World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:6, No:8, 2012

Positive periodic solutions for a predator-prey model with modified Leslie-Gower Holling-type II schemes and a deviating argument

Yanling Zhu and Kai Wang

Abstract— In this paper, by utilizing the coincidence degree theorem a predator-prey model with modified Leslie-Gower Holling-type II schemes and a deviating argument is studied. Some sufficient conditions are obtained for the existence of positive periodic solutions of the model.

Keywords— Predator-prey model, Holling II type functional response, Positive periodic solution, Coincidence degree theorem

I. Introduction

THE famous Leslie predator-prey system was introduced by Leslie in [1] as follows,

$$\begin{cases} \dot{x} = x(a - bx) - p(x)y, \\ \dot{y} = y(c - d\dfrac{y}{x}), \end{cases}$$

where the carrying capacity of the predator's environment is proportional to the number of prey, $x(t), y(t)$ stand for the population of the prey and the predator at time t respectively, and $p(x)$ is the so-called predator functional response to prey. This interesting formulation for the predator dynamics has been discussed by Leslie and Gower in [2] and by Pielou in [3]. When $p(x) = \frac{ex}{f+x}$, the functional response $p(x)$ is of type II, which was also proposed by Holling, so it is called a functional response of the predator of Holling type. The term $d\frac{y}{x}$ of the above equation is called Leslie-Gower term, which measures the loss in the predator population due to rarity (per capita y/x) of its favorite food. In the case of severe scarcity, y can switch over to other populations but its growth will be limited by the fact that its most favorite food x is not available in abundance. This situation can be taken care of by adding a positive constant k to the denominator, see [4]. Recently, there are several papers [5-9] on positive periodic solutions of the predator-prey model. Now we consider the following system

$$\begin{cases} \dot{x}(t) = x(t)\left[r_1(t) - b(t)x(t) - \dfrac{a_1(t)y(t)}{x(t)+k_1}\right], \\ \dot{y}(t) = y(t)\left[r_2(t) - \dfrac{a_2(t)y(t-r(t))}{x(t-r(t))+k_2}\right]. \end{cases} \qquad (1)$$

Yanling Zhu and Kai Wang are with the Department of Applied Mathematics and Institute of Dynamic Systems, School of Statistics and Applied Mathematics, Anhui University of Finance & Economics, Bengbu 233030, Anhui, PR China.
Emails: zhuyanling99@yahoo.com.cn(Y. Zhu)
wangkai050318@163.com(K. Wang)

In [10], by using Floquet theory of linear periodic impulsive equation the authors have studied the dynamic behaviors of the periodic predator-prey model without the deviating argument. In [11], the authors studied the global stability of the positive equilibrium when the delay is a constant in the system (1). Time delay plays an important role in many biological dynamical systems, being particularly relevant in ecology, where time delays have been recognized to contribute critically to the stable or unstable outcome of prey densities due to predation. Therefore, it is interesting and important to study the above delayed modified Leslie-Gower and Holling-type-II schemes. As far as we know, there are few papers for the existence of periodic solutions for above system with deviating arguments.

Stimulated by this reason, in this paper we consider the existence of positive periodic solutions for the predator-prey model (1) with modified Leslie-Gower Holling-type II schemes and a deviating argument. We assume that x is the size of the prey population, and y is the size of the predator population; r_1, r_2, a_1, a_2, b are positive T-periodic continuous functions, k_1, k_2 are positive constants with the ecology meaning as follows,

- r_1: the growth rate of prey;
- r_2: the growth rate of predator;
- a_1: the maximum value which per capita reduction rate of prey can attain;
- a_2: the maximum value which per capita reduction rate of predator can attain;
- b: the strength of competition among individuals of species;
- k_1: the extent to which environment provides protection to prey;
- k_2: the extent to which environment provides protection to predator.

