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On an Open Problem for Definable Subsets of Covering Approximation Spaces

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Abstract—Let $(U;\mathcal{D})$ be a G_r-covering approximation space $(U;\mathcal{C})$ with covering lower approximation operator $\underline{\mathcal{D}}$ and covering upper approximation operator $\overline{\mathcal{D}}$. For a subset X of U, this paper investigates the following three conditions: (1) X is a definable subset of $(U;\mathcal{D})$; (2) X is an inner definable subset of $(U;\mathcal{D})$; (3) X is an outer definable subset of $(U;\mathcal{D})$. It is proved that if one of the above three conditions holds, then the others hold. These results give a positive answer of an open problem for definable subsets of covering approximation spaces.

Keywords—Covering approximation space, covering approximation operator, definable subset, inner definable subset, outer definable subset.

I. Introduction

Pawlak approximation spaces, which were due to the classical rough set theory and were first proposed by Z. Pawlak in [8], are built on equivalence relation ([2], [6], [9], [10], [11], [12], [13]). D. Pei [14] generalized definable subsets of Pawlak approximation spaces to inner definable subsets and outer definable subsets. For a Pawlak approximation space $(U;\mathcal{C})$ with lower approximation operators $\underline{C}$ and covering upper approximation $\overline{C}$, D. Pei obtained that a subset X of U is a definable subset of $(U;\mathcal{C})$ if and only if X is an inner definable subset of $(U;\mathcal{C})$, if and only if X is an outer definable subset of $(U;\mathcal{C})$. However, equivalence relation imposes restrictions and limitations on many applications [26]. In the past years, with development of computer science, applications of rough-set theory have been extended from Pawlak approximation spaces to covering approximation spaces (see [1], [4], [7], [16], [18], [19], [20], [22], [23], [24], [25], [26], for example). This brings various covering approximation operators ([15], [17], [26]), and the following question arise naturally (see [5, Question 1.4]).

Question 1.1: For a covering approximation space with some covering lower approximation operators and covering upper approximation operators, what are relations among definable subset, inner definable subset and outer definable subset?

Taking Question 1.1 into account, X. Ge and Z. Li [5] investigate covering approximation spaces with ten covering approximation operators respectively, and give some answers of Question 1.1. However, The following question is still open (see [5, Question 4.5]).

Question 1.2: Let $(U;\mathcal{D})$ be a covering approximation space $(U;\mathcal{C})$ with lower approximation operators $\underline{\mathcal{D}}$ and covering upper approximation $\overline{\mathcal{D}}$. Whether there are some relations among definable subsets, inner definable subsets and outer definable subsets of $(U;\mathcal{D})$?

In this paper, we give some positive answers of Question 1.2 for G_r-covering approximation spaces.

II. Preliminaries

Definition 2.1 ([16]): Let U, the universe of discourse, be a finite set and $\mathcal{C}$ be a family of nonempty subsets of U.

(1) $\mathcal{C}$ is called a cover of U if $\bigcup\{K : K \in \mathcal{C}\} = U$.

(2) The pair $(U;\mathcal{C})$ is called a covering approximation space if $\mathcal{C}$ is a cover of U.

Notation 2.2: Let $(U;\mathcal{C})$ be a covering approximation space. Throughout this paper, we use the following notations, where $x \in U$ and $\mathcal{F} \subset \{X : X \subset U\}$.

(1) $\mathcal{C}_x = \{K : x \in K \in \mathcal{C}\}$.

(2) $N(x) = \bigcap\{K : K \in \mathcal{C}_x\}$.

(3) $D(x) = U - \bigcup\{K : K \in \mathcal{C} - \mathcal{C}_x\}$.

Definition 2.3 ([4]): Let $(U;\mathcal{C})$ be a covering approximation space. $(U;\mathcal{C})$ is called a G_r-covering approximation space if $x \in K \in \mathcal{C}$ implies $D(x) \subset K$.

Remark 2.4: G_r-covering approximation space in Definition 2.3 are denoted by S_r-covering approximation space in [3],

Definition 2.5 ([5]): Let $(U;\mathcal{C})$ be a covering approximation space and $X \subset U$. Put

$\underline{\mathcal{D}}(X) = \{x \in U : \exists u(u \in N(x) \bigwedge N(u) \subset X)\}$;

$\overline{\mathcal{D}}(X) = \{x \in U : \forall u(u \in N(x) \Longrightarrow N(u) \bigcap X \neq \emptyset)\}$.

