

# Some Algebraic Properties of Universal and Regular Covering Spaces

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**Abstract**—Let  $\tilde{X}$  be a connected space,  $X$  be a space, let  $p : \tilde{X} \rightarrow X$  be a continuous map and let  $(\tilde{X}, p)$  be a covering space of  $X$ . In the first section we give some preliminaries from covering spaces and their automorphism groups. In the second section we derive some algebraic properties of both universal and regular covering spaces  $(\tilde{X}, p)$  of  $X$  and also their automorphism groups  $A(\tilde{X}, p)$ .

**Keywords**—covering space, universal covering, regular covering, fundamental group, automorphism group.

## I. PRELIMINARIES

Let  $\tilde{X}$  be a connected space,  $X$  be a space and let

$$p : \tilde{X} \rightarrow X$$

be a continuous map. If for every  $x \in X$  has a path connected open neighborhood  $U$  such that  $p^{-1}(U)$  is open in  $\tilde{X}$  and each component of  $p^{-1}(U)$  is mapped topologically onto  $U$  by  $p$  then  $p$  is called a covering map (projection). In this case the pair  $(\tilde{X}, p)$  is called a covering space of  $X$ .

Let  $p : E \rightarrow B$  and  $f : X \rightarrow B$  be two maps. The lifting problem for  $f$  is to determine whether there is a continuous map

$$f^v : X \rightarrow E$$

such that

$$pf^v = f.$$

If there is such a map  $f^v$ , then  $f$  can be lifted to  $E$ , and we call  $f^v$  a lifting of  $f$ . A map  $p : E \rightarrow B$  is said to have homotopy lifting property with respect to the space  $X$  if given maps  $f^v : X \rightarrow E$  and  $F : X \times I \rightarrow B$  such that  $F(x, 0) = pf^v(x)$  for  $x \in X$ , there exists a map  $F^v : X \times I \rightarrow E$  such that  $F^v(x, 0) = f^v(x)$  for  $x \in X$  and  $pF^v = F$ . A map  $p : E \rightarrow B$  is called a fibration if  $p$  has the homotopy lifting property with respect to every space.  $E$  is called the total space and  $B$  the base space of the fibration. For  $b \in B$ ,  $p^{-1}(b)$  is called the fiber over  $b$  (see [1]).

Let  $(\tilde{X}, p)$  be a covering space of  $X$ ,  $\tilde{x} \in \tilde{X}$ , and  $p(\tilde{x}) = x$  for  $x \in X$ . Let  $\pi(X, x)$  be the fundamental group of  $X$  based  $x$ . For any point  $\tilde{x} \in p^{-1}(x)$  and any  $\alpha \in \pi(X, x)$  we define  $\tilde{x}\alpha \in p^{-1}(x)$  as follows: There exists a unique path class  $\tilde{\alpha}$  in  $\tilde{X}$  such that  $p_*(\tilde{\alpha}) = \alpha$  and the initial point of  $\tilde{\alpha}$  is the point  $\tilde{x}$ . Define  $\tilde{x}\alpha$  to be the terminal point of the path class  $\tilde{\alpha}$ . Then it is easily verify that  $(\tilde{x}\alpha)\beta = \tilde{x}(\alpha\beta)$  and  $\tilde{x}e = \tilde{x}$ . Thus  $\pi(X, x)$  be a group of right operators on the set  $p^{-1}(x)$ . Moreover,  $\pi(X, x)$  acts transitively on the set  $p^{-1}(x)$ , and hence  $p^{-1}(x)$  is a homogeneous right  $\pi(X, x)$ -space.

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From definition, we see that for any point  $\tilde{x} \in p^{-1}(x)$ , the isotropy subgroup corresponding to this point is precisely the subgroup  $p_*\pi(\tilde{X}, \tilde{x})$  of  $\pi(X, x)$ . Hence  $p^{-1}(x)$  is isomorphic to the space of cosets  $\pi(X, x)/p_*\pi(\tilde{X}, \tilde{x})$ , and the number of sheets of the covering is equal to the index of the subgroup  $p_*\pi(\tilde{X}, \tilde{x})$  in  $\pi(X, x)$ .

