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# Some properties of b-weakly compact operators on Banach lattice

Na Cheng and Zi-li Chen

*Abstract*—We investigate the sufficient condition under which each positive b-weakly compact operator is Dunford-Pettis. We also investigate the necessary condition on which each positive b-weakly compact operator is Dunford-Pettis. Necessary condition on which each positive b-weakly compact operator is weakly compact is also considered. We give the operator that is semi-compact, but it is not b-weakly. We present a necessary and sufficient condition under which each positive semi-compact operator is b-weakly compact.

*Keywords*—b-weakly compact, Dunford-Pettis operator, M-weakly compact operator, L-weakly compact operator, semi-compact operator, weakly sequentially continuous lattice operations, order continuous norm, positive Schur property.



## I. Introduction

RECALL that a subset $A$ of a Riesz space $E$ is called b-order bounded in $E$ if it is order bounded in $(E^{\sim})^{\sim}$. A Riesz space is said to have property (b) if $A \subset E$ is order bounded whenever $A$ is order bounded in $(E^{\sim})^{\sim}$. Note that every perfect Riesz space and therefore every order dual has property (b). Every reflexive Banach lattice has property (b). Every $KB$ space has property (b) and if $(E^{\sim})^{\sim}$ is retractable on $E$ then $E$ has property (b). On the other hand, by considering $A = \{e_n\}$ in $c_0$, we see that $c_0$ does not have property (b). An operator $T : E \to X$, mapping each b-order bounded subset of Banach lattice $E$ into a relatively weakly compact subset of Banach space $X$ is called a b-weakly compact operator. The collection of b-weakly compact operators will be denoted by $W_b(E, F)$. Then $W_b(E, F)$ is a closed subspace of $L(E, F)$, the vector space of all continuous operators from $E$ into $F$. Operators mapping order intervals into relatively weakly compact sets are called o-weakly operators and denoted by $W_o(E, F)$. The collection of weakly compact operators will be denoted by $W(E, F)$. Then $W(E, F) \subseteq W_b(E, F) \subseteq W_o(E, F)$, [9] gave examples to show that these inclusions may be proper.

An operator is said to be a Dunford-Pettis operator if it carries relatively weakly compact subsets onto norm totally bounded subsets. An operator $T$ from a Banach lattice $E$ into a Banach lattice $F$ is said to be M-weakly compact if each disjoint bounded sequence $(x_n)$ of $E$, we have $\lim_n \|T(x_n)\| = 0$. And an operator $T$ from a Banach lattice $E$ into a Banach lattice $F$ is called L-weakly compact if for each disjoint bounded sequence $(y_n)$, in the solid hull of $T(B_E)$, we have $\lim_n \|y_n\| = 0$ where $B_E$ is the closed unit ball of $E$.

In 2003, S.Alpay and B.Altin [9] studied the property (b). They proved that Banach lattice $E$ is a KB-space if and only if it has order continuous norm and property (b) [9, Theorem 2.1]. They also gave the definition of b-weakly compact operator. They characterized that $T : E \to X$ is b-weakly compact operator if and only if for each b-order bounded $A \subset E$ and disjoint sequence $(x_n)$ in $A$ satisfies $\lim_n \|T(x_n)\|$=0 [9, Theorem 2.8]. In 2006, S.Alpay and B.Altin [10] investigate Riesz spaces and Banach lattices enjoying property (b). They proved that if Banach $F$ is Dedekind complete, then the space of order bounded operators from Banach $E$ into $F$ has property (b) if and only if $F$ has property (b) [10, Theorem 2]. Every order closed Riesz subspace of a Dedekind complete Riesz space $E$ with property (b) has property (b) [10, Theorem 2]. In 2007, S.Alpay and B.Altin [11] characterized the b-weak compactness of $T$ in terms of its mapping properties [11, Theorem 1, Theorem 2, Theorem 4]. In 2007, B.Altin [13] investigated the order structure of b-weakly compact operator. In 2009, S.Alpay and B.Altin [12] gave characterized of KB-spaces in terms of b-weakly compact operators. A Banach lattice $F$ is KB-space if and only if for each Banach lattice $E$ and positive disjointness preserving operator $T : E \to F$ is b-weakly compact. In 2009, B. Aqzzouz and A. Elbou, and J. Hmichane [14] establish necessary and sufficient conditions under which b-weakly compact operators between Banach lattices have b-weakly compact adjoint or operators with b-weakly compact adjoint are themselves b-weakly compact. $T : E \to F$ between Banach lattices is a b-weakly compact operator, then its adjoint $T' : F' \to E'$ is b-weakly compact if and only if $F'$ or $E'$ is a KB-space. Each operator $T : E \to F$ is b-weakly compact whenever its adjoint $T' : F' \to E'$ is b-weakly compact if and only if $E$ or $F$ is a KB-space.