(1) $\overline{\mathcal{D}} : 2^U \longrightarrow 2^U$ is called covering upper approximation operator.

(2) $\underline{\mathcal{D}} : 2^U \longrightarrow 2^U$ is called covering lower approximation operator.

(3) $\overline{\mathcal{D}}(X)$ is called a covering upper approximation of X.

(4) $\underline{\mathcal{D}}(X)$ is called a covering lower approximation of X.

Remark 2.6: (1) Operators $\underline{\mathcal{D}}$ and $\overline{\mathcal{D}}$ in Definition 2.4 are denoted by $\underline{\mathcal{C}_3}$ and $\overline{C_3}$ respectively in [16], are denoted by $\underline{\mathcal{C}}_3$ and $\overline{C}^3$ respectively in [17], and are denoted by $\underline{\mathcal{C}_4}$ and $\overline{C_4}$ respectively in [21],

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(2) It is known that, for a covering approximation space $(U;\mathcal{D})$, the following implication does not hold in general ([16, Example 1]).

$$X \subset U \Longrightarrow \underline{\mathcal{D}}(X) \subset X \subset \overline{\mathcal{D}}(X).$$

We use $(U;\mathcal{D})$ to denote covering approximation space $(U;\mathcal{C})$ with covering lower approximation operator $\underline{\mathcal{D}}$ and covering upper approximation operator $\overline{\mathcal{D}}$.

Definition 2.7 ([5]): Let $(U;\mathcal{D})$ be a covering approximation space.
(1) X is called a definable subset of $(U;\mathcal{D})$ if $\underline{\mathcal{D}}(X) = \overline{\mathcal{D}}(X)$.
(2) X is called an inner definable subset of $(U;\mathcal{D})$ if $\underline{\mathcal{D}}(X) = X$.
(3) X is called an outer definable subset of $(U;\mathcal{D})$ if $\overline{\mathcal{D}}(X) = X$.

It is clear that the following proposition holds.

Proposition 2.8: Let $(U;\mathcal{D})$ be a covering approximation space and $X \subset U$. Consider the following conditions.
(1) X is a definable subset of $(U;\mathcal{D})$.
(2) X is an inner definable subset of $(U;\mathcal{D})$.
(3) X is an outer definable subset of $(U;\mathcal{D})$.
If two of the above three conditions hold, then the other holds.

Remark 2.9: It is known that a definable subset of $(U;\mathcal{D})$ need not to be an inner definable or outer definable subset of $(U;\mathcal{D})$ (see [5, Example 4.4]).

III. The Main Results

Lemma 3.1: Let $(U;\mathcal{C})$ be a covering approximation space and $x, y \in U$. Then the following are equivalent.
(1) $x \in N(y)$.
(2) $\mathcal{C}_y \subset \mathcal{C}_x$.
(3) $N(x) \subset N(y)$.
(4) $D(y) \subset D(x)$.
(5) $y \in D(x)$.

Proof: (1) $\Longrightarrow$ (2): Let $x \in N(y) = \bigcap\{K : K \in \mathcal{C}_y\}$. Then for each $K \in \mathcal{C}_y$, $x \in K$, and so $K \in \mathcal{C}_x$. Consequently, $C_y \subset \mathcal{C}_x$.
(2) $\Longrightarrow$ (3): Let $C_y \subset \mathcal{C}_x$. Then $N(x) = \bigcap\{K : K \in \mathcal{C}_x\} \subset \bigcap\{K : K \in \mathcal{C}_y\} = N(y)$.
(3) $\Longrightarrow$ (1): Let $N(x) \subset N(y)$. Then $x \in N(x) \subset N(y)$.
(2) $\Longrightarrow$ (4): Let $C_y \subset \mathcal{C}_x$. Then $\mathcal{C} - C_x \subset \mathcal{C} - \mathcal{C}_y$, and hence $\bigcup(\mathcal{C} - C_x) \subset \bigcup(\mathcal{C} - \mathcal{C}_y)$. So $D(y) = U - \bigcup(\mathcal{C} - C_y) \subset U - (\mathcal{C} - \mathcal{C}_x) = D(x)$.
(4) $\Longrightarrow$ (5): Let $D(y) \subset D(x)$. Then $y \in D(y) \subset D(x)$.
(5) $\Longrightarrow$ (2): Let $y \in D(x)$. Then $y \notin \bigcup(\mathcal{C} - \mathcal{C}_x)$. So $y \notin K$ for each $K \in \mathcal{C} - \mathcal{C}_x$, i.e., for each $K \in \mathcal{C}$, if $K \notin \mathcal{C}_x$, then $K \notin \mathcal{C}_y$. Consequently, if $K \in \mathcal{C}_y$, then $K \in \mathcal{C}_x$. This proves that $\mathcal{C}_y \subset \mathcal{C}_x$. ■

