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An Extension of the Kratzel Function and Associated Inverse Gaussian Probability Distribution Occurring in Reliability Theory

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***Abstract*—**In view of their importance and usefulness in reliability theory and probability distributions, several generalizations of the inverse Gaussian distribution and the Krätzel function are investigated in recent years. This has motivated the authors to introduce and study a new generalization of the inverse Gaussian distribution and the Krätzel function associated with a product of a Bessel function of the third kind $K_\nu(z)$ and a ω - Fox-Wright generalized hyper geometric function introduced in this paper. The introduced function turns out to be a unified gamma-type function. Its incomplete forms are also discussed. Several properties of this gamma-type function are obtained. By means of this generalized function, we introduce a generalization of inverse Gaussian distribution, which is useful in reliability analysis, diffusion processes, and radio techniques etc. The inverse Gaussian distribution thus introduced also provides a generalization of the Krätzel function. Some basic statistical functions associated with this probability density function, such as moments, the Mellin transform, the moment generating function, the hazard rate function, and the mean residue life function are also obtained.

***Keywords*—**Fox-Wright function, Inverse Gaussian distribution, Krätzel function & Bessel function of the third kind.

I. INTRODUCTION

THE Krätzel function $Z_\rho^\nu(x)$ is defined by

$$Z_\rho^\nu(x) = \int_0^\infty u^{\nu-1}\exp(-xu^\rho - \frac{1}{u})du\,,$$

$$\rho > 0, \nu \in C, x > 0 \qquad (1)$$

In particular, when $\rho = 1$ and $x = t^2/4$, then by virtue of [9], it gives

$$Z_1^\nu\left(\frac{t^2}{4}\right) = 2\left(\frac{t}{2}\right)^\nu K_\nu(t)$$

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where $K_\nu(t)$ is the Bessel function of the third kind or Macdonald function, as in [9].

For $\rho \geq 1$ the function (1) was introduced by [20] as a kernel of the integral transform

$$(K_\nu^\rho f)(x) = \int_0^\infty Z_\rho^\nu(xt)f(t)dt\ ;\ (x>0)$$

which is called by his name as the Krätzel function, as in [19], established asymptotic behavior of the function (1) for $\rho \geq 1$ together with the composition with a special differential operator. Reference [20] also defined a Bessel-type transform and obtained its properties by using the Mellin transform. This function is recently extended by [17] from $x>0$ to complex $z \in C$, and its representations in terms of the well-known H-function are established. The results obtained being different for $\rho > 0$ and $\rho < 0$ are applied to derive explicit forms of Krätzel function in terms of the Fox-Wright generalized hypergeometric function.

Note 1: We note that the integral (1) occurs in the study of astrophysical thermonuclear functions, which are derived on the basis of Boltzmann- Gibbs statistical mechanics. This integral has been evaluated by [21] by applying the statistical techniques. Recently a short and straightforward analytic proof of this integral is given by [26].

Definition of $\omega-$ *Fox-Wright generalized hypergeometric function:* We define the $\omega-$Fox-Wright generalized hypergeometric function by means of the Mellin-Barnes type integral in the form

$${}_pR_q(z) = {}_pR_q(a, a_2, ..., a_p, b_1, ..., b_q; \omega; z)$$

$$= \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\Gamma(a)\prod_{i=2}^{p}\Gamma(a_i)}\frac{1}{2\pi i}\int_L \frac{\Gamma(-s)\Gamma(a+s)\prod_{i=2}^{p}\Gamma(a_i+\omega s)}{\prod_{j=1}^{q}\Gamma(b_j+\omega s)}(-z)^s ds \qquad (2)$$

where $|\arg(-z)| < \pi$ $(i = (-1)^{1/2})$ and the poles of the integrand of (2) are assumed to be simple. The contour L is one of the contours $L= L_{-\infty}, L = L_{+\infty}$ and L= $L_{i\gamma\infty}$.These contours are explicitly defined in the monographs by [23], and [16]. We give the conditions for the representation of $\omega-$ Fox-Wright generalized hypergeometric function ${}_pR_q(z)$ to be represented by a Mellin – Barnes integral of the form (2). It may be noted that different conditions are obtained for the above three contours. Here we give conditions for the contour $L = L_{-\infty}$. For remaining contours the conditions of representation of the function ${}_pR_q(z)$ can be obtained from the conditions given by [18]. The result is given below in the form of a theorem.

Theorem1:

Let $a, a_i, b_j \in C, A_i, B_j \in R (i = 2,..., p; j = 1,..., q)$ be such that the conditions

$$\frac{a_i + k}{A_i} \neq -\nu, (i = 2,..., p; a_1 = a, A_1 = 1; k, \nu \in N_0) \quad (3)$$

and

$(a_i + k)A_j \neq (a_j + \nu)A_i$, $(i \neq j; i, j = 1,...., p, a_1 = a, A_1 = 1)$

are satisfied. Let $\mu = \omega(p - q + 1) - 1$ and either of the following conditions hold:

$\mu > -1, \quad z \neq 0; \mu = 1, \quad 0 < |z| < \beta;$

$\mu = -1, |z| = \beta, \mathrm{Re}(\delta) > 1/2$

Then the $\omega-$Fox-Wright generalized hypergeometric function has the Mellin-Barnes integral representation given by (2), where the path of integration $L = L_{-\infty}$ separates all the poles $s = \nu (\nu \in N_0)$ to the left and all the poles given by

$s = \frac{a_i + \nu}{\omega}$ where,

$\nu \in N_{0;}\ \omega \neq 0; a_1 = a$ with $\omega = 1; i = 1,..., p$ to the right.