To state our results, we need to fix some notation and recall some definitions. A Banach lattice is an AM-space if and only if the norm is additive on the positive cone of the dual. An element $e > 0$ in a Riesz space is said to be an order unit whenever for each $x$ there exists some $\lambda > 0$ with $|x| \leq \lambda e$. Now if a Banach lattice $E$ has an order unit $e > 0$, then $A_e = E$ holds, and the norm $\|x\|_\infty = \inf\{\lambda > 0 : |x| \leq \lambda e\}$ is equivalent to the original norm of $E$. In other words, if a Banach lattice $E$ has an order unit $e$, then $E$ can be renormed in such a way that it becomes an AM-space having $[-e, e]$ as its closed unit ball. A Banach lattice has order continuous norm if and only if every order bounded disjoint sequence id norm convergent to zero. A Banach lattice $E$ is said to be a KB-space, whenever every increasing norm bounded sequence of $E^+$ is norm convergent. For example, each reflexive Banach lattice is KB-space. Also, each KB-space has an order continuous norm, but a Banach lattice with an order continuous norm is not necessary a KB-

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space. In fact, the Banach lattice $c_0$ has an order continuous norm but it is not a KB-space. However, if $E$ is a Banach lattice, the topological dual $E'$ is a KB-space if and only if its norm is order continuous. The Banach lattice $E$ has the positive Schur property if each weakly null sequence with positive sequence in $E$ converges to zero in norm. A Banach lattice $E$ is said to have weakly sequentially continuous lattice operations whenever $x_n \xrightarrow{w} 0$ in $E$ implies $|x_n| \xrightarrow{w} 0$ in $E$. In an AM-space the lattice operations are weakly sequentially continuous. Also, every Banach lattice with the Schur property (i.e., $x_n \xrightarrow{w} 0$ implies $\|x_n\| \to 0$) has weakly sequentially continuous lattice operations. Thus, for example, the Banach lattice $C[0,1]$, $l_1$, $l_1 \oplus C[0,1]$ all have weakly sequentially continuous lattice operations.

The goal of this paper is to investigate the sufficient condition under which each positive b-weakly compact operator is Dunford-Pettis. We also investigate the necessary condition on which each positive b-weakly compact operator is Dunford-Pettis. Necessary condition on which each positive b-weakly compact operator is weakly compact is also considered. We give the operator that is semi-compact, but it is not b-weakly. We present a necessary and sufficient condition under which each positive semi-compact operator is b-weakly compact.

All notions concerning Banach lattices and not explained here are can find in [1] and [2].

## II. Properties of b-weakly compact operators

**Theorem 1:** For Banach lattice $F$, each positive b-weakly compact operator from AM-space into $F$ is Dunford-Pettis.

**Proof:** Let $\rho(x) = \|Tx\|$ for every $x \in E$, then $\rho$ is a continuous lattice seminorm on $E$. Suppose $T : E \to F$ is not a Dunford-Pettis operator, since AM-space has weakly sequentially continuous lattice operators, there exists a sequence $\{x_n\} \subset E_+$ with $x_n \xrightarrow{w} 0$, and $\|Tx_n\| \geq 1$.

Corollary 2.3.5 of [2] shows that for every $0 < c < 1$, there exists a subsequence $(k(n))_{n=1}^{\infty} \subset N$ and a disjoint sequence $\{y_n\} \subset E_+$ such that

$$y_n \leq x_{k(n)}, \|Ty_n\| \geq c$$

for all $n \in N$. Since $y_n \leq x_{k(n)}$ and $x_n \xrightarrow{w} 0$, the uniform boundness theorem implies that the sequence $y_n$ is bounded.