**Lemma 1.1:** Let  $E$  be a homogen  $G$ -space and let  $H$  be a isotropy subgroup of  $e \in E$ . Then the automorphism group of  $E$  is isomorphic to the quotient space  $N[H]/H$ , where  $N[H]$  denotes the normalization of  $H$ . [4]

Let  $(\tilde{X}, p)$  be a covering space of  $X$ . If  $X$  is simply connected, then the fundamental group  $\pi(X, x)$  is trivial and the index of  $p_*\pi(\tilde{X}, \tilde{x})$  in  $\pi(X, x)$  is 1. So  $(\tilde{X}, p)$  is an one-sheeted covering of  $X$  and therefore  $p$  is a homeomorphism. Similarly, if  $\tilde{X}$  is simply connected, then  $\pi(\tilde{X}, \tilde{x})$  is trivial and the index of  $p_*\pi(\tilde{X}, \tilde{x})$  in  $\pi(X, x)$  is equal to the order of  $\pi(X, x)$ .

Let  $(\tilde{X}_1, p_1)$  and  $(\tilde{X}_2, p_2)$  be two covering spaces of  $X$ . Then a homomorphism from  $(\tilde{X}_1, p_1)$  into  $(\tilde{X}_2, p_2)$  is a continuous map

$$h : \tilde{X}_1 \rightarrow \tilde{X}_2$$

such that

$$p_2h = p_1.$$

Let  $h$  be a homomorphism from  $(\tilde{X}_1, p_1)$  into  $(\tilde{X}_2, p_2)$ . Then  $h$  is called an isomorphism, if there exists a homomorphism  $k$  from  $(\tilde{X}_2, p_2)$  into  $(\tilde{X}_1, p_1)$  such that both compositions  $hk$  and  $kh$  are identity maps. A homomorphism of covering spaces is an isomorphism if and only if it is a homeomorphism in the usual sense.

**Lemma 1.2:** Let  $(\tilde{X}_1, p_1)$  and  $(\tilde{X}_2, p_2)$  be two simply connected covering spaces of  $X$ . Then there exists a unique homeomorphism  $h : (\tilde{X}_1, \tilde{x}_1) \rightarrow (\tilde{X}_2, \tilde{x}_2)$  such that  $p_2h = p_1$ . [2]

A covering transformation of a covering space  $(\tilde{X}, p)$  of  $X$  is a homeomorphism

$$h : \tilde{X} \rightarrow \tilde{X}$$

such that

$$ph = p.$$

The set of all covering transformations of  $(\tilde{X}, p)$  form a group denoted by  $A(\tilde{X}, p)$  called the automorphism group.

**Lemma 1.3:** Let  $(\tilde{X}_1, p_1)$  and  $(\tilde{X}_2, p_2)$  be two covering spaces of  $X$  such that  $p_1(\tilde{x}_1) = p_2(\tilde{x}_2) = x$ . Then there exists

a homomorphism  $\varphi$  from  $(\tilde{X}_1, p_1)$  into  $(\tilde{X}_2, p_2)$  such that  $\varphi(\tilde{x}_1) = \tilde{x}_2$  if and only if  $p_{1*}\pi(\tilde{X}_1, \tilde{x}_1) \subset p_{2*}\pi(\tilde{X}_2, \tilde{x}_2)$ . [4]