$r(t)$ is nonnegative, bounded and continuous function on $[0, +\infty)$. If set $r = \sup\{r(t) : t \in [0, +\infty)\}$, obviously we obtain $r \in [0, +\infty)$. Considering the application of model (1) to population dynamic systems, we assume that all positive solutions of model (1) satisfy the initial conditions as follows,

$$\begin{cases} x(\theta) = \varphi(\theta),\ \theta \in [-r, 0],\ \varphi(0) = \varphi_0 > 0, \\ y(\theta) = \psi(\theta),\ \psi \in [-r, 0],\ \psi(0) = \psi_0 > 0. \end{cases}$$

where φ and ψ are given nonnegative and bounded continuous functions on $[-r, 0]$.

The remainder of this paper is organized as follows. In the following section some useful lemma and notations are given. By using some analysis techniques and the coincidence degree theorem, we estimate a *priori bounds* of positive periodic solutions for system (1) in section 3. The significance of this paper is that the conditions are easy to be verified.

II. Lemma and Notations

In order to present sufficient conditions for guaranteeing the existence of positive periodic solutions for system (1), we firstly introduce the coincidence degree theorem.

Let X and Y be two Banach spaces, $L : \mathrm{Dom}L \subset X \to Y$ is linear map, and $N : X \to Y$ is continuous map. If $dim\,\mathrm{Ker}L = codim\,\mathrm{Im}L < +\infty$ and $\mathrm{Im}L \in Y$ is closed, then we call operator L a Fredholm operator with index zero [12]. And if L is a Fredholm operator with index zero and exist continuous projects $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I - Q)$, then $L|_{\mathrm{Dom}L\cap\mathrm{Ker}P} : (I - P)X \to \mathrm{Im}L$ exists inverse function, we set it as K_p. Assume that $\Omega \in X$ is any open set, if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N(\overline{\Omega}) \in X$ is relative compact, then we call that $N \in \overline{\Omega}$ is L-compact. For $\mathrm{Im}Q$ and $\mathrm{Ker}L$ are isomorphism, there exists an isomorphism map $J : \mathrm{Im}L \to \mathrm{Ker}L$.

Lemma 2.1.(*See [12]*)(Coincidence degree theorem)

Let X and Y be two Banach spaces, $L : Dom(L) \subset X \to Y$ be a Fredholm operator with index zero, $\Omega \subset Y$ be an open bounded set, and $N : \overline{\Omega} \to X$ be L-compact on $\overline{\Omega}$. If all the following conditions hold

[C_1] $Lx \neq \lambda Nx$, for $x \in \partial\Omega \cap Dom(L)$, $\lambda \in (0, 1)$;

[C_2] $Nx \notin ImL$, for $x \in \partial\Omega \cap KerL$;

[C_3] $deg\{JQN, \Omega\cap KerL, 0\} \neq 0$, where $J : ImQ \to KerL$ is an isomorphism.

Then equation $Lx = Nx$ has at least one solution on $\overline{\Omega} \bigcap Dom(L)$.

For convenience, we use the following notations. For a T-periodic continuous function f, we denote

$$\overline{f} = \frac{1}{T}\int_0^T f(t)\,dt, \quad f^L = \min_{t\in[0,T]} f(t),$$

$$f^M = \max_{t\in[0,T]} f(t), \quad |f|_0 = \max_{t\in[0,T]} |f(t)|.$$

III. Main result

In this section, we will establish some sufficient conditions for the existence of positive T-periodic solutions for the predator-prey model (1).

Theorem 3.1. *Suppose that the following conditions are satisfied*

[D_1] $\left[k_1\, r_1 - a_1 e^{A_5}\right]^L > 0$, and

[D_2] $k_1\overline{r}_1 - \overline{a}_1 e^{B_2} > 0$, where A_5 and B_2 are defined in the proof of the theorem.

