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# A new preconditioned AOR method for Z-matrices

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*Abstract*—In this paper, we present a preconditioned AOR-type iterative method for solving the linear systems $Ax = b$, where $A$ is a Z-matrix. And give some comparison theorems to show that the rate of convergence of the preconditioned AOR-type iterative method is faster than the rate of convergence of the AOR-type iterative method.

*Keywords*—Z-matrix, AOR-type iterative method, precondition, comparison.



## I. Introduction

FOR solving linear system

$$Ax = b, \tag{1}$$

where $A$ is an $n \times n$ square matrix, and $x$ and $b$ are n-dimensional vectors, the basic iterative method is

$$Mx^{k+1} = Nx^k + b, \; k = 0, 1, \cdots, \tag{2}$$

where $A = M - N$ and $M$ is nonsingular. Thus (2) can be written as

$$x^{k+1} = Tx^k + c, \; k = 0, 1, ...,$$

where $T = M^{-1}N$, $c = M^{-1}b$.

Assuming $A$ has unit diagonal entries and let $A = I - L - U$ where $I$ is the identity matrix, $-L$ and $-U$ are strictly lower and strictly upper triangular parts of $A$, respectively. Then,
(I) the iteration matrix of the classical Gauss-Seidel-type method is given by

$$T = (I - L)^{-1}U \tag{3}$$

(II) the iteration matrix of the classical SOR-type method is given by

$$L_r = (I - rL)^{-1}[(1 - r)I + rU] \tag{4}$$

where $r \neq 0$ is a parameter called the relaxation parameter.
(III) the iteration matrix of the classical AOR-type method is given by

$$L_{r,w} = (I - L)^{-1}[(1 - w)I + (w - r)L + wU] \tag{5}$$

where $w$ and $r$ are real parameters and $w \neq 0$.

Transform the original system (1) into the preconditioned form

$$PAx = Pb.$$

Then, we can define the basic iterative scheme:

$$M_p x^{k+1} = N_p x^k + Pb, \; k = 0, 1, \ldots,$$

where $PA = M_p - N_p$ and $M_p$ is nonsingular. Thus the equation above can also be written as

$$x^{k+1} = Tx^k + c, \; k = 0, 1, \ldots,$$

where $T = M_p^{-1}N_p, \; c = M_p^{-1}Pb$.

In paper [1], Meijun Wu et al. presented the preconditioned AOR-type iterative method with

$$P_\alpha = I + S_\alpha = \begin{pmatrix} 1 & -\alpha_1 a_{12} & & & \\ & 1 & -\alpha_2 a_{23} & & \\ & & \ddots & \ddots & \\ & & & 1 & -\alpha_{n-1}a_{n-1,n} \\ & & & & 1 \end{pmatrix} \tag{6}$$

and $\alpha_i (i = 1, 2, \cdots, n-1)$ are nonnegative real numbers, and obtained some comparison results.

In this paper, we will present the preconditioned AOR-type iterative method with

$$P_\beta = I + K_\beta = \begin{pmatrix} 1 & & & & \\ -\beta_1 a_{12} & 1 & & & \\ & -\beta_2 a_{23} & \ddots & & \\ & & \ddots & 1 & \\ & & & -\beta_{n-1}a_{n-1,n} & 1 \end{pmatrix} \tag{7}$$

and $\beta_i(\; i = 1, 2, \cdots, n-1)$ are nonnegative real numbers.

In the following, we consider three splittings for $\tilde{A}$:

$$\tilde{A} = \begin{cases} (\tilde{D} - \tilde{L}) - \tilde{U} \\ \frac{1}{r}(\tilde{D} - r\tilde{L}) - \frac{1}{r}[(1 - r)\tilde{D} + r\tilde{U}] \\ \frac{\tilde{D} - r\tilde{L}}{w} - \frac{1}{w}[(1 - w)\tilde{D} + (w - r)\tilde{L} + w\tilde{U}] \end{cases} \tag{8}$$

where $\tilde{D}$ , $-\tilde{L}$ and $-\tilde{U}$ are diagonal, strictly lower and strictly upper triangular parts of $\tilde{A}$, respectively.

In view of (8), the iteration matrices associated with $\tilde{A}$ are:

$$\tilde{T} = (\tilde{D} - \tilde{L})^{-1}\tilde{U} \tag{9}$$

$$\tilde{L}_r = (\tilde{D} - r\tilde{L})^{-1}[(1 - r)\tilde{D} + r\tilde{U}] \tag{10}$$

$$\tilde{L}_{r,w} = (\tilde{D} - r\tilde{L})^{-1}[(1 - w)\tilde{D} + (w - r)\tilde{L} + w\tilde{U}] \tag{11}$$

In this paper, we will discuss the preconditioned iterative methods with the preconditioner $P_\beta$ for solving Z-matrices linear systems and present comparison theorems of these methods.