Let  $(\tilde{X}, p)$  be a covering space of  $X$  and  $p(\tilde{x}_1) = p(\tilde{x}_2) = x$  for  $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$  and  $x \in X$ . Let consider the homomorphisms

$$p_* : \pi(\tilde{X}, \tilde{x}_1) \rightarrow \pi(X, x)$$

and

$$p_* : \pi(\tilde{X}, \tilde{x}_2) \rightarrow \pi(X, x).$$

Let  $\{\gamma_i : i \in I\}$  be a path class in  $\tilde{X}$  with initial point  $\tilde{x}_1$  and terminal point  $\tilde{x}_2$ . Define

$$u_* : \pi(\tilde{X}, \tilde{x}_1) \rightarrow \pi(\tilde{X}, \tilde{x}_2)$$

to be

$$u_*(\alpha) = \gamma^{-1}\alpha\gamma$$

for  $\gamma \in \{\gamma_i : i \in I\}$ . Then

$$v_* : \pi(X, x) \rightarrow \pi(X, x)$$

is defined by

$$v_*(\beta) = (p_*\gamma)^{-1}\beta(p_*\gamma).$$

Since  $p_*\gamma$  is a closed it is a path in  $\pi(X, x)$ . So the images of the fundamental groups  $\pi(\tilde{X}, \tilde{x}_1)$  and  $\pi(\tilde{X}, \tilde{x}_2)$  under  $p_*$  are conjugate subgroups of  $\pi(X, x)$  (for further details on covering spaces see [3,4]).

*Lemma 1.4:* Let  $(\tilde{X}, p)$  be path connected covering space of a locally pathwise connected space  $X$ . Then  $p$  is a homeomorphism if and only if  $p_*\pi(\tilde{X}, \tilde{x}) = \pi(X, x)$ . [5]

From Lemmas 1.3 and 1.4, we say that if  $(\tilde{X}_1, p_1)$  and  $(\tilde{X}_2, p_2)$  are two covering spaces of a locally pathwise connected space  $X$ , then these two coverings are isomorphic if and only if  $p_{1*}\pi(\tilde{X}_1, \tilde{x}_1)$  and  $p_{2*}\pi(\tilde{X}_2, \tilde{x}_2)$  are conjugate subgroups of  $\pi(X, x)$ .

Let  $X$  be a connected space. The category of connected spaces of  $X$  has objects which are covering projections  $p : \tilde{X} \rightarrow X$ , where  $\tilde{X}$  is connected, and morphisms  $p_1 : \tilde{X}_1 \rightarrow X, p_2 : \tilde{X}_2 \rightarrow X$  and  $f : \tilde{X}_1 \rightarrow \tilde{X}_2$  such that  $p_2f = p_1$ .

Let  $X$  be a connected space and  $\tilde{X}$  be a locally path connected space. A universal covering space of  $X$  is an object  $p : \tilde{X} \rightarrow X$  of the category of connected covering spaces of  $X$  such that for any object  $p_1 : \tilde{X}_1 \rightarrow X$  there is a morphism  $f : \tilde{X} \rightarrow \tilde{X}_1$  such that  $p_1f = p$ .

*Lemma 1.5:* If  $(\tilde{X}, p)$  is a universal covering space of  $X$ , then the automorphism group  $A(\tilde{X}, p)$  is isomorphic to  $\pi(X, x)$ . Moreover, the order of the fundamental group  $\pi(X, x)$  is equal to the number of the sheets of the covering  $(\tilde{X}, p)$  of  $X$ . [4]

Let  $\tilde{X}$  be be a pathwise connected space and let  $(\tilde{X}, p)$  be a covering space of a locally pathwise connected space  $X$ . Then  $(\tilde{X}, p)$  is a regular covering space of  $X$  if and only if  $p_*\pi(\tilde{X}, \tilde{x})$  is a normal subgroup of  $\pi(X, x)$ .

*Lemma 1.6:* Let  $p : \tilde{X} \rightarrow X$  be a covering map such that  $p(\tilde{x}_1) = p(\tilde{x}_2)$  for  $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$ . Then  $p$  is regular if and only if  $p_*\pi(\tilde{X}, \tilde{x}_1) = p_*\pi(\tilde{X}, \tilde{x}_2)$ . [5]

The connection between universal and regular covering is given by the following Lemma.