Then system (1) *has at least one positive T-periodic solution.*

Proof Let $x(t) = e^{u(t)}$, $y(t) = e^{v(t)}$, then from (1) we have

$$\begin{cases} \dot{u}(t) = r_1(t) - b(t)e^{u(t)} - \dfrac{a_1(t)}{k_1 + e^{u(t)}}e^{v(t)}, \\ \dot{v}(t) = r_2(t) - \dfrac{a_2(t)}{k_2 + e^{u(t-r(t))}}e^{v(t-r(t))}. \end{cases}$$

Let $X = Y = \{z(t) = (u(t), v(t))^\top \in C(R, R^2) : z(t + T) \equiv z(t)\}$ equipped with the norm

$$||z|| = ||(u(t), v(t))^\top|| = \max_{t\in[0,T]} |u(t)| + \max_{t\in[0,T]} |v(t)|,$$

then X and Y are Banach spaces. We define operators L, P and Q as follows, respectively

$$L : \mathrm{Dom}L \cap X \to Y,\ Lz = \frac{dz}{dt},$$

$$P(z) = z(0),\ Q(z) = \frac{1}{T}\int_0^T z(t)dt,$$

where $\mathrm{Dom}L = \{z \in X : z(t) \in C^1(R, R^2)\}$, and define $N : X \to Y$ by the form

$$Nz = \left[\begin{array}{c} r_1(t) - b(t)e^{u(t)} - \dfrac{a_1(t)}{k_1 + e^{u(t)}}e^{v(t)} \\ r_2(t) - \dfrac{a_2(t)}{k_2 + e^{u(t-r(t))}}e^{v(t-r(t))} \end{array}\right].$$

Then it follows that $\mathrm{Ker}L = R^2$, $\mathrm{Im}L = \{z \in Y : \int_0^T z(t)dt = 0\}$ is closed in Y. $dim\mathrm{Ker}L = codim\mathrm{Im}L$, and P, Q are continuous projectors such that

$$\mathrm{Im}P = \mathrm{Ker}L,\ \mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I - Q),$$

thus L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) K_P: $\mathrm{Im}L \to \mathrm{Ker}P \cap \mathrm{Dom}L$ is given by the following form

$$K_p(z) = \int_0^t z(s)ds - \frac{1}{T}\int_0^T\int_0^t z(s)\,ds\,dt.$$

Therefore,

$$QNz = \left[\begin{array}{c} \dfrac{1}{T}\displaystyle\int_0^T \left[r_1(s) - b(s)e^{u(s)} - \dfrac{a_1(s)}{k_1 + e^{u(s)}}e^{v(s)}\right]ds \\ \dfrac{1}{T}\displaystyle\int_0^T \left[r_2(s) - \dfrac{a_2(s)}{k_2 + e^{u(s-r(s))}}e^{v(s-r(s))}\right]ds \end{array}\right]$$

and

$$K_p(I-Q)Nz$$
$$= \begin{bmatrix} \int_0^t \left[r_1(s) - b(s)e^{u(s)} - \frac{a_1(s)}{k_1 + e^{u(s)}} e^{v(s)} \right] ds \\ \int_0^t \left[r_2(s) - \frac{a_2(s)}{k_2 + e^{u(s-r(s))}} e^{v(s-r(s))} \right] ds \end{bmatrix}$$
$$- \begin{bmatrix} \frac{1}{T} \int_0^T \int_0^t \left[r_1(s) - b(s)e^{u(s)} - \frac{a_1(s)e^{v(s)}}{k_1 + e^{u(s)}} \right] dsdt \\ \frac{1}{T} \int_0^T \int_0^t \left[r_2(s) - \frac{a_2(s)e^{v(s-r(s))}}{k_2 + e^{u(s-r(s))}} \right] dsdt \end{bmatrix}$$
$$+ \begin{bmatrix} \frac{T-2t}{2T} \int_0^t \left[r_1(s) - b(s)e^{u(s)} - \frac{a_1(s)e^{v(s)}}{k_1 + e^{u(s)}} \right] ds \\ \frac{T-2t}{2T} \int_0^t \left[r_2(s) - \frac{a_2(s)e^{v(s-r(s))}}{k_2 + e^{u(s-r(s))}} \right] ds \end{bmatrix}.$$

Obviously, QN and $K_P(I-Q)N$ are continuous, it is not difficult to know that $K_P(I-Q)N(\overline{\Omega})$ is compact for any open bounded set $\Omega \subset X$ by employing *Arzela-Ascoli* theorem. Moreover, $QN(\overline{\Omega})$ is clearly bounded. So N is L-compact on $\overline{\Omega}$.