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## II. Comparison results of preconditioned AOR-type methods with preconditioner $P_\beta$

We need the following definitions and results.

**Definition 2.1** (Young [3]). $A$ is a Z-matrix if $a_{ij} \leq 0$, for all $i, j = 1, 2, \ldots n$, $i \neq j$.

**Lemma 2.2** (Young [3]). Let $A \geq 0$ be an irreducible matrix. Then

(1) $A$ has a positive real eigenvalue equals to its spectral radius;

(2) To $\rho(A)$ there corresponds an eigenvector $x > 0$;

(3) $\rho(A)$ is a simple eigenvalue of $A$.

**Lemma 2.3** (Varga [4]). Let $A$ be a nonnegative matrix. Then

(1) If $\alpha x \leq Ax$ for some nonnegative vector $x$, $x \neq 0$, then $\alpha \leq \rho(A)$;

(2) If $Ax \leq \beta x$ for some positive vector $x$, then $\rho(A) \leq \beta$. Moreover, if $A$ is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector $x$, then $\alpha \leq \rho(A) \leq \beta$ and $x$ is a positive vector.

**Lemma 2.4** ([5]). Let $A = M - N$ be an M-splitting of $A$. Then $\rho(M^{-1}N) < 1$ if and only if $A$ is a nonsingular M-matrix.

**Lemma 2.5** ([6]). Let $A$ be a Z-matrix. Then $A$ is a nonsingular M-matrix if and only if there is a positive vector $x$ such that $Ax \geq 0$.

For the linear system (1), we consider its preconditioned form

$$P_\beta A x = P_\beta b$$

with the preconditioner $P_\beta = I + K_\beta$ in this section.

We apply the AOR method to it and have the corresponding preconditioned AOR iteration matrix

$$\hat{L}_{r,w} = [D_\beta - rL_\beta]^{-1}[(1-w)D_\beta + (w-r)L_\beta + wU_\beta], \quad (12)$$

where $D_\beta$ , $-L_\beta$ and $-U_\beta$ are diagonal, strictly lower and strictly upper triangular parts of $A_\beta = P_\beta A$, respectively.

Now we give the main results as follows.

**Theorem 2.1** Let $A = I - L - U \in R^{n\times n}$ be a nonsingular Z-matrix, $L_{r,w}$and $\hat{L}_{r,w}$ be the iteration matrices given by (5) and (12). Assume that $0 < r < w < 1$, and $0 < \beta_i < 1$, $i = 1, 2, \ldots, n-1$.

(I) If $\rho(L_{r,w}) < 1$, then

$$\rho(\hat{L}_{r,w}) \leq \rho(L_{r,w}) < 1$$

(II) Let $A$ be irreducible. Assume that

$$a_{i,i-1}a_{i-1,i} < 1, \ \ i = 2, \ldots, n.$$

then

(1) $\rho(\hat{L}_{r,w}) > \rho(L_{r,w}) \quad if\ \rho(L_{r,w}) > 1$;
(2) $\rho(\hat{L}_{r,w}) = \rho(L_{r,w}) \quad if\ \rho(L_{r,w}) = 1$;
(3) $\rho(\hat{L}_{r,w}) < \rho(L_{r,w}) \quad if\ \rho(L_{r,w}) < 1$.

**Proof.** Let

$$\begin{aligned}
M &= \tfrac{1}{w}(I - rL)\\
N &= \tfrac{1}{w}[(1-w)I + (w-r)L + wU]\\
E_\beta &= \tfrac{1}{w}(D_\beta - rL_\beta)\\
F_\beta &= \tfrac{1}{w}[(1-w)D_\beta + (w-r)L_\beta + wU_\beta]\\
M_\beta &= \tfrac{1}{w}(I + K_\beta)(I - rL)\\
N_\beta &= \tfrac{1}{w}(I + K_\beta)[(1-w)I + (w-r)L + wU]
\end{aligned}$$

Then, we have

$$A = M - N, \quad A_\beta = E_\beta - F_\beta = M_\beta - N_\beta$$

(I) Since $A$ is a nonsingular Z-matrix and $0 < r < w < 1$, $w \neq 0$, it is clear that $M = \frac{1}{w}(I - rL)$ is a nonsingular M-matrix and the splitting

$$A = M - N = \frac{1}{w}(I - rL) - \frac{1}{w}[(1-w)I + (w-r)L + wU]$$

is an M-splitting. Since $\rho(L_{r,w}) < 1$, it follows from Lemma 2.4 that $A$ is a nonsingular M-matrix. Then by Lemma 2.5, there is a positive vector $x$ such that $Ax \geq 0$ , so $A_\beta x = (I + K_\beta)Ax \geq 0$.