*Lemma 1.7:* Universal covering is regular. [6]

*Lemma 1.8:* Let  $\tilde{X}$  be a pathwise connected space and let  $(\tilde{X}, p)$  is a regular covering space of a locally pathwise connected space  $X$ . Then  $X$  homeomorphic to the quotient space  $\tilde{X}/A(\tilde{X}, p)$ . [3]

Let  $G$  be a group of homeomorphisms of  $X$ . If for every  $x \in X$ , there exists a neighborhood  $V$  of  $x$  such that  $gV \cap V = \emptyset$ , for all  $g \in G$  different from the unity of  $G$ , then we say  $G$  acts discontinuously on  $X$ .

*Lemma 1.9:* Let  $G$  be a discontinuous proper group of homeomorphisms of a locally pathwise connected space  $X$  and let  $q : X \rightarrow X/G$  be a naturel projection defined by  $q(x) = [x]$ . Then  $(\tilde{X}, q)$  is a regular covering space of  $X/G$  and  $A(\tilde{X}, q)$  is isomorphic to  $G$ . [5]

## II. UNIVERSAL AND REGULAR COVERING SPACES.

In [6], we consider covering spaces  $(\tilde{X}, p)$  of a pathwise connected space  $X$  and also consider the automorphism group  $A(\tilde{X}, p)$  of this covering. In the present paper we will derive some algebraic properties of universal and regular covering spaces  $(\tilde{X}, p)$  of a pathwise connected space  $X$ .

*Theorem 2.1:* Let  $(\tilde{X}_1, p_1)$  and  $(\tilde{X}_2, p_2)$  be two covering spaces of  $X$  such that  $p_1(\tilde{x}_1) = p_2(\tilde{x}_2) = x$  for  $\tilde{x}_1 \in \tilde{X}_1, \tilde{x}_2 \in \tilde{X}_2$ . Then there exists an isomorphism  $\varphi$  from  $(\tilde{X}_1, p_1)$  into  $(\tilde{X}_2, p_2)$  such that  $\varphi(\tilde{x}_1) = \tilde{x}_2$  if and only if  $p_{1*}\pi(\tilde{X}_1, \tilde{x}_1) = p_{2*}\pi(\tilde{X}_2, \tilde{x}_2)$ .

*Proof:* We know from Lemma 1.3 that if  $(\tilde{X}_1, p_1)$  and  $(\tilde{X}_2, p_2)$  are two covering spaces of  $X$  such that  $p_1(\tilde{x}_1) = p_2(\tilde{x}_2) = x$ , then there exists a homomorphism  $\varphi$  from  $(\tilde{X}_1, p_1)$  into  $(\tilde{X}_2, p_2)$  such that  $\varphi(\tilde{x}_1) = \tilde{x}_2$  if and only if  $p_{1*}\pi(\tilde{X}_1, \tilde{x}_1) \subset p_{2*}\pi(\tilde{X}_2, \tilde{x}_2)$ . Therefore there exists another homomorphism  $\psi$  from  $(\tilde{X}_2, p_2)$  into  $(\tilde{X}_1, p_1)$  such that  $\psi(\tilde{x}_2) = \tilde{x}_1$  if and only if  $p_{2*}\pi(\tilde{X}_2, \tilde{x}_2) \subset p_{1*}\pi(\tilde{X}_1, \tilde{x}_1)$ . Hence  $\psi$  is the invers homomorphism of  $\varphi$  since both compositions  $\varphi\psi$  and  $\psi\varphi$  are identity maps. Therefore there exists an isomorphism  $\varphi$  from  $(\tilde{X}_1, p_1)$  into  $(\tilde{X}_2, p_2)$  such that  $\varphi(\tilde{x}_1) = \tilde{x}_2$  if and only if  $p_{1*}\pi(\tilde{X}_1, \tilde{x}_1) = p_{2*}\pi(\tilde{X}_2, \tilde{x}_2)$ . ■

From Theorem 2.1 the following two corollaries can be given.

