World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:4, No:7, 2010

Generalized inverse eigenvalue problems for symmetric arrow-head matrices

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Abstract—In this paper, we first give the representation of the general solution of the following inverse eigenvalue problem (IEP): Given $X \in \mathbf{R}^{n\times p}$ and a diagonal matrix $\Lambda \in \mathbf{R}^{p\times p}$, find nontrivial real-valued symmetric arrow-head matrices A and B such that $AX\Lambda = BX$. We then consider an optimal approximation problem: Given real-valued symmetric arrow-head matrices $\tilde{A}$, $\tilde{B} \in \mathbf{R}^{n\times n}$, find $(\hat{A},\hat{B}) \in \mathcal{S}_E$ such that $\|\hat{A}-\tilde{A}\|^2 + \|\hat{B}-\tilde{B}\|^2 = \min_{(A,B)\in\mathcal{S}_E}(\|A-\tilde{A}\|^2+\|B-\tilde{B}\|^2)$, where $\mathcal{S}_E$ is the solution set of IEP. We show that the optimal approximation solution $(\hat{A},\hat{B})$ is unique and derive an explicit formula for it.

Keywords—partially prescribed spectral information, symmetric arrow-head matrix, inverse problem, optimal approximation.

I. INTRODUCTION

THROUGHOUT this paper, we denote the real $m \times n$ matrix space by $\mathbf{R}^{m\times n}$, the set of all symmetric matrices in $\mathbf{R}^{n\times n}$ by $\mathbf{SR}^{n\times n}$, the transpose and the Moore-Penrose generalized inverse of a real matrix A by A^T and A^+, respectively. I_n represents the identity matrix of size n. For $A, B \in \mathbf{R}^{m\times n}$, an inner product in $\mathbf{R}^{m\times n}$ is defined by $(A,B) = \text{trace}(B^TA)$, then $\mathbf{R}^{m\times n}$ is a Hilbert space. The matrix norm $\|\cdot\|$ induced by the inner product is the Frobenius norm. Given two matrices $A = [a_{ij}] \in \mathbf{R}^{m\times n}$ and $B = [b_{ij}] \in \mathbf{R}^{p\times q}$, the Kronecker product of A and B is defined by $A \otimes B = [a_{ij}B] \in \mathbf{R}^{mp\times nq}$. Also, for an $m \times n$ matrix $A = [a_1, a_2, \cdots, a_n]$, where $a_i, i = 1, \cdots, n$, is the i-th column vector of A, the stretching function $\text{Vec}(A)$ is defined by $\text{Vec}(A) = [a_1^T, a_2^T, \cdots, a_n^T]^T$.

Definition 1 An $n \times n$ matrix A is called an arrow-head matrix if

$$A = \begin{bmatrix} a_1 & b_1 & b_2 & \cdots & b_{n-1} \\ c_1 & a_2 & 0 & \cdots & 0 \\ c_2 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & 0 & 0 & \cdots & a_n \end{bmatrix}.$$

If $b_i = c_i,\ i = 1, \cdots, n-1$, then A is a symmetric arrow-head matrix.

The application background and the computations of the eigenvalues and eigenvectors of this kind of matrices can see [1, 2, 3, 4]. The inverse problem of constructing the symmetric arrow-head matrix from spectral data has been investigated by Peng et al. [5], and Borges et al. [6]. In this paper we will further consider generalized inverse eigenvalue problems for symmetric arrow-head matrices, which can be described as follows:

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Problem IEP. Given $X \in \mathbf{R}^{n\times p}$ and a diagonal matrix $\Lambda \in \mathbf{R}^{p\times p}$, find nontrivial real-valued symmetric arrow-head matrices A and B such that

$$AX\Lambda = BX. \qquad (1)$$

Problem II. Given real-valued symmetric arrow-head matrices $\tilde{A}$, $\tilde{B} \in \mathbf{R}^{n\times n}$, find $(\hat{A},\hat{B}) \in \mathcal{S}_E$ such that

$$\|\hat{A}-\tilde{A}\|^2+\|\hat{B}-\tilde{B}\|^2 = \min_{(A,B)\in\mathcal{S}_E}(\|A-\tilde{A}\|^2+\|B-\tilde{B}\|^2), \qquad (2)$$

where $\mathcal{S}_E$ is the solution set of IEP.