The theorem readily follows from the result given in [18].

It is interesting to observe that for $p = 2$ and $q = 1$, the function ${}_pR_q(z)$ defined by (2) reduces to Dotsenko function, as in [7], and [8], ${}_2R_1(a, b; c; \omega; z)$, defined by

$$_2R_1(a,b;c;\omega;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k\omega) z^k}{\Gamma(c+k\omega)(k)!} \quad (4)$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} {}_2\Psi_1 \left[\begin{matrix} (a,1),(b,\omega) \\ (c,\omega) \end{matrix} \Big| z \right]$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(-s)\Gamma(a+s)\Gamma(b+\omega s)(-z)^s\, ds}{\Gamma(c+\omega s)} \quad (5)$$

where $|\arg(-z)| < \pi$ and the poles of the integrand of (5) are assumed to be simple . Here ${}_2\psi_1(.)$ is a special case of the Fox-Wright generalized hypergeometric function defined by [22], [29], and [30]. It may be noted that the result (4) can be obtained from (5) by calculating the residues at the poles of $\Gamma(-s)$ at the points given by $s = \nu \in N_0$. Reference [8] also obtained the inversion formula for the integral transform and the exact solution of a Fredholm integral equation of the first kind involving such a function in the kernel. It is interesting to observe that for $\omega = 1$, (4) reduces to a Gauss hypergeometric function ${}_2F_1(a, b; c; z)$. When z is replaced by z/a and a tends to infinity, then (4) yields its confluent form as

$$_1R_1(b;c;\omega,\mu;z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(b+\frac{\omega k}{\mu}) z^k}{\Gamma(c+\frac{\omega k}{\mu})(k)!}$$

where $b, c \in C$;

$\mathrm{Re}(c) > \mathrm{Re}(b) > 0, \omega \in R^+;\ b + \frac{\omega k}{\mu}, c + \frac{\omega k}{\mu} \neq 0, -1, -2, ..$

A generalization of the Krätzel function is introduced and studied by [3]. In the same paper, a generalized inverse Gaussian distribution is also defined and its various statistical properties are investigated. A generalization of Kratzel function and associated probability distributions are studied by [25]. In a recent paper, [24] has introduced a generalization of the Krätzel function and inverse Gaussian distribution in the form

$$S_{\omega,\rho}^{a,b,c;\lambda,\nu,p}(z) = \left(\frac{2p}{\pi}\right)^{1/2} \int_0^{\infty} t^{\lambda-1} K_\nu(pt) dt \times {}_2R_1(a,b,c;\omega; -\frac{z}{t^\rho}) d$$

where $z, \omega, \rho > 0, \mathrm{Re}(p) > 0$

$$\mathrm{Re}(\lambda + |\nu| + \frac{a\rho}{\omega}) > 0, \mathrm{Re}(\lambda + |\nu| + \frac{b\rho}{\omega}) > 0; \omega \neq 0$$

The object of this paper is to consider a further generalization of the Krätzel function and inverse Gaussian distribution by using the product of ω-generalized Fox-Wright hypergeometric function introduced in the next section and the well known Bessel function of the third kind $K_\nu(x)$ in [1]. This function turns out to be a generalized gamma-type function. Corresponding incomplete functions are introduced and some of their properties are investigated. These functions are employed to define and study a new probability density function. Special cases of the parameters in the result (14) give rise to certain well-known densities. Some statistical properties are also derived. A study of this new function defined by (7) in the next section will give deeper, general and useful results in the theory of special functions, integral transforms and probability distributions, which are useful in reliability theory and diffusion problems.

The following generalized inverse Gaussian distribution is due to [11].

$$f(t)=A(\alpha,a,b)t^{a-1}\exp(-at-\frac{b}{t})\ ,\ a,\ b,\ t>0,\quad -\infty<\alpha<\infty \tag{6}$$

where, $A(\alpha;a,b)=\left[\int_0^\infty t^{\alpha-1}\exp(-at-\frac{b}{t})dt\right]^{-1}$

The inverse Gaussian distribution arises as the density of the first passage time of the Brownian motion with positive drift. Such models are used in reliability theory, theory of demographic rates, as in [13], [14]. Applications of the distribution defined by (6) are discussed by [14] in several applied problems, such as fractures of air-conditioning equipment and traffic data etc. A comprehensive account of the inverse Gaussian distribution with applications can be found in [6].