*Corollary 2.2:* If  $(\tilde{X}, p)$  is a covering space of  $X$  such that  $p(\tilde{x}_1) = p(\tilde{x}_2) = x$  for  $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$ , then there exists an automorphism  $\varphi \in A(\tilde{X}, p)$  such that  $\varphi(\tilde{x}_1) = \tilde{x}_2$  if and only if  $p_*\pi(\tilde{X}, \tilde{x}_1) = p_*\pi(\tilde{X}, \tilde{x}_2)$ .

**Corollary 2.3:** If  $(\tilde{X}_1, p_1)$  and  $(\tilde{X}_2, p_2)$  are two covering spaces of  $X$  such that  $p_1(\tilde{x}_1) = p_2(\tilde{x}_2) = x$  for  $\tilde{x}_1 \in \tilde{X}_1, \tilde{x}_2 \in \tilde{X}_2$ , then these two coverings are isomorphic if and only if  $p_{1*}\pi(\tilde{X}_1, \tilde{x}_1)$  and  $p_{2*}\pi(\tilde{X}_2, \tilde{x}_2)$  are belong the same conjugate class in  $\pi(X, x)$ .

**Theorem 2.4:** Let  $(\tilde{X}_1, p_1)$  and  $(\tilde{X}_2, p_2)$  be two universal covering spaces of  $X$  such that  $p_1(\tilde{x}_1) = p_2(\tilde{x}_2) = x$  for  $\tilde{x}_1 \in \tilde{X}_1, \tilde{x}_2 \in \tilde{X}_2$ . Then  $p_{1*}\pi(\tilde{X}_1, \tilde{x}_1) = p_{2*}\pi(\tilde{X}_2, \tilde{x}_2)$ .

*Proof:*  $\tilde{X}_1$  and  $\tilde{X}_2$  are simply connected since  $(\tilde{X}_1, p_1)$  and  $(\tilde{X}_2, p_2)$  are universal covering space of  $X$ . From Lemma 1.2, there exists a homeomorphism  $h : \tilde{X}_1 \rightarrow \tilde{X}_2$  such that  $p_2h = p_1$ . Therefore  $(\tilde{X}_1, p_1)$  and  $(\tilde{X}_2, p_2)$  are isomorphic and hence  $p_{1*}\pi(\tilde{X}_1, \tilde{x}_1) = p_{2*}\pi(\tilde{X}_2, \tilde{x}_2)$  by Theorem 2.1. ■

From Theorem 2.4, the following corollary can be given.

**Corollary 2.5:** If  $(\tilde{X}_1, p_1)$  and  $(\tilde{X}_2, p_2)$  are two universal covering spaces of  $X$  such that  $p_1(\tilde{x}_1) = p_2(\tilde{x}_2) = x$  for  $\tilde{x}_1 \in \tilde{X}_1, \tilde{x}_2 \in \tilde{X}_2$ , then

- 1) these coverings are isomorphic;
- 2)  $p_{1*}\pi(\tilde{X}_1, \tilde{x}_1)$  and  $p_{2*}\pi(\tilde{X}_2, \tilde{x}_2)$  are conjugate subgroups of  $\pi(X, x)$  if and only if there exists an automorphism  $\varphi \in A(\tilde{X}, p)$  such that  $\varphi(\tilde{x}_1) = \tilde{x}_2$ .

**Theorem 2.6:** If  $(\tilde{X}, p)$  is a universal covering space of  $X$ , then the index of the subgroup  $p_*\pi(\tilde{X}, \tilde{x})$  in  $\pi(X, x)$  is equal to the order of the automorphism group  $A(\tilde{X}, p)$ .