Let $(U, \mathcal{C})$ be a covering approximation space. In this paper, we call $\{N(x) : x \in U\}$ forms a partition of U, if for each pair $x, y \in U$, $N(x) = N(y)$ or $N(x) \bigcap N(y) = \emptyset$.

Lemma 3.2: Let $(U;\mathcal{D})$ be a covering approximation space. Then the following are equivalent.
(1) $(U;\mathcal{D})$ is a G_r-covering approximation space.
(2) $\forall x, y \in U (x \in N(y) \Longrightarrow y \in N(x))$.
(3) $\{N(x) : x \in U\}$ forms a partition of U.
(4) $\forall x, y \in U (x \in N(y) \Longrightarrow N(x) = N(y))$.

Proof: (1) $\Longrightarrow$ (2): Assume that $(U;\mathcal{D})$ is a G_r-covering approximation space. Let $x, y \in U$ and $x \in N(y)$. By Lemma 3.1, $y \in D(x)$. Let $x \in K \in \mathcal{C}$. Since $(U;\mathcal{D})$ is a G_r-covering approximation space, $D(x) \subset K$, and so $y \in D(x) \subset K$. It follows that $y \in N(x)$.
(2) $\Longrightarrow$ (1): Assume that (2) holds. Let $x \in K \in \mathcal{C}$. If $y \in D(x)$, then $x \in N(y)$ by Lemma 3.1, and hence $y \in N(x) \subset K$. It follows that $D(x) \subset K$.
(2) $\Longrightarrow$ (3): It holds by [16, Lemma 2].
(3) $\Longrightarrow$ (4): Assume that (3) holds. Let $x, y \in U$ and $x \in N(y)$. Then $x \in N(x) \bigcap N(y) \neq \emptyset$. So $N(x) = N(y)$.
(4) $\Longrightarrow$ (2): Assume that (4) holds. Let $x, y \in U$ and $x \in N(y)$. Then $N(x) = N(y)$. It follows that $y \in N(y) = N(x)$. ■

Now we give the main results as follows.

Theorem 3.3: Let $(U;\mathcal{D})$ be a G_r-covering approximation space and $X \subset U$. If X is a definable subset of $(U;\mathcal{D})$, then X is a both inner definable and outer definable subset of $(U;\mathcal{D})$.

Proof: Let X be a definable subset of $(U;\mathcal{D})$. Then $\underline{\mathcal{D}}(X) = \overline{\mathcal{D}}(X)$. By Proposition 2.8, it suffices to prove that X is an outer definable subset of $(U;\mathcal{D})$. Let $x \in X$. For each $u \in U$, if $u \in N(x)$, then $N(u) = N(x)$ by Lemma 3.2. Note that $x \in N(x) \bigcap X \neq \emptyset$. So $N(u) \bigcap X = N(x) \bigcap X \neq \emptyset$. It follows that $x \in \overline{\mathcal{D}}(X)$. This proves that $X \subset \overline{\mathcal{D}}(X)$. On the other hand, let $x \in \overline{\mathcal{D}}(X)$. Since $\underline{\mathcal{D}}(X) = \overline{\mathcal{D}}(X)$, $x \in \underline{\mathcal{D}}(X)$, and so there exists $u \in N(x)$ such that $N(u) \subset X$. By Lemma 3.2, $x \in N(x) = N(u) \subset X$. This proves that $\overline{\mathcal{D}}(X) \subset X$. Consequently, $\overline{\mathcal{D}}(X) = X$, i.e., X is an outer definable subset of $(U;\mathcal{D})$. ■

Theorem 3.4: Let $(U;\mathcal{D})$ be a G_r-covering approximation space and $X \subset U$. If X is an inner definable subset of $(U;\mathcal{D})$, then X is a both definable and outer definable subset of $(U;\mathcal{D})$.