II. A UNIFIED GAMMA-TYPE (KRATZEL) FUNCTION

Definition 1: A unified gamma-type function is defined as

$$S_{\omega,\rho}^{a,a_2,...,a_p;b_1,...,b_q;\omega;\lambda,\nu,p}(z)=\left(\frac{2p}{\pi}\right)^{1/2}\times$$

$$\int_0^\infty t^{\lambda-1}K_\nu(pt)\ {}_pR_q(a,a_2,...,a_p;b_1,.../,b_q;\omega;-\frac{z}{t^\rho})dt \tag{7}$$

where, $z,\omega,\rho>0,\mathrm{Re}(p)>0$

$$\mathrm{Re}(\lambda+|\nu|+\frac{a\rho}{\omega})>0,$$

$$\mathrm{Re}(\lambda+|\nu|+\frac{\rho a_i}{\omega})>0(i=2,...,p);\omega\neq 0.$$

When p = 2 and q = 1, the above result reduces to the following one given by [25]

$$S_{\omega,\rho}^{a,b,c;\lambda,\nu,p}(z)=\left(\frac{2p}{\pi}\right)^{1/2}\times$$

$$\int_0^\infty t^{\lambda-1}K_\nu(pt)\ {}_2R_1(a,b,c;\omega;-\frac{z}{t^\rho})dt \tag{8}$$

where, $z,\omega,\rho>0,\mathrm{Re}(p)>0$

$\mathrm{Re}(\lambda+|\nu|+\frac{a\rho}{\omega})>0,\mathrm{Re}(\lambda+|\nu|+\frac{b\rho}{\omega})>0;\omega\neq 0.$ When z is replaced by z/a and a tends to infinity in (8), then we arrive at the following result associated with ω-confluent hypergeometric function

$$S_{\omega,\rho}^{b,c;\lambda,\nu,p}(z)=\left(\frac{2p}{\pi}\right)^{1/2}\times$$

$$\int_0^\infty t^{\lambda-1}K_\nu(pt)\ {}_1R_1(b,c;\omega;-\frac{z}{t^\rho})dt \tag{9}$$

where, $z,\omega,\rho>0,\mathrm{Re}(p)>0,\omega\neq 0$;

$b+\omega k,c+\omega k\neq 0,-1,-2,....$

If we further set $\nu=1/2,\lambda$ is replaced by $\lambda+1/2,$ then by virtue of the identity

$$K_{\pm 1/2}(x)=\left(\frac{\pi}{2x}\right)^{1/2}\exp(-x),$$

We obtain the generalized gamma-type function recently studied by [3] in the following form:

$$S_{\omega,\rho}^{b,c;\lambda+1/2,1/2,p}(z)=S_{\omega,\rho}^{b,c;\lambda+1/2,p}(z)=B_{\omega,\rho}^{b,c;\lambda,p}(2\sqrt{z})$$

$$=\int_0^\infty t^{\lambda-1}\exp(-pt)_1R_1(b,c;\omega;-\frac{z}{t^\rho})dt\ , \tag{10}$$

where $z,\omega,\rho>0,\mathrm{Re}(p)>0,$

$\mathrm{Re}(\lambda+\frac{b\rho}{\omega})>0;\omega\neq 0;c+\omega k,b+\omega k\neq 0,-1,-2,...$ and

$B_{\omega,\rho}^{b.c;\lambda,p}(.)$ is the notation for generalized gamma-type function defined by [3].

Next, if we set $b=c,\omega=p=1,$ then (10) reduces to the Krätzel function in the notation $\eta(\rho,1+\lambda;z)$ for the Krätzel function employed by [3] as:

$$Z_\rho^\lambda(z)=\eta(\rho,1+\lambda;z)=\int_0^\infty t^{-\lambda-1}\exp(-t-zt^{-\rho})dt$$

where, $\mathrm{Re}(\rho)>0,\mathrm{Re}(z)>0.$

Special cases of the generalized function (7):

A. If we set $\nu=1/2,$ we obtain the generalized gamma function $S_{\omega,\rho}^{a,a_2,...,a_pb_1,...,b_q;\lambda,1/2,p}(z)=$

$$\int_0^\infty t^{\lambda-1}\exp(-pt)\ {}_pR_q(a,a_2,...,a_p;b_1,...,b_q;\omega;-\frac{z}{t^\rho})dt \tag{11}$$

which for p = 2 and q = 1 gives

$$S_{\omega,\rho}^{a,b,c;\lambda,1/2,p}(z)=\int_0^\infty t^{\lambda-1}\exp(-pt)\ {}_2R_1(a,b,c;\omega;-\frac{z}{t^\rho})dt \tag{12}$$

where Re(p)>0, Re(z)>0,

$\mathrm{Re}(\lambda-a\rho)>0,\mathrm{Re}(\lambda-b\rho)>0,c\neq 0,-1,-2,....$

If we take $p=q=\rho=1$, $a=b,b_1=c$ in (11), then we obtain the gamma-type function studied by [2] :

$$S_{\omega,1}^{b,c;\lambda+1/2,1/2,p}(z)={}_\omega\Gamma_z(b,c;\lambda,p)=$$

$$\int_0^\infty t^{\lambda-1}\exp(-pt)_1R_1(b,c;\omega;-\frac{z}{t})dt \tag{13}$$

Note 2: The result (12) also gives the Laplace transform of the Dotsenko function defined by (3).