*Proof:* Let  $(\tilde{X}, p)$  be a universal covering space of  $X$ . Then the automorphism group  $A(\tilde{X}, p)$  is isomorphic to  $\pi(X, x)$  by Lemma 1.5, and the number of the sheets of the covering of  $X$  is equal to the order of the automorphism group  $A(\tilde{X}, p)$ . On the other hand, we know that the number of sheets of covering is equal to the index of  $p_*\pi(\tilde{X}, \tilde{x})$  in  $\pi(X, x)$ . Therefore the index of the subgroup  $p_*\pi(\tilde{X}, \tilde{x})$  in  $\pi(X, x)$  is equal to the order of the automorphism group  $A(\tilde{X}, p)$ . ■

Let  $(\tilde{X}, p)$  be a covering space of  $X$  and let

$$N[p_*\pi(\tilde{X}, \tilde{x})] = \left\{ \alpha \in \pi(X, x) : \alpha p_*\pi(\tilde{X}, \tilde{x}) \alpha^{-1} = p_*\pi(\tilde{X}, \tilde{x}) \right\} \quad (1)$$

be the normalizer of  $p_*\pi(\tilde{X}, \tilde{x})$  in  $\pi(X, x)$ . Then the automorphism group  $A(\tilde{X}, p)$  is isomorphic to the quotient group  $N[p_*\pi(\tilde{X}, \tilde{x})]/p_*\pi(\tilde{X}, \tilde{x})$  by Lemma 1.1.

**Theorem 2.7:** If  $(\tilde{X}, p)$  is a regular covering space of  $X$ , then the automorphism group  $A(\tilde{X}, p)$  is isomorphic to the quotient group  $\pi(X, x)/p_*\pi(\tilde{X}, \tilde{x})$ .

*Proof:* Let  $(\tilde{X}, p)$  be a regular covering space of  $X$ . Then from definition  $p_*\pi(\tilde{X}, \tilde{x})$  is a normal subgroup of  $\pi(X, x)$  and therefore the normalizer of  $p_*\pi(\tilde{X}, \tilde{x})$  is equal

to  $\pi(X, x)$ , i.e.  $N[p_*\pi(\tilde{X}, \tilde{x})] = \pi(X, x)$ . Therefore  $A(\tilde{X}, p)$  is isomorphic to the quotient group  $\pi(X, x)/p_*\pi(\tilde{X}, \tilde{x})$  by (1) and Lemma 1.1. ■

**Theorem 2.8:** If  $(\tilde{X}, p)$  is a regular covering space of  $X$  such that  $p(\tilde{x}_1) = p(\tilde{x}_2) = x$ , for  $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$  and  $x \in X$ , then  $p_*\pi(\tilde{X}, \tilde{x}_1)$  and  $p_*\pi(\tilde{X}, \tilde{x}_2)$  are conjugate subgroups of  $\pi(X, x)$ .

*Proof:* Let  $(\tilde{X}, p)$  is a regular covering space of  $X$ . Then  $p_*\pi(\tilde{X}, \tilde{x}_1) = p_*\pi(\tilde{X}, \tilde{x}_2)$  by Lemma 1.6. Hence from Corollary 2.2, there exists an automorphism  $\varphi \in A(\tilde{X}, p)$  such that  $\varphi(\tilde{x}_1) = \tilde{x}_2$ . Therefore  $p_*\pi(\tilde{X}, \tilde{x}_1)$  and  $p_*\pi(\tilde{X}, \tilde{x}_2)$  are conjugate subgroups of  $\pi(X, x)$  by Corollary 2.3. ■

**Theorem 2.9:** If  $(\tilde{X}, p)$  is a universal covering space of  $X$ , then the automorphism group of the set  $p^{-1}(x)$ , which is a right  $\pi(X, x)$ -space, is isomorphic to the normalizer of  $p_*\pi(\tilde{X}, \tilde{x})$  in  $\pi(X, x)$ .