Proof: Let X be an inner definable subset of $(U;\mathcal{D})$. Then $\underline{\mathcal{D}}(X) = X$. By Proposition 2.8, it suffices to prove that X is an outer definable subset of $(U;\mathcal{D})$. Let $x \in X$. For each $u \in U$, if $u \in N(x)$, then $N(u) = N(x)$ by Lemma 3.2. Note that $x \in N(x) \bigcap X \neq \emptyset$. So $N(u) \bigcap X = N(x) \bigcap X \neq \emptyset$. It follows that $x \in \overline{\mathcal{D}}(X)$. This proves that $X \subset \overline{\mathcal{D}}(X)$. On the other hand, let $x \in \overline{\mathcal{D}}(X)$. Since $x \in N(x)$, $N(x) \bigcap X \neq \emptyset$, so there exists $y \in N(x) \bigcap X$. Note that $y \in X = \underline{\mathcal{D}}(X)$. So there exists $z \in N(y)$ such that $N(z) \subset X$. Since $y \in N(x)$ and $z \in N(y)$, by Lemma 3.2, $x \in N(x) = N(y) = N(z) \subset X$. This proves that $\overline{\mathcal{D}}(X) \subset X$. Consequently, $\overline{\mathcal{D}}(X) = X$, i.e., X is an outer definable subset of $(U;\mathcal{D})$. ■

Theorem 3.5: Let $(U;\mathcal{D})$ be a G_r-covering approximation space and $X \subset U$. If X is an outer definable subset of $(U;\mathcal{D})$, then X is a both definable and inner definable subset of $(U;\mathcal{D})$.

Proof: Let X be an outer definable subset of $(U;\mathcal{D})$. Then $\overline{\mathcal{D}}(X) = X$. By Proposition 2.8, it suffices to prove that X is an inner definable subset of $(U;\mathcal{D})$. Let $x \in X$. We claim that $N(x) \subset X$. In fact, if $N(x) \not\subset X$, then there exists $y \in N(x)$ such that $y \notin X$. Since $\overline{\mathcal{D}}(X) = X$, $y \notin \overline{\mathcal{D}}(X)$, hence there exists $z \in N(y)$ such that $N(z) \bigcap X = \emptyset$. Since $y \in N(x)$ and $z \in N(y)$, by Lemma 3.2, $N(x) = N(y) = N(z)$, hence $N(x) \bigcap X = N(z) \bigcap X = \emptyset$. This contradicts that $x \in N(x) \bigcap X \neq \emptyset$. So $N(x) \subset X$. Note that $x \in N(x)$. It follows that $x \in \underline{\mathcal{D}}(X)$. This proves that $X \subset \underline{\mathcal{D}}(X)$. On the other hand, let $x \in \underline{\mathcal{D}}(X)$. Then there exists $u \in N(x)$ such that $N(u) \subset X$. By Lemma 3.2, $x \in N(x) = N(u) \subset X$. This proves that $\underline{\mathcal{D}}(X) \subset X$. Consequently, $\underline{\mathcal{D}}(X) = X$, i.e., X is an inner definable subset of $(U;\mathcal{D})$. ■

Remark 3.6: G_1-covering approximation spaces, G_2-covering approximation spaces and G_3-covering approximation spaces are investigated in [3], [4], and it is known that G_3-covering approximation spaces $\Longrightarrow$ G_2-covering approximation spaces $\Longrightarrow$ G_1-covering approximation spaces $\Longrightarrow$ G_r-covering approximation spaces ([4, Theorem 4.1]). In addition, Each Pawlak approximation space is a G_r-covering approximation spaces ([3, Remark 3.4]). So conclusion in Theorem 3.3 (resp. Theorem 3.4, Theorem 3.5) holds for G_r-covering approximation spaces (resp. G_1-covering approximation spaces, G_2-covering approximation spaces, G_3-covering approximation spaces).