B. When b or z tends to zero in (13), it reduces to a well-known gamma function. Further, if we set $\rho=\omega=1,c=b,\nu=1/2,\lambda$ is replaced by $\lambda+1/2$ then (9) yields a known result given in [10], pp.146 (2):

$$S_{1,1}^{b,b;\lambda+1/2,1/2,p}(z)=\int_0^\infty t^{\lambda-1}\exp(-pt)\exp(-\frac{z}{t})dt$$

$$= 2\left(\frac{z}{4p}\right)^{\lambda/2} K_\nu (z^{1/2} p^{1/2})$$

where Re(p)>0, Re(z)>0

DERIVATIVES OF THE S-FUNCTION: As a consequence of the definition of the S-function defined by (7) and differential properties of $\omega -$Fox-Wright generalized hypergeometric function defined by (2), the following results are easily derived.

A. $$\frac{d}{dz}\left[z^\sigma S_{\omega,\rho}^{a,a_2,\ldots,a_p;b_1,\ldots,b_q;\lambda,\nu,p}(z)\right]$$

$$= \sigma z^{\sigma-1} S_{\omega,\rho}^{a,a_2,\ldots,a_p;\lambda,\nu,p}(z)$$

$$- \frac{a\prod_{j=1}^{q}\Gamma(b_j)\prod_{i=2}^{p}\Gamma(a_i+\omega)}{\prod_{i=2}^{p}\Gamma(a_i)\prod_{j=1}^{q}\Gamma(b_j+\omega)} z^\sigma \times$$

$$S_{\omega,\rho}^{a+1,a_2+\omega,\ldots,a_p+\omega;b_1+\omega,\ldots,b_q+\omega;\lambda-\rho,\nu,p}(z)$$

B. $$\frac{d}{dz}\left[z^{-\sigma} S_{\omega,\rho}^{a,a_2,\ldots,a_p;b_1,\ldots,b_q;\lambda,\nu,p}(z)\right]$$

$$= -\sigma z^{-\sigma-1} S_{\omega,\rho}^{a,a_2,\ldots,a_p;\lambda,\nu,p}(z)$$

$$- \frac{a\prod_{j=1}^{q}\Gamma(b_j)\prod_{i=2}^{p}\Gamma(a_i+\omega)}{\prod_{i=2}^{p}\Gamma(a_i)\prod_{j=1}^{q}\Gamma(b_j+\omega)} z^{-\sigma} \times$$

$$S_{\omega,\rho}^{a+1,a_2+\omega,\ldots,a_p+\omega;b_1+\omega,\ldots,b_q+\omega;\lambda-\rho,\nu,p}(z)$$

C. $$z\frac{d}{dz}\left[S_{\omega,\rho}^{a,a_2,\ldots,a_p;b_1,\ldots,b_q;\lambda,\nu,p}(z)\right]$$

$$= \left[\begin{array}{l} S_{\omega,\rho}^{a+1,a_2,\ldots,a_p;b_1,\ldots,b_q;\lambda,\nu,p}(z) - \\ S_{\omega,\rho}^{a,a_2,\ldots,a_p;b_1,\ldots,b_q;\lambda,\nu,p}(z) \end{array}\right]$$

D. $$z\frac{d}{dz}\left[S_{\omega,\rho}^{a,a_2,\ldots,a_p;b_1,\ldots,b_q;\lambda,\nu,p}(z)\right]$$

$$= -az \frac{\prod_{j=1}^{q}\Gamma(b_j)\prod_{i=2}^{p}\Gamma(a_i+\omega)}{\prod_{j=1}^{q}\Gamma(b_j+\omega)} \times \qquad (14)$$

$$S_{\omega,\rho}^{a+1,a_2,\ldots,a_p,b_1,\ldots,b_q;\lambda-\rho,\nu,p}(z)$$

III. ASSOCIATED FAMILIES OF INCOMPLETE S-FUNCTIONS

Definition 2: A generalized incomplete gamma function corresponding to the S-function investigated here is defined by

$$S_0^x\left[\begin{array}{l} a,a_2,\ldots,a_p;b_1,\ldots,b_q;\lambda,\nu,p \\ \omega,\rho \end{array}(z,x)\right]$$

$$= \left(\frac{2p}{\pi}\right)^{1/2} \int_0^x t^{\lambda-1} K_\nu(pt) \times$$

$$ {}_pR_q(a,a_2,\ldots,a_p;b_1,\ldots,b_q;\omega;-\frac{z}{t^\rho})dt \qquad (15)$$

where, $z,\omega,\rho > 0, \mathrm{Re}(p) > 0$,

$$\mathrm{Re}(\lambda + |\nu| + \frac{a\rho}{\omega}) > 0,$$

and

$$\mathrm{Re}(\lambda + |\nu| + \min_{2\le i\le p} \frac{a_i\rho}{\omega}) > 0; \omega \ne 0. c \ne 0,-1,-2,.....$$

$z \ge 0$. When p = 2 and q = 1, the (15) reduces to one involving Dotsenko function defined by (5) as:

$$C_0^x\left[{}_{\omega,\rho}^{a,b,c;\lambda,\nu,p}(z,x)\right] = \left(\frac{2p}{\pi}\right)^{1/2} \int_0^x t^{\lambda-1} K_\nu(pt) \times \qquad (16)$$

$$ {}_2R_1(a,b,c;\omega;-\frac{z}{t^\rho})dt$$

where

$z,\omega,\rho > 0, \mathrm{Re}(p) > 0$

$$\mathrm{Re}(\lambda + |\nu| + \frac{a\rho}{\omega}) > 0,$$

$$\mathrm{Re}(\lambda + |\nu| + \frac{b\rho}{\omega}) > 0; \omega \ne 0, c \ne 0,-1,-2,.....; z \ge 0$$