Now we consider the operator equation

$$Lz = \lambda Nz, \ \lambda \in (0,1).$$

We can get

$$\dot{u}(t) = \lambda \left(r_1(t) - b(t)e^{u(t)} - \frac{a_1(t)}{k_1 + e^{u(t)}} e^{v(t)} \right), \tag{2}$$

$$\dot{v}(t) = \lambda \left(r_2(t) - \frac{a_2(t)}{k_2 + e^{u(t-r(t))}} e^{v(t-r(t))} \right). \tag{3}$$

Let

$$u(\xi_1) = \max_{t\in[0,T]} u(t), \ \ u(\eta_1) = \min_{t\in[0,T]} u(t),$$
$$v(\xi_2) = \max_{t\in[0,T]} v(t), \ \ v(\eta_2) = \min_{t\in[0,T]} v(t).$$

One easily know that

$$\dot{u}(\xi_1) = \dot{u}(\eta_1) = 0, \ \dot{v}(\xi_2) = \dot{v}(\eta_2) = 0.$$

Then from (2) and (3) we know

$$r_1(\xi_1) = b(\xi_1)e^{u(\xi_1)} + \frac{a_1(\xi_1)}{k_1 + e^{u(\xi_1)}} e^{v(\xi_1)}. \tag{4}$$

$$r_2(\xi_2) = \frac{a_2(\xi_2)}{k_2 + e^{u(\xi_2 - r(\xi_2))}} e^{v(\xi_2 - r(\xi_2))}. \tag{5}$$

We get from (4) that

$$e^{u(\xi_1)} \leqslant \frac{r_1(\xi_1)}{b(\xi_1)} \leqslant \left[\frac{r_1}{b}\right]^M,$$

i.e.,

$$u(\xi_1) \leqslant \ln \left[\frac{r_1}{b}\right]^M := A_1. \tag{6}$$

Similarly, from (5) we have

$$e^{v(\xi_2 - r(\xi_2))} \leqslant \frac{(k_2 + e^{A_1}) r_2(\xi_2)}{a_2(\xi_2)} \leqslant (k_2 + e^{A_1}) \left[\frac{r_2}{a_2}\right]^M,$$

i.e.,

$$v(\xi_2 - r(\xi_2)) \leqslant \ln \left[(k_2 + e^{A_1}) \left[\frac{r_2}{a_2}\right]^M \right] := A_2. \tag{7}$$

On the other hand, it follows from (3) that

$$e^{v(\eta_2 - r(\eta_2))} = \frac{(k_2 + e^{u(\eta_2 - r(\eta_2))}) r_2(\eta_2)}{a_2(\eta_2)} \geqslant k_2 \left[\frac{r_2}{a_2}\right]^L,$$

i.e.,

$$v(\eta_2 - r(\eta_2)) \geqslant \ln k_2 \left[\frac{r_2}{a_2}\right]^L := A_3 \tag{8}$$

By the continuity of function v, (7) and (8), there is a constant $\xi^* \in [0,T]$ such that

$$|v(\xi^*)| \leq \max\{|A_2|, |A_3|\} := A_4,$$

which yields that

$$|v|_0 \leqslant |v(\xi^*)| + \int_0^T \left| r_2(t) - \frac{a_2(t)}{k_2 + e^{u(t-r(t))}} e^{v(t-r(t))} \right| dt \leqslant A_4 + 2T\overline{r}_2 := A_5. \tag{9}$$