The paper is organized as follows. In Section 2, using the Kronecker product and stretching function $\text{Vec}(\cdot)$ of matrices, we give an explicit representation of the solution set $\mathcal{S}_E$ of Problem IEP. In Section 3, we show that there exists a unique solution in Problem II and present the expression of the unique solution $(\hat{A},\hat{B})$ of Problem II. Finally, in Section 4, a numerical algorithm to acquire the optimal approximation solution under the Frobenius norm sense is described and a numerical example is provided.

II. THE SOLUTION OF PROBLEM IEP

To begin with, we introduce two lemmas.

Lemma 1: [7] If $L \in \mathbf{R}^{m\times q},\ b \in \mathbf{R}^m$, then $Ly = b$ has a solution $y \in \mathbf{R}^q$ if and only if $LL^+b = b$. In this case, the general solution of the equation can be described as $y = L^+b + (I_q - L^+L)z$, where $z \in \mathbf{R}^q$ is an arbitrary vector.

Lemma 2: [8] Let $D \in \mathbf{R}^{m\times n}, H \in \mathbf{R}^{n\times l}, J \in \mathbf{R}^{l\times s}$. Then

$$\text{Vec}(DHJ) = (J^T \otimes D)\text{Vec}(H).$$

Let S_0 be the set of all real-valued symmetric arrow-head matrices, then S_0 is a linear subspace of $\mathbf{SR}^{n\times n}$, and the dimension of S_0 is $d = 2n-1$.

Define Y_{ij} as

$$Y_{ij} = \begin{cases} \frac{\sqrt{2}}{2}(e_ie_j^T + e_je_i^T),\ i = 1,\ j = 2, \cdots, n; \\ e_ie_i^T, \quad i = j = 1, \cdots, n, \end{cases} \qquad (3)$$

where $e_i,\ i = 1, \cdots, n$, is the i-th column vector of the identity matrix I_n. It is easy to verify that $\{Y_{ij}\}$ forms an orthonormal basis of the subspace S_0, that is,

$$(Y_{ij}, Y_{kl}) = \begin{cases} 0, & i \neq k \text{ or } j \neq l, \\ 1, & i = k \text{ and } j = l. \end{cases} \tag{4}$$

Now, if $A,\ B \in \mathbf{R}^{n\times n}$ are symmetric arrow-head matrices, then $A,\ B$ can be expressed as

$$A = \sum_{i,j} \alpha_{ij} Y_{ij},\ B = \sum_{i,j} \beta_{ij} Y_{ij}, \tag{5}$$

where the real numbers $\alpha_{ij},\ \beta_{ij},\ \begin{cases} i = 1,\ j = 2, \cdots, n; \\ i = j = 1, \cdots, n, \end{cases}$ are yet to be determined. Substituting (5) into (1), we have

$$\sum_{i,j} \alpha_{ij} Y_{ij} X \Lambda - \sum_{i,j} \beta_{ij} Y_{ij} X = 0. \tag{6}$$

Let

$$\alpha = [\alpha_{11}, \cdots, \alpha_{n,n}, \alpha_{12}, \cdots, \alpha_{1,n}]^T,$$

$$\beta = [\beta_{11}, \cdots, \beta_{n,n}, \beta_{12}, \cdots, \beta_{1,n}]^T,$$

$$\begin{aligned} G = [&\text{Vec}(Y_{11}), \cdots, \text{Vec}(Y_{n,n}), \\ &\text{Vec}(Y_{12}), \cdots, \text{Vec}(Y_{1,n})] \in \mathbf{R}^{n^2\times d} \end{aligned} \tag{7}$$

