $x = x_0 + x_1$ and $y = y_0 + y_1$ for $[\omega_1, \omega_2, \tau, \alpha_1, \alpha_2, \varepsilon, x(0), \dot{x}(0), y(0), \dot{y}(0)] = [1, 3, 5, 1, 1, 0.1, 3, 0, 1, 0]$.
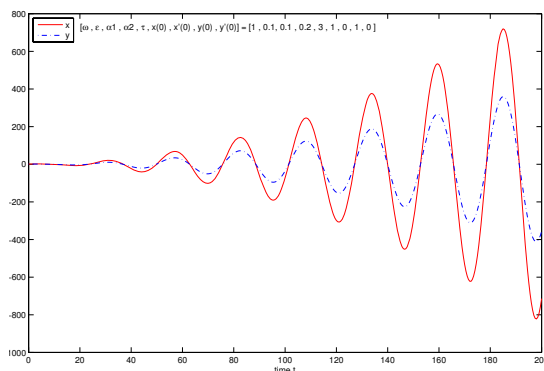

Fig. 2.   Plots of $x = x_0 + x_1$ and $y = y_0 + y_1$ for $[\omega, \varepsilon, \alpha_1, \alpha_2, \tau, x(0), \dot{x}(0), y(0), \dot{y}(0)] = [1, 0.1, 0.1, 0.2, 3, 1, 0, 1, 0]$.

Figures 1-3 show the plots of approximate values for x and y for different values of parameters of system (1) and different initial values of x and y.

IV. CONCLUSIONS

In this paper, we studied the dynamics of a system of two van der Pol oscillators with delayed position and velocity. We obtained approximate values for frequencies of the system. On account of, He's parameter-expansion method is an efficient
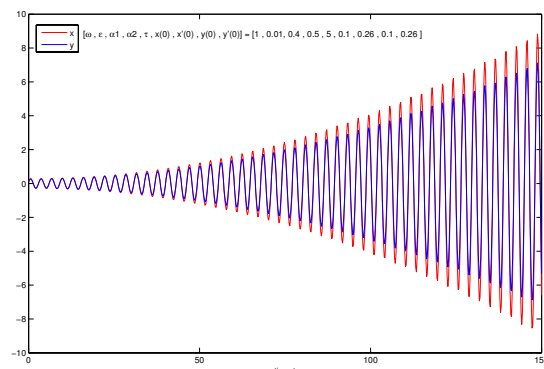

Fig. 3.   Plots of $x = x_0 + x_1$ and $y = y_0 + y_1$ for $[\omega, \varepsilon, \alpha_1, \alpha_2, \tau, x(0), \dot{x}(0), y(0), \dot{y}(0)] = [1, 0.01, 0.4, 0.5, 5, 0.1, 0.26, 0.1, 0.26]$.

method to solve such systems. The procedure of looking for a solution is also very simple and straightforward.

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