Observing that $(y_1 + \cdots + y_n)_1^{\infty}$ is a monotone norm bounded sequence, there exists $x'' \in E''_+$ such that

$$0 \leq y_1 + \cdots + y_n \leq x''$$

together with the fact that $T$ is b-weakly compact, it follows that

$$\|Ty_n\| \to 0 (n \to \infty)$$

This gives a contradiction. □

**Theorem 2:** Let $E$ and $F$ be two Banach lattices, if every positive b-weakly compact operator $T : E \to F$ is Dunford-Pettis, then the norm of $F$ is order continuous or the lattice operations of $E$ are weakly sequentially continuous.

**Proof:** If the norm of $F$ is not order continuous and the lattice operations of $E$ are not weakly sequentially continuous, A.W.Wickstead constructed in the proof of Theorem 2 of [4] two positive operators $S, T : E \to F$ such that $0 \leq S \leq T$ and $T$ is compact and hence it is b-weakly compact, Proposition 2.2 of [6] implies $S$ is b-weakly compact, but it is not Dunford-Pettis. □

**Theorem 3:** Let $E$ and $F$ be two Banach lattices, if every positive b-weakly compact operator $T : E \to F$ is weakly compact, then one of the following statements is valid:

(a) The norm of the topological dual $E'$ is order continuous.

(b)$F$ is reflexive.

**Proof:** Suppose that neither the norm of $E'$ is order continuous nor $F$ is reflexive, then there exist a sublattice $H$ of $E$ which is isomorphic to $l_1$ and a positive projection $P : E \to l_1$.

On the other hand, since the closed unit ball $B_F$ of $F$ is not weakly compact, there exists a sequence $(e_n)$ in $B_F$ which does not have any weakly convergent subsequence.

Consider the operator $S : l_1 \to F$ defined by

$$S(x_n) = \Sigma_{n=1}^{\infty} x_n e_n$$

It is easy to see that $S \cdot P$ is o-weakly compact, since $l_1$ is a KB-space, it is b-weakly compact, but it is not weakly compact. □

**Theorem 4:** Let $E$ and $F$ be two Banach lattices, if each positive o-weakly compact operator $T : E \to F$ is L-weakly compact, then one of the following conditions holds.

(a) $F$ are KB-spaces.

(b) $E'$ has the positive Schur property.

**Proof:** Suppose $F$ is not a KB-space, Theorem 2.4.12 of [4] implies that $F$ contains a sublattice isomorphism to $c_0$. Applying Theorem 3.1 of [3] it suffices to show each disjoint weak null sequence $(x'_n)_1^{\infty} \subset E'_+$ is norm convergent to 0.

For each $x \in E$ define $T : E \to c_0$ by

$$Tx = (x'_n(x))_1^{\infty}$$

Theorem 17.5 of [1] implies $T$ is a weakly compact operator, hence it is o-weakly compact, it is L-weakly compact. Theorem 18.13 of [1] implies

$$T' : l_1 \to E'$$

is M-weakly compact. As

$$T'(e_n) = x'_n$$

for all $n \in N$, where $e_n$ is the sequence with n'th entry equals to 1 and all others are zero, we conclude that

$$\|x'_n\| \to 0 (n \to \infty)$$

□

Recall that A continuous operator $T : E \to F$ is said to be semi-compact if for each $\epsilon > 0$, there exists some $u \in F^+$ such that $T(U) \subset [-u, u] + \epsilon V$ where $U$, $V$ denote the closed unit balls of $E$ and $F$, respectively. Each compact operator, M-weakly compact (L-weakly compact) operator between Banach lattice is semi-compact. However, a semi-compact operator need not be compact, weakly compact, M-weakly compact (L-weakly compact). For instance, the identity operator $I : \ell_\infty \to \ell_\infty$ is semi-compact, but it does not have any one of the above mentioned compactness properties.

**Theorem 5:** Let $E$ and $F$ be nonzero Banach lattices such that $F$ is $\sigma$-Dedekind complete. Then the following statements are equivalent.

1) Each positive semi-compact operator $T : E \to F$ is b-weakly compact.