*Proof:* We know that the automorphism group of the set  $p^{-1}(x)$  is isomorphic to the automorphism group  $A(\tilde{X}, p)$ . On the other hand this covering is regular since universal covering is regular by Lemma 1.7, and from definition  $p_*\pi(\tilde{X}, \tilde{x})$  is a normal subgroup of  $\pi(X, x)$ . Therefore

$$N[p_*\pi(\tilde{X}, \tilde{x})] = \pi(X, x).$$

From Lemma 1.5,  $A(\tilde{X}, p)$  is isomorphic to  $\pi(X, x)$ . Hence the automorphism group of the set  $p^{-1}(x)$  is isomorphic to the normalizer of  $p_*\pi(\tilde{X}, \tilde{x})$  in  $\pi(X, x)$ . ■

**Theorem 2.10:** If  $(\tilde{X}, p)$  is a universal covering spaces of  $X$ , then  $N[p_*\pi(\tilde{X}, \tilde{x})]$  is isomorphic to  $\pi(X, x)/p_*\pi(\tilde{X}, \tilde{x})$ .

*Proof:* Let  $(\tilde{X}, p)$  be a universal covering space of  $X$ . Then this covering is regular by Lemma 1.7. Therefore from definition  $p_*\pi(\tilde{X}, \tilde{x})$  is a normal subgroup of  $\pi(X, x)$ , and  $N[p_*\pi(\tilde{X}, \tilde{x})] = p_*\pi(\tilde{X}, \tilde{x})$ . From Lemma 1.5,  $A(\tilde{X}, p)$  is isomorphic to  $\pi(X, x)$  since this covering is universal. Therefore  $A(\tilde{X}, p)$  is isomorphic to  $N[p_*\pi(\tilde{X}, \tilde{x})]$ . Moreover  $A(\tilde{X}, p)$  is isomorphic to  $\pi(X, x)/p_*\pi(\tilde{X}, \tilde{x})$  by Theorem 2.7. Hence  $N[p_*\pi(\tilde{X}, \tilde{x})]$  and  $\pi(X, x)/p_*\pi(\tilde{X}, \tilde{x})$  are isomorphic. ■

Let  $X$  be a locally pathwise connected space and let  $G$  be the discontinuous group of homeomorphisms of  $X$ . Now consider the naturel map  $q : X \rightarrow X/G$  defined by  $q(x) = [x]$ . Then  $(X, q)$  is a regular covering space of  $X/G$  and the automorphism group  $A(X, q)$  is isomorphic to  $G$  by Lemma 1.9.

**Theorem 2.11:** Let  $X$  be a locally pathwise connected, and let  $G$  be the discontinuous group of homeomorphisms of  $X$ . Let  $q : X \rightarrow X/G$  be the naturel map such that  $q(x_1) = q(x_2)$  for  $x_1, x_2 \in X$ . Then there exists an automorphism

$\varphi \in A(X, q)$  such that  $\varphi(x_1) = x_2$ . Moreover  $A(X, q)$  is isomorphic to the quotient group  $\pi(X/G, [x])/q_*\pi(X, x)$ .

*Proof:* The map  $q : X \rightarrow X/G$  is a regular map since  $G$  is the discontinuous group of homeomorphisms of  $X$  by Lemma 1.9. Moreover  $q_*\pi(X, x_1) = q_*\pi(X, x_2)$  for  $x_1, x_2 \in X$  by Lemma 1.6, and hence there exists an automorphism  $\varphi \in A(X, q)$  such that  $\varphi(x_1) = x_2$  by Theorem 2.1. It is clear from Theorem 2.7 that  $A(X, q)$  is isomorphic to the quotient group  $\pi(X/G, [x])/q_*\pi(X, x)$  since  $(X, q)$  is a regular covering of  $X/G$ . ■