and

$$M = (\Lambda^T X^T \otimes I_n)G,\ N = (X^T \otimes I_n)G. \tag{8}$$

Using Lemma 2, we see that the equation of (6) is equivalent to

$$M\alpha - N\beta = 0. \tag{9}$$

It follows from Lemma 1 that the equation of (9) with unknown vector α has a solution if and only if

$$E_M N\beta = 0, \tag{10}$$

where $E_M = I_{np} - MM^+$. Using Lemma 1 again, we know that the equation of (10) with respect to β is always solvable and the general solution to the equation is

$$\beta = (I_d - (E_M N)^+ E_M N)u, \tag{11}$$

where $u \in \mathbf{R}^d$ is an arbitrary vector.
Substituting (11) into (9) and applying Lemma 1, we obtain

$$\alpha = M^+ N(I_d - (E_M N)^+ E_M N)u + F_M v, \tag{12}$$

where $F_M = I_d - M^+ M$, and $v \in \mathbf{R}^d$ is an arbitrary vector.

In summary of above discussion, we have proved the following result.

Theorem 1: Suppose that $X \in \mathbf{R}^{n\times p}, \Lambda \in \mathbf{R}^{p\times p}$, and Λ is a diagonal matrix. Let $\{Y_{ij}\},\ G, M, N$ be given as in (3), (7) and (8). Write $d = 2n - 1,\ E_M = I_{np} - MM^+,\ F_M = I_d - M^+ M$. Then the solution set $\mathcal{S}_E$ of Problem IEP can be expressed as

$$\begin{aligned} \mathcal{S}_E = \big\{&(A, B) \in \mathbf{SR}^{n\times n} \times \mathbf{SR}^{n\times n} : \\ &A = K(\alpha \otimes I_n),\ B = K(\beta \otimes I_n)\big\}, \end{aligned} \tag{13}$$

where

$$K = [Y_{11}, \cdots, Y_{n,n}, Y_{12}, \cdots, Y_{1,n}] \in \mathbf{R}^{n\times nd}, \tag{14}$$

α, β are, respectively, given by (12) and (11) with $u, v \in \mathbf{R}^d$ being arbitrary vectors.

III. The solution of Problem II

It follows from Theorem 1 that the set $\mathcal{S}_E$ is always nonempty. It is easy to verify that $\mathcal{S}_E$ is a closed convex subset of $\mathbf{SR}^{n\times n} \times \mathbf{SR}^{n\times n}$. From the best approximation theorem [9], we know there exists a unique solution $(\hat{A}, \hat{B})$ in $\mathcal{S}_E$ such that (2) holds.

We now focus our attention on seeking the unique solution $(\hat{A}, \hat{B})$ in $\mathcal{S}_E$. For the real-valued symmetric arrow-head matrices $\tilde{A}$ and $\tilde{B}$, it is easily seen that $\tilde{A}, \tilde{B}$ can be expressed as the linear combinations of the orthonormal basis $\{Y_{ij}\}$, that is,

$$\tilde{A} = \sum_{i,j} \gamma_{ij} Y_{ij},\ \tilde{B} = \sum_{i,j} \delta_{ij} Y_{ij}, \tag{15}$$

where $\gamma_{ij},\ \delta_{ij},\ \begin{cases} i = 1,\ j = 2, \cdots, n; \\ i = j = 1, \cdots, n, \end{cases}$ are uniquely determined by the elements of $\tilde{A}$ and $\tilde{B}$. Let

$$\gamma = [\gamma_{11}, \cdots, \gamma_{n,n}, \gamma_{12}, \cdots, \gamma_{1,n}]^T, \tag{16}$$

$$\delta = [\delta_{11}, \cdots, \delta_{1,n}, \delta_{12}, \cdots, \delta_{1,n}]^T. \tag{17}$$