Furthermore, a complement of the incomplete S-function is defined analogously by

$$S_x^\infty\left[{}_{\omega,\rho}^{a,a_2,\ldots,a_p;b_1,\ldots,b_q;\lambda,\nu,p}(z,x)\right]$$

$$= \left(\frac{2p}{\pi}\right)^{1/2} \int_x^\infty t^{\lambda-1} K_\nu(pt) \times \qquad (17)$$

$$ {}_pR_q(a,a_2,\ldots,a_p;b_1,\ldots,b_q;\omega;-\frac{z}{t^\rho})dt$$

where $z,\omega,\rho > 0, \mathrm{Re}(p) > 0$

$$\mathrm{Re}(a) > 0, \mathrm{Re}(a_j) > 0\,(j = 2,....,p\ ), \mathrm{Re}(\lambda) > 1;$$

$$\omega \ne 0, c \ne 0,-1,-2,.....; z \ge 0$$

which for p = 2 and q = 1 gives

$$D_x^\infty\left[{}_{\omega,\rho}^{a,b;c;\lambda,\nu,p}(z,x)\right] = \left(\frac{2p}{\pi}\right)^{1/2} \int_x^\infty t^{\lambda-1} K_\nu(pt) \times$$

$$ {}_2R_1(a,b,c;\omega;-\frac{z}{t^\rho})dt \qquad (18)$$

where $z, \omega, \rho > o$, Re(a)>0, Re(b)>0, Re(p)>0, $\mathrm{Re}(\lambda) > 1, c \neq 0,-1,-2,...$ x>0, and $z \geq 0$

Then clearly, from (7), (15), and (17) we find that

$$S_{\omega,\rho}^{a,a_2,...,a_p b_1,...,b_q;\lambda,\nu,p}(z)$$
$$= S_0^x\left[{}^{a,a_1,...,a_p;b_1,...,b_q;\lambda,\nu,p}_{\omega,\rho}(z,x)\right] +$$
$$S_x^\infty\left[{}^{a,a_1,...,a_p;b_1,...,b_q;\lambda,\nu,p}_{\omega,\rho}(z,x)\right]$$

Remark 1: If in (16), we set $p = \rho = 1, \nu = 1/2$, replace z by z/a, replace λ by $b + 1/2$ and take the limit as a tends to infinity, we obtain the following incomplete functions introduced by [27]

$$\lim_{a\to\infty} C_0^x\left[{}^{a,b,c;b+\frac{1}{2},\frac{1}{2},1}_{\omega,1}(\frac{z}{a},x)\right]$$
$$= C_0^x\left[{}^{b,c;b+1/2,1/2,1}_{\omega,1}(z,x)\right]$$
$$= C\left[{}^{b,c;b,1}_{\omega,1}(2z^{1/2},x)\right] = {}_\omega\gamma_z(b,c;x)$$
$$= \int_0^x t^{b-1}\exp(-t)\,{}_1R_1(b,c;\omega;-\frac{z}{t})dt \qquad (19)$$

and

$$\lim_{a\to\infty} D_x^\infty\left[{}^{a,b,c;b+1/2,1/2,1}_{\omega,1}(z,x)\right]$$
$$= D_x^\infty\left[{}^{b,c;b+1/2,1/2,1}_{\omega,1}(z,x)\right]$$
$$= D_{\omega,1}^{b,c;b,1}(2z^{1/2},x) = {}_\omega\Gamma_z(b,c;x)$$
$$= \int_x^\infty t^{b-1}\exp(-t)\,{}_1R_1(b,c;\omega;-\frac{z}{t})dt \qquad (20)$$

where, $C\left[{}^{b,c;b,1}_{\omega,1}(z,x)\right]$ and $D\left[{}^{b,c;b,1}_{\omega,1}(z,x)\right]$ are the incomplete functions discussed by [3] and ${}_\omega\gamma_z$ and ${}_\omega\Gamma_z$ are the incomplete functions studied by [27].

As $z \to 0$ in (19), and (20), we arrive at the following interesting results:

$$C_0^x\left[{}^{b,c;\lambda+1/2,1/2,1}_{\omega,1}(o,x)\right] = \int_0^x t^{\lambda-1}\exp(-t)dt = \gamma(\lambda,x)$$

and

$$D_x^\infty\left[{}^{b,c;\lambda+1/2,1/2,1}_{\omega,1}(o,x)\right] = \int_x^\infty t^{\lambda-1}\exp(-t)dt = \Gamma(\lambda,x)$$

where $\gamma(\lambda,x)$ and $\Gamma(\lambda,x)$ denote the well-known incomplete gamma function and its complement respectively.

It is interesting to note that the incomplete functions considered by [4], [5] follow as special cases of (19) and (20) by giving suitable values to the parameters.