*Theorem 2.12:* Let  $(\tilde{X}, p)$  be the universal covering space of a locally pathwise connected space  $X$  and let  $A(\tilde{X}, p)$  be the discontinuous group of homeomorphisms of  $\tilde{X}$ . Then the fundamental groups  $\pi(X, x)$  and  $\pi(\tilde{X}/A(\tilde{X}, p), [\tilde{x}])$  are isomorphic for the natural map  $q : \tilde{X} \rightarrow \tilde{X}/A(\tilde{X}, p)$  defined by  $q(\tilde{x}) = [\tilde{x}]$ .

*Proof:* Note that

$$q : \tilde{X} \rightarrow \tilde{X}/A(\tilde{X}, p)$$

is a regular map since  $A(\tilde{X}, p)$  is the discontinuous group of homeomorphisms of  $\tilde{X}$  by Lemma 1.9, and the automorphism group  $A(\tilde{X}, q)$  is isomorphic to the automorphism group  $A(\tilde{X}, p)$ , i.e.  $A(\tilde{X}, q) = A(\tilde{X}, p)$ . Moreover  $(\tilde{X}, q)$  is a universal covering space of  $\tilde{X}/A(\tilde{X}, p)$  since  $(\tilde{X}, p)$  is a universal covering space of  $X$ , i.e.  $\tilde{X}$  is simply connected. Therefore from Lemma 1.5, the automorphism group  $A(\tilde{X}, p)$  is isomorphic to the fundamental group  $\pi(X, x)$  and the automorphism group  $A(\tilde{X}, q)$  is isomorphic to the fundamental group  $\pi(\tilde{X}/A(\tilde{X}, p), [\tilde{x}])$ . Hence  $\pi(X, x)$  and  $\pi(\tilde{X}/A(\tilde{X}, p), [\tilde{x}])$  are isomorphic. ■

From Theorem 2.12, the following corollary can be obtained.

*Corollary 2.13:* If  $(\tilde{X}, p)$  is a universal covering space of a locally pathwise connected space  $X$  and  $A(\tilde{X}, p)$  is the discontinuous group of homeomorphisms of  $\tilde{X}$ , then  $\pi(\tilde{X}/A(\tilde{X}, p), [\tilde{x}])$  and  $A(\tilde{X}, p)$  are isomorphic.

*Theorem 2.14:* If  $(\tilde{X}, p)$  is the universal covering space of  $X$ , then the order of the automorphism group  $A(\tilde{X}, p)$  is equal to the number of the elements in the orbit  $[\tilde{x}]$  of  $\tilde{x} \in p^{-1}(x)$ .

*Proof:* Let  $(\tilde{X}, p)$  be the universal covering space of  $X$ . Then the automorphism group  $A(\tilde{X}, p)$  is isomorphic to the fundamental group  $\pi(X, x)$  by Lemma 1.5, and the order of  $\pi(X, x)$  is equal to the number of the sheets of covering. We proved in [4] that the number of the elements in the orbit  $[\tilde{x}]$  of the point  $\tilde{x} \in p^{-1}(x)$  is equal to the number of the sheets of the covering  $(\tilde{X}, p)$  of  $X$ . Therefore the order of the automorphism group  $A(\tilde{X}, p)$  is equal to the number of the elements in the orbit  $[\tilde{x}]$  of  $\tilde{x} \in p^{-1}(x)$ . ■

From Theorem 2.14, the following corollary can be given.

*Corollary 2.15:* If  $(\tilde{X}, p)$  is a universal covering space of a locally pathwise connected space  $X$ , then the number of the elements in the orbit  $[\tilde{x}]$  is equal to the number of the sheets of the covering  $(\tilde{X}, p)$  of  $X$  for the natural map  $q : \tilde{X} \rightarrow \tilde{X}/A(\tilde{X}, p)$  defined by  $q(x) = [x]$ .

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