Then, for any pair of matrices $(A, B) \in \mathcal{S}_E$ in (13), by the relations of (4) and (15) we see that

$$\begin{aligned} f &= \|A - \tilde{A}\|^2 + \|B - \tilde{B}\|^2 \\ &= \|\sum_{i,j}(\alpha_{ij} - \gamma_{ij})Y_{ij}\|^2 + \|\sum_{i,j}(\beta_{ij} - \delta_{ij})Y_{ij}\|^2 \\ &= (\sum_{i,j}(\alpha_{ij} - \gamma_{ij})Y_{ij}, \sum_{i,j}(\alpha_{ij} - \gamma_{ij})Y_{ij}) \\ &+ (\sum_{i,j}(\beta_{ij} - \delta_{ij})Y_{ij}, \sum_{i,j}(\beta_{ij} - \delta_{ij})Y_{ij}) \\ &= \sum_{i,j}(\alpha_{ij} - \gamma_{ij})(Y_{ij}, \sum_{i,j}(\alpha_{ij} - \gamma_{ij})Y_{ij}) \\ &+ \sum_{i,j}(\beta_{ij} - \delta_{ij})(Y_{ij}, \sum_{i,j}(\beta_{ij} - \delta_{ij})Y_{ij}) \\ &= \sum_{i,j}(\alpha_{ij} - \gamma_{ij})^2 + \sum_{i,j}(\beta_{ij} - \delta_{ij})^2 \\ &= \|\alpha - \gamma\|^2 + \|\beta - \delta\|^2. \end{aligned}$$

Substituting (11) and (12) into the relation of f, we have

$$\begin{aligned} f &= \|M^+NWu + F_Mv - \gamma\|^2 + \|Wu - \delta\|^2 \\ &= u^TWN^T(MM^T)^+NWu - 2\gamma^TM^+NWu \\ &- 2\gamma^TF_Mv + v^TF_Mv + \gamma^T\gamma \\ &+ u^TWu - 2u^TW\delta + \delta^T\delta, \end{aligned}$$

where $W = I_d - (E_MN)^+E_MN$. Therefore,

$$\begin{aligned} \frac{\partial f}{\partial u} &= 2WN^T(MM^T)^+NWu \\ &\quad -2WN^T(M^+)^T\gamma + 2Wu - 2W\delta, \\ \frac{\partial f}{\partial v} &= 2F_Mv - 2F_M\gamma. \end{aligned}$$

Clearly, $\|A - \tilde{A}\|^2 + \|B - \tilde{B}\|^2 = \min$ if and only if

$$\frac{\partial f}{\partial u} = 0, \quad \frac{\partial f}{\partial v} = 0$$

which yields

$$Wu = (I_d + WN^T(MM^T)^+NW)^{-1}W(\delta + N^T(M^+)^T\gamma), \tag{18}$$

$$F_Mv = F_M\gamma. \tag{19}$$

Upon substituting (18) and (19) into (11) and (12), we obtain

$$\hat{\alpha} = M^+NW(I_d + WN^T(MM^T)^+NW)^{-1} W(\delta + N^T(M^+)^T\gamma) + F_M\gamma, \tag{20}$$

$$\hat{\beta} = W(I_d + WN^T(MM^T)^+NW)^{-1}W(\delta + N^T(M^+)^T\gamma). \tag{21}$$

By now, we have proved the following result.

Theorem 2: Let the real-valued symmetric arrow-head matrices $\tilde{A}$ and $\tilde{B}$ be given. Then Problem II has a unique solution and the unique solution of Problem II can be expressed as

$$\hat{A} = K(\hat{\alpha} \otimes I_n), \tag{22}$$

$$\hat{B} = K(\hat{\beta} \otimes I_n), \tag{23}$$

where $\hat{\alpha},\ \hat{\beta}$ are given by (20) and (21), respectively.

IV. A NUMERICAL EXAMPLE

Based on Theorem 1 and Theorem 2 we can describe an algorithm for solving Problem IEP and Problem II as follows.

Algorithm 1.