2) At least one of the following conditions holds:

a) The norm of $E$ is order continuous.

b) The norm of $F$ is order continuous.

**Proof:** $2) - a) \Rightarrow 1)$ Suppose that $E$ has order continuous norm and $T : E \to F$ is a positive semi-compact operator. Theorem 12.9 of [1] implies that each order interval of Banach lattice $E$ is weakly compact, together with the fact that $T$ is a positive semi-compact operator, it follows that $T$ is weakly compact. Hence, $T$ is b-weakly compact.

$2) - b) \Rightarrow 1)$ Suppose that $F$ has order continuous norm and $T : E \to F$ is a positive semi-compact operator. For each $\epsilon > 0$ there exists some $u \in F^+$ such that

$$T(U) \subseteq [-u, u] + \epsilon V$$

$U$ and $V$ denote the closed unit balls of $E$ and $F$, respectively. Theorem 12.9 of [1] implies that the order interval $[-u, u]$ in $F$ is weakly compact, combined with Theorem 10.17 of [1] show that $T(U)$ is relatively weakly compact, it follows that $T$ is weakly compact. Hence, $T$ is b-weakly compact.

$1) \Rightarrow 2)$ Assume by way of contradiction that neither $E$ nor $F$ has an order continuous norm. To finish the proof, we have to construct a positive semi-compact operator $T : E \to F$ which is not b-weakly compact.

Since the norm on $E$ is not order continuous, applying Theorem 12.13 of [1] that there exists some $x \in E^+$ and a sequence $(x_n) \subset [0, y]$ which does not converge to zero in norm. We may assume that $\|x_n\| = 1$ for all $n$.

Hence, by lemma 2.1 of [15] there exists a positive disjoint sequence $(g_n)$ of $E'$ with $\|g_n\| \leq 1$ such that

$g_n(x_n) = 1$ for all $n$ and $g_n(x_m) = 0$ for $n \neq m$.

For all $x \in E$, define the positive operator $R : E \to \ell_\infty$ by

$$R(x) = (g_1(x), g_2(x), \cdots)$$

Note that $R(B_E) \subset B_{\ell_\infty}$.

On the other hand, as the norm on $F$ is not order continuous, applying Theorem 12.13 of [1] that there exists some $y \in F^+$ and a sequence $(y_n) \subset [0, y]$ which does not converge to zero in norm. We may assume that $\|y_n\| = 1$ for all $n$.

Since $\sum_{i=1}^{n} y_i \leq y$ holds for all $n$, and $F$ is $\sigma$-Dedekind complete, for all $(\alpha_1, \alpha_2, \cdots) \in \ell_\infty$, define the positive operator $S : \ell_\infty \to F$ by

$$S(\alpha_1, \alpha_2, \cdots) = \lim \sum_{i=1}^{n} \alpha_i y_i$$

Defines a lattice isomorphism from $\ell_\infty$ into $F$ where $\lim \sum_{i=1}^{n} \alpha_i y_i$ denotes the order limit of the partial sum $\sum_{i=1}^{n} \alpha_i y_i$.

Since the sequence $(y_n)$ is order bounded and disjoint, for each $(\alpha_1, \alpha_2, \cdots) \in B_{\ell_\infty}$, we see that

$$|S(\alpha_1, \alpha_2, \cdots)| = \lim \sum_{i=1}^{n} |\alpha_i| y_i \leq (\sup |\alpha_i|) \cdot y \leq y$$

Hence $S(\alpha_1, \alpha_2, \cdots) \in [-y, y]$, and we have $S(B_{\ell_\infty}) \subset [-y, y]$.

Now consider the operator $T = S \circ R : E \to F$ by

$$T(x) = \lim \sum_{i=1}^{n} g_i(x) y_i$$

it is positive, and we have

$$T(B_E) = S(R(B_E)) \subset S(B_{\ell_\infty}) \subset [-y, y]$$

It is clear that $T$ is semi-compact.