Derivatives of the Incomplete S-Functions:

We have

$$\frac{d}{dz}\left\{S_0^x\left[{}^{a,a_2,...,a_p;b_1,...,b_q;\lambda,\nu,p}_{\omega,\rho}(z,x)\right]\right\}$$
$$= \sigma z^{\sigma-1} S_0^x\left[{}^{a,a_2,...,a_p;b_1,...,b_q;\lambda,\nu,p}_{\omega,\rho}(z,x)\right]$$
$$- \frac{a\prod_{j=1}^q\Gamma(b_j)\prod_{i=2}^p\Gamma(a_j+\omega)}{\prod_{i=2}^p\Gamma(a_j)\prod_{j=1}^q\Gamma(b_j+\omega)} \times \qquad (21)$$
$$S_0^x\left[{}^{a+1,a_2+\omega,...,a_p+\omega,b_1+\omega,...,b_q+\omega;\lambda-\rho,\nu,p}_{\omega,\rho}(z,x)\right]$$

When p = 2 and q = 1, (21) gives

$$\frac{d}{dz}\left\{C_0^x\left[{}^{a,b,c;\lambda,\nu,p}_{\omega,\rho}(z,x)\right]\right\} =$$
$$\sigma z^{\sigma-1} C_0^x\left[{}^{a,b,c;\lambda,\nu,p}_{\omega,\rho}(z,x)\right]$$
$$- \frac{a\Gamma(c)\Gamma(b+\omega)}{\Gamma(b)\Gamma(c+\omega)} C_0^x\left[{}^{a+1,b+\omega,c+\omega;\lambda-\rho,\nu,p}_{\omega,\rho}(z,x)\right]$$

where the incomplete function $C_0^x(.)$ is defined in (16).

Next, we have

$$\frac{d}{dz}\left[S_x^\infty\left[{}^{a,a_2,...,a_p,b_1,...,b_q;\lambda,\nu,p}_{\omega,\rho}(z,x)\right]\right]$$
$$= \sigma z^{\sigma-1} S_x^\infty\left[{}^{a,a_2,...,a_p,b_1,...,b_q;\lambda,\nu,p}_{\omega,\rho}(z,x)\right]$$
$$- \frac{a\prod_{j=1}^q\Gamma(b_J)\prod_{i=2}^p\Gamma(a_i+\omega)}{\prod_{i=2}^p\Gamma(a_i))\prod_{j=1}^q\Gamma(b_j+\omega)}$$

which for p = 2 and q = 1 yields

$$\frac{d}{dz}\left[D_x^\infty\left[{}^{a,b,c;\lambda,\nu,p}_{\omega,\rho}(z,x)\right]\right]$$
$$= \sigma z^{\sigma-1} D_x^\infty\left[{}^{a,b,c;\lambda,\nu,p}_{\omega,\rho}(z,x)\right]$$
$$- \frac{a\Gamma(c)\Gamma(b+\omega)}{\Gamma(b)\Gamma(c+\omega)} D_x^\infty\left[{}^{a+1,b+\omega,c+\omega;\lambda-\rho,\nu,p}_{\omega,\rho}(z,x)\right]$$

where the incomplete function $D_x^\infty(.)$ is defined in (18).

Considering

$$\frac{d}{dt}[t^{\lambda-1}K_\nu(pt)\ {}_pR_q(a,a_2,...,a_p;b_1,...,b_q;\omega,-z/t^\rho)]$$
$$= (\lambda-1)t^{\lambda-2}K_\nu(pt)$$
$$\times {}_pR_q(a,a_2,...,a_p;b_1,...,b_q;\omega,-z/t^\rho)$$
$$+ p\left[-\frac{\lambda}{t}K_\nu(pt) - K_{\nu-1}(pt)\right]t^{\lambda-1} \times$$
$${}_pR_q(a,a_2,...,a_p;b_1,...,b_q;\omega,-z/t^\rho) +$$

$$\frac{\prod_{j=1}^{q}\Gamma(b_j)\Gamma(a+1)\prod_{i=2}^{p}\Gamma(a_i+\omega)}{\Gamma(a)\prod_{i=2}^{p}\Gamma(a_i)\prod_{j=1}^{q}\Gamma(b_j+\omega)} t^{\lambda-\rho-1}K_\nu(pt))$$

$$\times {}_pR_q(a+1, a_2+\omega,\ldots,a_p+\omega; b_1+\omega,\ldots,b_q+\omega;\omega,-z/t^\rho),$$

and integrating from x to ∞ and making certain adjustments, it yields

$$\frac{a\prod_{j=1}^{q}\Gamma(b_j)\prod_{i=2}^{p}\Gamma(a_i+\omega)}{\prod_{i=2}^{p}\Gamma(a_i)\prod_{j=1}^{q}\Gamma(b_j+\omega)}\times$$

$$S_x^\infty\left[\begin{matrix}a+1,a_2+\omega,\ldots,a_p+\omega{:}b_1+\omega,\ldots,b_q+\omega;\lambda-\rho,\nu,p\\ \omega,\rho\end{matrix}(z,x)\right]+$$

$$(\lambda-1)S_x^\infty\left[\begin{matrix}a,a_2,\ldots,a_p;b_1,\ldots,b_q;\lambda-1.\nu,p\\ \omega,\rho\end{matrix}(z,x)\right]-$$

$$\lambda p S_x^\infty\left[\begin{matrix}a,a_2,\ldots,a_p;b_1,\ldots,b_q;\lambda-1.\nu,p\\ \omega,\rho\end{matrix}(z,x)\right]$$