IV. Conclusion

In this paper, we give some positive answers of Question 1.2 for G_r-covering approximation spaces. However, we do not know that whether "G_r-" can be omitted. More precisely, the following question is still open.

Question 4.1: Let $(U;\mathcal{D})$ be a covering approximation space and $X \subset U$.
(1) Does $\underline{\mathcal{D}}(X) = X$ imply $\overline{\mathcal{D}}(X) = X$?
(2) Does $\overline{\mathcal{D}}(X) = X$ imply $\underline{\mathcal{D}}(X) = X$?

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References

[1] Z. Bonikowski, E. Bryniarski and U. Wybraniec, Extensions and intentions in the rough set theory, Information Sciences, 107(1998), 149-167.
[2] X. Ge, Further investigations on higher mathematics scores for Chinese university students, International Journal of Computational and Mathematical Sciences, 3(2009), 181-185.
[3] X. Ge, An application of covering approximation spaces on network security, Computers and Mathematics with Applications, 60(2010), 1191-1199.
[4] X. Ge, J. Li and Y. Ge, Some separations in covering approximation, International Journal of Computational and Mathematical Sciences, 4(2010), 156-160.
[5] X. Ge and Z. Li, Definable subsets in covering approximation spaces, International Journal of Computational and Mathematical Sciences, 5(2011), 31-34.
[6] X. Ge and J. Qian, Some investigations on higher mathematics scores for Chinese university students, International Journal of Computer and Information Engineering, 3(2009), 209-212.
[7] E. Lashin, A. Kozae, A. Khadra and T. Medhat, Rough set theory for topological spaces, International Journal of Approximate Reasoning, 40(2005), 35-43.
[8] Z. Pawlak, Rough sets, International journal of computer and information science, 11(1982), 341-356.
[9] Z. Pawlak, Rough Sets: Theoretical Aspects of Reasoning About Data, Kluwer Academic Publishers, 1991.
[10] Z. Pawlak, Rough classification, International journal of Human-computer Studies, 51(1999), 369-383.
[11] Z. Pawlak and A. Skowron, Rudiments of rough sets, Information Sciences, 177(2007), 3-27.
[12] Z. Pawlak and A. Skowron, Rough sets: Some extensions, Information Sciences, 177(2007), 28-40.
[13] Z. Pawlak and A. Skowron, Rough sets and Boolean reasoning, Information Sciences, 177(2007), 41-73.
[14] D. Pei, On definable concepts of rough set models, Information Sciences, 177(2007), 4230-4239.
[15] J. A. Pomykala, Approximation operations in approximation spaces, Bull. Pol. Acad. Sci., 35(1987), 653-662.
[16] K. Qin, Y. Gao and Z. Pei, *On covering rough sets*, Lecture Notes in Artificial Intelligence, 4481(2007), 34-41.
[17] P. Samanta and M. Chakraborty, Covering based approaches to rough sets and implication lattices, Lecture Notes in Artificial Intelligence, 5908(2009), 127-134.
[18] Y. Yao, Views of the theory of rough sets in finite universes, International Journal of Approximate Reasoning, 15(1996), 291-317.
[19] Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, Information Sciences, 111(1998), 239-259.
[20] Y. Yao, On generalizing rough set theory, Lecture Notes in AI, 2639(2003), 44-51.
[21] Z. Yun, Xun Ge and X. Bai, Axiomatization and conditions for neighborhoods in a covering to form a partition, Information Sciences, (2011), doi:10.1016/j.ins.2011.01.013.
[22] W. Zakowski, Approximations in the Space (u, Π), Demonstratio Mathematica, 16(1983), 761-769.
[23] W. Zhu, Topological approaches to covering rough sets, Information Sciences, 177(2007), 1499-1508.
[24] W. Zhu, Relationship between generalized rough sets based on binary relation and covering, Information Sciences, 179(2009), 210-225.
[25] W. Zhu and F. Wang, Reduction and axiomatization of covering generalized rough sets, Information Sciences, 152(2003), 217-230.
[26] W. Zhu and F. Wang, On Three Types of Covering Rough Sets, IEEE Transactions on Knowledge and Data Engineering, 19(2007), 1131-1144.