On the other hand, for all $n$, we have

$$T(x_n) = \lim \sum_{i=1}^{n} g_i(x_n) y_i = y_n$$

It follows that $\|T(x_n)\| = \|u_n\| = 1$. As the sequence $(x_n)$ is order bounded and disjoint in $E$, it is clear that $T$ is not order weakly compact. Hence, $T$ is not b-weakly compact. $\square$

**Theorem 6:** Let $E$ and $F$ be nonzero Banach lattices. Then the following statements are equivalent.

1) Each positive semi-compact operator $T' : F' \to E'$ is b-weakly compact.

2) At least one of the following conditions holds:

a) The norm of $E'$ is order continuous.

b) The norm of $F'$ is order continuous.

**Proof:** $1) \Rightarrow 2)$ Assume by way of contradiction that neither $E'$ nor $F'$ has an order continuous norm. To finish the proof, we have to construct a positive semi-compact operator $T' : F' \to E'$ which is not b-weakly compact.

Since the norm on $E'$ is not order continuous, applying Theorem 2.6 of [15] that there exists a disjoint sequence $\{x_n\} \subset E^+$ with $\|x_n\| \leq 1$ for all $n$ and there exists some $0 \leq x' \in E'$ with $x'(x_n) = 1$ for all $n$. Moreover, the components $x'_n$ of $x'$, in the carrier $C_{x_n}$ from an order bounded disjoint sequence in $(E')^+$ such that

$x'_n(x_n) = x'(x_n) = 1$ for all $n$ and $x'_n(x_m) = 0$ for $n \neq m$.

Note that $0 \leq x'_n \leq x'$ holds for all $n$.

For all $x \in E$, define the positive operator $R : E \to \ell_1$ by

$$R(x) = (x'_n(x))_{n=1}^{\infty}$$

Since $\sum_{i=1}^{\infty} |x'_n(x)| \leq \sum_{i=1}^{\infty} x'_n(|x|) \leq x'(|x|)$ holds for each $x \in E$, the operator $R$ is well defined.

On the other hand, as the norm on $F'$ is not order continuous, applying Theorem 12.13 of [1] that there exists some $f' \in F'_+$ and a disjoint sequence $(f'_n) \subset [0, f']$ which does not converge to zero in norm. We may assume that $\|f'_n\| = 1$ for all $n$. Hence, for each $n$, we can choose $f_n \in F_+$ with $\|f_n\| = 1$ and $f'_n(f_n) \geq \frac{1}{2}\|f_n\| = \frac{1}{2}$.

For all $(\lambda_n) \in \ell_\infty$ consider the positive operator $S : \ell_\infty \to F$ defined by

$$S(\lambda_n) = \sum_{n=1}^{\infty} \lambda_n f_n$$

Since $(\lambda_n) \in \ell_\infty$ and $\sum_{n=1}^{\infty} \|\lambda_n f_n\| = \sum_{n=1}^{\infty} |\lambda_n|$, it follows that $S$ is well defined.

Now, for all $x \in E$, consider the operator $T = S \circ R : E \to F$ defined by

$$T(x) = \sum_{n=1}^{\infty} x'_n(x) f_n$$

Its adjoint $T' : F' \to E'$ defined by

$$T'(g') = \sum_{n=1}^{\infty} g'_n(f_n) x'_n$$

for all $g' \in F'$. Since $\ell_\infty$ is an AM-space with unit, it follows that $R'$ is semi-compact, hence $T'$ is semi-compact.

On the other hand, note that the sequence $f'_n$ is order bounded and disjoint, and

$$\|T'(f'_n)\| = \| \sum_{i=1}^{\infty} \|f'_n(f_n)x'_i\|$$

$$\geq \|f'_n(f_n)x'_n\| \geq \frac{1}{2}\|x'_n\|$$

$$\geq \frac{1}{2}x'_n(x_n) \geq \frac{1}{2}$$

Hence, $T'$ is not o-weakly compact, it is not b-weakly compact. $\square$

## III. Conclusions

In this paper, we investigate the sufficient condition under which each positive b-weakly compact operator is Dunford-Pettis. We also investigate the necessary condition on which each positive b-weakly compact operator is Dunford-Pettis. Necessary condition on which each positive b-weakly compact operator is weakly compact is also considered. We give the operator that is semi-compact, but it is not b-weakly. We present a necessary and sufficient condition under which each positive semi-compact operator is b-weakly compact.

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