In addition, from (2) we can easily get that

$$r_1(\eta_1) - b(\eta_1)e^{u(\eta_1)} \leqslant \frac{1}{k_1} a_1(\eta_1) e^{v(\eta_1)},$$

i.e.,

$$u(\eta_1) \geqslant \ln \left[\frac{k_1 r_1 - a_1 e^{A_5}}{k_1 b}\right]^L := A_6. \tag{10}$$

Now by (6), (9) and (10) we obtain the *priori bounds* of $z(t), \ \forall\, t \in [0,T]$, that is

$$A_6 \leqslant u(t) \leqslant A_1 \ \text{ and } \ -A_5 \leqslant v(t) \leqslant A_5, \ \text{ for } t \in [0,T].$$

Set $M_1 = \max\{|A_1|, |A_6|\}$ and take $M = M_1 + A_5 + \varepsilon$, where ε is a positive constant which is large enough such that every solution $(u^*, v^*)^\top$ (if the system has at least one solution) of the following system of algebraic equation set

$$\begin{cases} \overline{r}_1 - \overline{b}\, e^u - \dfrac{\overline{a}_1\, e^v}{k_1 + e^u} = 0, \\ \overline{r}_2 - \dfrac{\overline{a}_2\, e^v}{k_2 + e^u} = 0, \end{cases} \tag{11}$$

satisfies

$$|u^*| + |v^*| < M$$

and

$$\max(|B_1|, |B_4|) + \max(|B_2|, |B_3|) < M,$$

where

$$B_1 = \ln \frac{\overline{r}_1}{\overline{b}}, \ B_2 = \ln \frac{\overline{r}_2 (k_2 + e^{B_1})}{\overline{a}_2},$$

$$B_3 = \ln \frac{k_2 \overline{r}_2}{\overline{a}_2} \text{ and } B_4 = \ln \frac{k_1 \overline{r}_1 - \overline{a}_1 e^{B_2}}{k_1 \overline{b}}.$$

Now let $\Omega = \{z \in X : ||z|| < M\}$, then Ω satisfies condition $[C_1]$ in Lemma 2.1, $\forall z = (u^*, v^*)^\top \in (\partial\Omega \cap \mathrm{Ker}L)$, then $||z|| = M$. If the system of algebraic equation set (11) has at least one solution, we conclude that

$$QNz = \begin{bmatrix} \overline{r}_1 - \overline{b}\, e^u - \dfrac{\overline{a}_1\, e^v}{k_1 + e^u} \\ \overline{r}_2 - \dfrac{\overline{a}_2\, e^v}{k_2 + e^u} \end{bmatrix} \neq 0.$$

If the system of algebraic equation set (11) has no solution, one can directly obtain $QNz \neq 0$. Finally in order to prove $[C_3]$ in Lemma 2.1 we define a homomorphism mapping

$$J : \mathrm{Im}Q \to \mathrm{Ker}L,\ Jz \equiv z,$$

and

$$H : \mathrm{Dom}X \times [0, 1],$$

$$H(u, v, \mu) = \begin{bmatrix} \overline{r}_1 - \overline{b}\, e^u \\ \overline{r}_2 - \dfrac{\overline{a}_2\, e^v}{k_2 + e^u} \end{bmatrix} + \mu \begin{bmatrix} -\dfrac{\overline{a}_1\, e^v}{k_1 + e^u} \\ 0 \end{bmatrix},$$

where μ is a parameter. Now we claim that

$$H(u, v, \mu) \neq 0, \forall\, (u, v, \mu) \in (\partial\Omega \cap \mathrm{Ker}L) \times [0, 1].$$