1) Input $\tilde{A},\ \tilde{B},\ \Lambda,\ X$.
2) Form the orthonormal basis $\{Y_{ij}\}$ by (3).
3) Compute $G,\ M,\ N$ according to (7) and (8), respectively.
4) Compute $E_M = I_{np} - MM^+,\ F_M = I_d - M^+M,\ W = I_d - (E_MN)^+E_MN$.
5) Form vectors $\gamma,\ \delta$ by (15), (16) and (17).
6) Compute $K,\ \hat{\alpha},\ \hat{\beta}$ by (14), (20) and (21), respectively.
7) Compute the unique solution $(\hat{A}, \hat{B})$ of Problem II by (22) and (23).

Example 1. Given

$$\tilde{A} = \begin{pmatrix} -4 & 2 & 5 & 1 & 2 & 11 \\ 2 & -3 & 0 & 0 & 0 & 0 \\ 5 & 0 & -6 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 & -4 & 0 \\ 11 & 0 & 0 & 0 & 0 & -44 \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} -7 & 2 & 19 & 9 & 3 & 15 \\ 2 & -13 & 0 & 0 & 0 & 0 \\ 19 & 0 & -8 & 0 & 0 & 0 \\ 9 & 0 & 0 & -6 & 0 & 0 \\ 3 & 0 & 0 & 0 & -3 & 0 \\ 15 & 0 & 0 & 0 & 0 & -28 \end{pmatrix}$$

and

$$\begin{aligned} \Lambda &= \text{diag}\{\lambda_1,\ \lambda_2,\ \lambda_3\} \\ &= \text{diag}\{2.1709,\ 0.84882,\ 0.73245\}, \end{aligned}$$

$$\begin{aligned} X &= [x_1,\ x_2,\ x_3] \\ &= \begin{bmatrix} -0.27362 & 0.019321 & -0.090308 \\ 0.071468 & 0.00087224 & -0.0056621 \\ 0.70165 & 0.085116 & -0.3485 \\ -0.91 & 0.035512 & -0.16064 \\ -0.050465 & -0.91 & -0.39781 \\ -0.030195 & -0.024079 & 0.91 \end{bmatrix}. \end{aligned}$$

Using Algorithm 1, we obtain the unique solution of Problem II as follows.

$$\hat{A} = \left[\begin{matrix} -3.8904 & 1.8001 & 4.7078 \\ 1.8001 & -2.7001 & 0 \\ 4.7078 & 0 & -5.6494 \\ 0.90628 & 0 & 0 \\ 1.8874 & 0 & 0 \\ 10.383 & 0 & 0 \end{matrix}\right. \left.\begin{matrix} 0.90628 & 1.8874 & 10.383 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1.8126 & 0 & 0 \\ 0 & -3.7749 & 0 \\ 0 & 0 & -41.531 \end{matrix}\right],$$

$$\hat{B} = \left[\begin{matrix} -6.6982 & 2.0161 & 20.037 \\ 2.0161 & -13.105 & 0 \\ 20.037 & 0 & -8.4365 \\ 9.1354 & 0 & 0 \\ 3.1709 & 0 & 0 \\ 15.857 & 0 & 0 \end{matrix}\right.$$

$$\begin{bmatrix} 9.1354 & 3.1709 & 15.857 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -6.0902 & 0 & 0 \\ 0 & -3.1709 & 0 \\ 0 & 0 & -29.6 \end{bmatrix},$$

We define the residual as

$$\text{res}(\lambda_i, x_i) = \|(\lambda_i \hat{A} - \hat{B})x_i\|,$$

and the numerical results shown as follows.

(λ_i, x_i)	(λ_1, x_1)	(λ_2, x_2)	(λ_3, x_3)
$\text{Res}(\lambda_i, x_i)$	1.7468e-014	1.0116e-014	3.6636e-014

Furthermore, we can figure out

$$\|\hat{A} - \tilde{A}\| = 2.7319, \quad \|\hat{B} - \tilde{B}\| = 2.57.$$

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