$$-pS_x^\infty\left[\begin{matrix}a,a_2,\ldots,a_p,b_1,\ldots,b_q;\lambda,\nu-1,p\\ \omega,\rho\end{matrix}(z,x)\right]+$$

$$\left(\frac{2p}{\pi}\right)^{1/2} t^{\lambda-1}K_\nu(pt)\times$$

$${}_pR_q(a,a_2,\ldots,a_p;b_1,\ldots,b_q;\omega;-z/t^\rho)=0$$

IV. A CLASS OF PROBABILITY DENSITY FUNCTIONS

In this section, we will discuss a generalized inverse Gaussian distribution and employ for Λ the function

$$\Lambda=\mu\left(\frac{2p}{\pi}\right)^{1/2}\left[S^{a,a_2,\ldots,a_p;b_1,\ldots,b_q;\frac{\lambda}{\mu},\nu,p}_{\omega,\frac{\rho}{\mu}}(z)\right]^{-1}$$

where p and z denote the scalar parameters, whereas λ,ρ represent the shape parameters.

Definition 3: A unified form of the inverse Gaussian distribution is defined by

$$f(x)=\begin{cases}\Lambda x^{\lambda-1}K_\nu(px^\mu)\times\\ {}_pR_q(a,a_2,\ldots,a_p;b_1,\ldots,b_q;\omega;-z/x^\rho), & for\ x>0\\ 0, & elsewhere\end{cases} \quad (22)$$

where

$$\lambda\pm\mu\nu+\frac{\rho a}{\omega}>0, \lambda\pm\mu\nu+\frac{\rho a_i}{\omega}>0 (i=2,\ldots,p);$$

$$\mu,\omega,p,\rho>0, z>0,\ b_j\neq 0,-1,-2,\ldots(j=1,\ldots,q)$$

The parameters and variable appearing in (22) are so restricted that $f(x)$ remains positive for $x>0$.
It readily follows that

$$\int_{-\infty}^{\infty} f(x)dx=1.$$

Further from the definition (22), we infer that $f(0)=0$ and $\lim_{x\to\infty} f(x)=0.$

Differentiation of the expression (22) gives

$$f'(x)$$

$$=\left[\begin{matrix}\frac{\lambda-1}{x}-\frac{p\mu x^{\mu-1}K_\nu'(px^\mu)}{K_\nu(px^\mu)}+\\ +\frac{az\rho K_\nu(px^\mu)\Gamma(c)\prod_{j=1}^{q}\Gamma(b_j)\prod_{i=2}^{p}\Gamma(a_i+\omega)}{t^{\rho+1}\prod_{i=2}^{p}\Gamma(a_j)\prod_{j=1}^{q}\Gamma(b_j+\omega)}\\ {}_pR_q(a+1.a_2+\omega,\ldots,a_p+\omega;b_1+\omega,\ldots,b_q\\ +\omega:-z/t^\rho)\end{matrix}\right]f(x)$$

Special cases of the generalized inverse Gaussian distribution: For p = 2 and q = 1, the probability density model (22) reduces to one studied by [25]

$$f(x)=\begin{cases}\Lambda_1 x^{\lambda-1}K_\nu(px^\mu)\\ {}_2R_1(a,b;c;\omega;-z/x^\rho), for \quad x>0.\\ 0,\quad elsewhere\end{cases} \quad (23)$$

where

$$\lambda\pm\mu\nu+\frac{\rho a}{\omega}>0, \lambda\pm\mu\nu+\frac{\rho b}{\omega}>0; \mu,\omega,p,\rho>0, z>0.$$

$c\neq 0,-1,-2,\ldots$ and

$$\Lambda_1=\mu\left(\frac{2p}{\pi}\right)^{1/2}\left[S^{a,b;c;\frac{\lambda}{\mu},\nu,p}_{\omega,\frac{\rho}{\mu}}(z)\right]^{-1}$$

If $\nu=1/2$, the density defined by (23) reduces to the Gaussian density function associated with Dotsenko function (5) in the interesting form

$f(x)=\Lambda_2 x^{\lambda-1}\exp(-px^\mu)\,{}_2R_1(a,b;c;\omega,-z/x^\rho)$, $x>0$
= 0, elsewhere

where $$\Lambda_2=\mu\left[S^{a,b,c;((\lambda+\frac{\mu}{2})/\mu),\frac{1}{2}.p}_{\omega,\rho/\mu}(z)\right]^{-1}$$

Replacing z by z/a in the above expression and taking the limit as $a\to\infty$, we arrive at the density function recently studied by [3], $f(x)=\Lambda_3 x^{\lambda-1}\exp(-px^\mu)\,{}_1R_1(b;c;\omega,-z/x^\rho)$, $x>0$
= 0, elsewhere (24)

where
$z\geq 0$, $p>0$, $\mu>0$; $z,\omega,\rho>0$, $\mathrm{Re}(\lambda)>1$, , $c\neq 0,-1,-2,\ldots$ and

$$\Lambda_3 = \mu\left[S_{\omega,\rho/\mu}^{b,c;((\lambda+\frac{\mu}{2})/\mu),\frac{1}{2}.p}(z) \right]^{-1}$$