If not, then there exist $z = (u, v)^\top \in \partial\Omega \cap \mathrm{Ker}L, \mu \in [0, 1]$ with $|u| + |v| = M$ such that

$$H(u, v, \mu) = 0,$$

i.e.,

$$\overline{r}_1 - \overline{b}\, e^u - \frac{\mu\overline{a}_1\, e^v}{k_1 + e^u} = 0, \tag{12}$$

$$\overline{r}_2 - \frac{\overline{a}_2\, e^v}{k_2 + e^u} = 0. \tag{13}$$

By (12) we easily see

$$u \leqslant \ln \frac{\overline{r}_1}{\overline{b}} = B_1.$$

Similarly, from (13) we obtain

$$v \leqslant \ln \frac{\overline{r}_2(k_2 + e^{B_1})}{\overline{a}_2} = B_2$$

and

$$v \geqslant \ln \frac{k_2\overline{r}_2}{\overline{a}_2} = B_3.$$

Also from (12) we can get

$$u \geqslant \ln \frac{k_1\overline{r}_1 - \overline{a}_1 e^{B_2}}{k_1\overline{b}} = B_4.$$

Therefore,

$$|u| + |v| \leqslant \max(|B_1|, |B_4|) + \max(|B_2|, |B_3|) < M,$$

which leads to a contradiction. Hence by a direct calculation we have

$$\begin{aligned} & deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \\ =& deg\{H(u, v, 1), \Omega \cap \mathrm{Ker}L, 0\} \\ =& deg\{H(u, v, 0), \Omega \cap \ker L, 0\} \\ =& deg\Big\{(\overline{r}_1 - \overline{b}\, e^u, \overline{r}_2 - \frac{\overline{a}_2\, e^v}{k_2 + e^u})^\top, \Omega \cap \ker L, 0\Big\} \\ >& 0. \end{aligned}$$

So $[C_3]$ in Lemma 2.1 is satisfied. By applying Lemma 2.1, we conclude that system (1) has at least one positive T-periodic solution. The proof is now finished. □

ACKNOWLEDGMENT

This research was supported by the NSF of Anhui (No.10040606Q01, 1208085QA11), the NSF of the Department of Education of Anhui (No.KJ2011B003) and the Youth Talent Foundation of the Department of Education of Anhui (No.2012SQRL077).

REFERENCES

[1] P.H. Leslie, *Some further notes on the use of matrices in population mathematics*, Biometrica **35** (1948) 213-245.
[2] P.H. Leslie, J.C. Gower, *The properties of a stochastic model for the predatorCprey type of interaction between two species*, Biometrica **47** (1960) 219-234.
[3] E.C. Pielou, *An Introduction to Mathematical Ecology*, Wiley-Interscience, New York, 1969.
[4] M.A. Aziz-Alaoui, *Study of a Leslie-Gower-type tritrophic population*, Chaos Solitons & Fractals **14** (2002) 1275-1293.
[5] Z.D. Teng, L.S. Chen, *Necessary and sufficient conditions for existence of positive periodic solutions of periodic predator-prey systems*, Acta. Math. Scientia **18** (4) (1998) 402-406(in Chinese).
[6] C. Chen, F.D. Chen, *Conditions for global attractivity of multispecise ecological Competition-Predator system with Holling III type functional response*, J. Biomathematics **19** (2) (2004) 136-140.
[7] F.D. Chen, *The permanence and global attractivity of Lotka-Volterra competition system with feedback controls*, Nonlinear Anal. Real World Appl. **7** (2006) 133-143.
[8] K. Wang, Y.L. Zhu, *Global attractivity of positive periodic solution for a Volterra model*. Appl. Math. Comput. **203** (2) (2008) 493-501.
[9] K. Wang, *Existence and global asymptotic stability of positive periodic solution for a Predator-Prey system with mutual interference*, Nonlinear Anal. Real World Appl. **10** (5) (2009) 2774-2783.
[10] X.Y. Song, Y.F. Li, *Dynamic behaviors of the periodic predator-prey model with modified Leslie-Gower Holling-type II schemes and impulsive effect*, Nonlinear Anal. Real World Appl. **9** (2008) 64-79.
[11] A.F. Nindjin, M.A. Aziz-Alaoui, M. Cadivel, *Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with time delay*, Nonlinear Anal. Real World Appl. **7** (2006) 1104-1118.
[12] R.E. Gaines, J.L. Mawhin, *Cioncidence Degree and Nonlinear Differential Equations*, Springer-Verlag, Berlin, 1977.