By Lemma 2.5, $A_\beta$ is also a nonsingular M-matrix.

Obviously, we can get $D_\beta$ is an positive diagonal matrix, and $L_\beta$ is nonnegative. From $r > 0$ we know that $E_\beta$ is a Z-matrix. Since $rD_\beta^{-1}L_\beta \geq 0$ is a strictly lower triangular matrix so that $\rho(rD_\beta^{-1}L_\beta) = 0 < 1$, we have $(I - rD_\beta^{-1}L_\beta)^{-1} \geq 0$. Then

$$E_\beta = (I - rD_\beta^{-1}L_\beta)^{-1}D_\beta^{-1} \geq 0$$

Hence $E_\beta$ is an nonsingular M-matrix.

By $F_\beta \geq 0$ we can prove that $A_\beta = E_\beta - F_\beta$ is an M-splitting. It follows from Lemma 2.4 that

$$\rho(\hat{L}_{r,w}) = \rho(E_\beta^{-1}F_\beta) < 1.$$

Since $A_\beta = E_\beta - F_\beta$ and $A = M - N$ are both M-splitting and $M_\beta^{-1}N_\beta = M^{-1}N$, two splittings $A_\beta = E_\beta - F_\beta = M_\beta - N_\beta$ are nonnegative.

On the other hand,

$$\begin{aligned}
&M_\beta - E_\beta\\
&= \tfrac{1}{w}(I + K_\beta)(I - rL) - \tfrac{1}{w}(D_\beta - rL_\beta)\\
&= \tfrac{1}{w}(I + K_\beta - rL - rK_\beta L - D_\beta + rL_\beta)\\
&= \tfrac{1}{w}(I + K_\beta - rL - rK_\beta L - D_\beta\\
&\quad + r(D_\beta - I + L - K_\beta + K_\beta L))\\
&= \tfrac{1}{w}(I + K_\beta - rL - rK_\beta L - D_\beta\\
&\quad + rD_\beta - rI + rL - rK_\beta + rK_\beta L)\\
&= \tfrac{1}{w}(I + K_\beta - D_\beta + rD_\beta - rI - rK_\beta)\\
&= \tfrac{1}{w}[(I - r)(I - D_\beta) + (1 - r)K_\beta]\\
&\geq 0
\end{aligned}$$

which implies

$$A_\beta^{-1}M_\beta - A_\beta^{-1}E_\beta = A_\beta^{-1}(M_\beta - E_\beta) \geq 0,$$

Therefore, $A_\beta^{-1}M_\beta \geq A_\beta^{-1}E_\beta \geq 0$. So we have $\rho(E_\beta^{-1}F_\beta) \leq \rho(M_\beta^{-1}N_\beta)$, that is

$$\rho(\hat{L}_{r,w}) \leq \rho(L_{r,w}) < 1.$$

(II) Let $A = I - L - U$ be irreducible. Since

$$\begin{aligned}
L_{r,w} &= (I - rL)^{-1}[(1-w)I + (w-r)L + wU]\\
&= (1-w)I + w(1-r)L + wU + Q
\end{aligned}$$

with $Q = (I - rL)^{-1}rL[w(1-r)L + wU] \geq 0$

We have $L_{r,w}$ is a nonnegative and irreducible matrix. According to Lemma 2.2, there exits a positive vector $x$, such that

$$L_{r,w}x = \lambda x,$$

From the expression of $L_{r,w}$ we obtain the following equality

$$[(1-w)I + (w-r)L + wU]x = \lambda(I - rL)x$$

which is equivalent to

$$[(1-w-r)I + (w-r+\lambda r)L + wU]x = 0, \qquad (13)$$

and

$$(\lambda-1)(I-rL)xw(L+U-I)x \qquad (14)$$

Let $K_\beta U = K_1 + K_2$, where $K_1,\ K_2$ are the diagonal and lower triangular parts of $K_\beta U$, respectively. So

$$\begin{aligned} A_\beta &= D_\beta - L_\beta - U_\beta \\ &= (I-K_1) - (L - K_\beta + K_\beta L) - (U + K_2) \end{aligned}$$

where $D_\beta = I - K_1$, $L_\beta = L - K_\beta + K_\beta L$, $U_\beta = U + K_2$

By (13) and (14), we have

$$\begin{aligned} &\hat{L}_{r,w}x - \lambda x \\ &= (D_\beta - rL_\beta)^{-1}[(1-w)D_\beta + (w-r)L_\beta + wU_\beta \\ &\quad - \lambda(D_\beta - rL_\beta)]x \\ &= (D_\beta - rL_\beta)^{-1}[(1-w-\lambda)D_\beta \\ &\quad + (w-r+\lambda r)L_\beta + wU_\beta]x \\ &= (D_\beta - rL_\beta)^{-1}[(1-w-\lambda)(I-K_1) \\ &\quad + (w-r+\lambda r)(L - K_\beta + K_\beta L) + w(U+K_2)]x \\ &= (D_\beta - rL_\beta)^{-1}\left\{[(1-w-\lambda)I + (w-r+\lambda r)L + wU]\right. \\ &\quad + [-(1-w-\lambda)K_1 \\ &\quad \left.+(w-r+\lambda r)(-K_\beta + K_\beta L) + wK_2]\right\} x \\ &= (D_\beta - rL_\beta)^{-1}[-(1-w-\lambda)K_1 \\ &\quad + (w-r+\lambda r)(-K_\beta + K_\beta L) + wK_2]x \\ &= (D_\beta - rL_\beta)^{-1}[(\lambda-1)K_1 + r(\lambda-1)(K_\beta L - K_\beta) \\ &\quad + wK_\beta(L+U-I)]x \\ &= (D_\beta - rL_\beta)^{-1}[(\lambda-1)K_1 + r(\lambda-1)(K_\beta L - K_\beta) \\ &\quad + (\lambda-1)(I-rL)K_\beta]x \\ &= (D_\beta - rL_\beta)^{-1}[(\lambda-1)K_1 - r(\lambda-1)K_\beta + (\lambda-1)K_\beta]x \\ &= (\lambda-1)(D_\beta - rL_\beta)^{-1}[K_1 + (1-r)K_\beta]x \end{aligned}$$

Here $(D_\beta - rL_\beta)^{-1} \geq 0, \quad K_1 \geq 0, \quad (1-r)K_\beta \geq 0$

(1) If $\lambda > 1$, then $\hat{L}_{r,w} \geq 0$ but not equal to 0. Therefore

$$\hat{L}_{r,w} \geq \lambda x.$$

By Lemma 2.3, we get $\rho(\hat{L}_{r,w}) > \lambda = \rho(L_{r,w})$.

(2) If $\lambda = 1$, then $\hat{L}_{r,w} = 0$but not equal to 0. Therefore

$$\hat{L}_{r,w} = \lambda x.$$

By Lemma 2.3, we get $\rho(\hat{L}_{r,w}) = \lambda = \rho(L_{r,w})$.

(3) If $\lambda < 1$, then $\hat{L}_{r,w} \leq 0$but not equal to 0. Therefore

$$\hat{L}_{r,w} \leq \lambda x.$$

By Lemma 2.3, we get $\rho(\hat{L}_{r,w}) < \lambda = \rho(L_{r,w})$.

**Corollary 2.2** Let $A = I - L - U \in R^{n\times n}$ be a nonsingular Z-matrix, $L_r$ and $\hat{L}_r$ be the iteration matrices of classical SOR-type methods and preconditioned SOR-type methods with preconditioner $P_\beta$, respectively. Assume that $0 < r < 1$, and $0 < \beta_i < 1$, $i = 1, 2, \ldots, n-1$.

(I) If $\rho(L_r) < 1$, then $\rho(\hat{L}_r) \leq \rho(L_r) < 1$;

(II) Let $A$ be irreducible. Assume that $a_{i,i-1}a_{i-1,i} < 1,\ i = 2, \ldots, n$, then

(1) $\rho(\hat{L}_{r,w}) > \rho(L_{r,w}) \quad if\ \rho(L_{r,w}) > 1$;

(2) $\rho(\hat{L}_{r,w}) = \rho(L_{r,w}) \quad if\ \rho(L_{r,w}) = 1$;

(3) $\rho(\hat{L}_{r,w}) < \rho(L_{r,w}) \quad if\ \rho(L_{r,w}) < 1$.

**Corollary 2.3** Let $A = I - L - U \in R^{n\times n}$ be a nonsingular Z-matrix, $T$ and $\hat{T}$ be the iteration matrices of classical Gauss-Seidel-type methods and preconditioned Gauss-Seidel-type methods with preconditioner $P_\beta$, respectively. $0 < \beta_i < 1$, $i = 1, 2, \ldots, n-1$.

(I) If $\rho(T) < 1$, then $\rho(\hat{T}) \leq \rho(T) < 1$;

(II) Let $A$ be irreducible. Assume that $a_{i,i-1}a_{i-1,i} < 1,\ i = 2, \ldots n$, then

(1) $\rho(\hat{T}) > \rho(T)$ if $\rho(T) > 1$;

(2) $\rho(\hat{T}) = \rho(T)$ if $\rho(T) = 1$;

(3) $\rho(\hat{T}) < \rho(T)$ if $\rho(T) < 1$.

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