If we take $\rho = \mu$ in (24) it reduces to the inverse Gaussian density discussed by [2] in the form

$$f(x) = \Lambda_4 x^{\lambda-1} \exp(-px^{\mu})\,_1R_1(b;c;\omega,-z/x^{\mu}),\ x>0$$
$$= 0, \text{ elsewhere} \qquad (25)$$

where $z \geq 0$, $p>0$; $z,\omega,\mu > 0$, $\mathrm{Re}(\lambda) > 1, \mathrm{Re}(b) > 0, c \neq 0,-1,-2,...$ and

$$\Lambda_4 = \mu\left[S_{\omega,1}^{b,c;((\lambda+\frac{\mu}{2})/\mu),\frac{1}{2}.p}(z) \right]^{-1}$$

$$= \mu\left[{}_{\omega}\Gamma_z(b;c;\frac{\lambda}{\mu};p) \right]^{-1}$$

We note that the notation given on the right of the above equation is due to [2].

For $c = b = \omega = p = 1$ in (25), we obtain

$$f(x) = \frac{\mu x^{\lambda-1} \exp(-x^{\mu} - \frac{z^2}{4x^{\rho}})}{S_{1,1}^{b,b;((\lambda+\frac{\mu}{2})/\mu),\frac{1}{2},1}\left(\frac{z^2}{4}\right)}$$

$$= \frac{\mu x^{\lambda-1} \exp(-x^{\mu} - \frac{z^2}{4x^{\rho}})}{Z_{(\rho/\mu\}}^{(-\lambda/\mu)}(\frac{z^2}{4})}$$

where $Z_{\rho}^{\lambda}(x)$ is the Krätzel function defined by (1).

For $\mu = \rho = 1$, $p = 2$, $q = 1$, (22) becomes

$$f(x) = \Lambda_5 x^{\lambda-1} K_{\nu}(px)\ {}_2R_1(a,b;c;\omega;-z/x),\ x>0$$
$$= 0, \text{ elsewhere} \qquad (26)$$

where, $\Lambda_5 = \left(\frac{2p}{\pi}\right)^{1/2} \left[S_{\omega,1}^{a,b,c;\lambda,\nu,p}(z) \right]^{-1}$

If we further take $\nu = 1/2$ and replace z by z/a and take the limit as $a \to \infty$, then (26) gives the following density function

$$f(x) = \Lambda_6 x^{\lambda-1} \exp(-px)\,_1R_1(b;c;\omega;-z/x) \qquad (27)$$

where, $\Lambda_6 = \left[S_{\omega,1}^{b,c;\lambda+1/2,1/2,p}(z) \right]^{-1}$ which for $b = c$, $\omega = 1$ yields

$$f(x) = \frac{x^{\lambda-1} \exp(-px - \frac{z}{x})}{2\left(\frac{z}{p}\right)^{\frac{\lambda}{2}} K_{\lambda}(2\sqrt{pz})} \qquad (28)$$

where $K_{\lambda}(z)$ is the modified Bessel function of the third kind or Macdonald function. The density represented by (28) is the inverse Gaussian density studied by [11].

V. A SET OF STATISTICAL FUNCTIONS

For the statistical density defined by (22), the following statistical functions are derived.

The k^{th} -moment:

The k^{th} – moment μ_k' about the origin of a continuous real random variable x with density function $f(x)$ is defined by [21]

$$\mu_k' = \int_{-\infty}^{\infty} x^k f(x)dx.$$

Theorem 2: For the density function $f(x)$ defined by (22), the k^{th} – moment is given by

$$\mu_k' = \frac{S_{\omega,\frac{\rho}{\mu}}^{a,a_2,...,a_p;b_1,...,b_q;\frac{\lambda+k}{\mu},\nu,p}(z)}{S_{\omega,\frac{\rho}{\mu}}^{a,a_2,...,a_p;b_1,...,b_q;\frac{\lambda}{\mu},\nu,p}(z)} \qquad (29)$$

which for $p = 2$ and $q = 1$ reduces to the following result given by [24]

$$\mu_k' = \frac{S_{\omega,\frac{\rho}{\mu}}^{a,b;c;\frac{\lambda+k}{\mu},\nu,p}(z)}{S_{\omega,\frac{\rho}{\mu}}^{a,b;c;\frac{\lambda}{\mu},\nu,p}(z)}$$

On taking $p = 2$, $q = 1$ and $\nu = 1/2$ in (29), replacing z by z/a and taking the limit as a tends to infinity, we obtain the following result obtained by [3] as:

$$\mu_k' = \frac{S_{\omega,\frac{\rho}{\mu}}^{b,c;\frac{\lambda+\frac{\mu}{2}+k}{\mu},\nu,p}(z)}{S_{\omega,\frac{\rho}{\mu}}^{b,c;\frac{\lambda+\frac{\mu}{2}}{\mu},\nu,p}(z)} = \frac{B_{\omega,\rho/\mu}^{b,c;\frac{\lambda+k}{\mu},p}(2\sqrt{z})}{B_{\omega,\rho/\mu}^{b,c;\frac{\lambda}{\mu},p}(2\sqrt{z})}$$

If we further take $\mu = \rho$ and replace z by $z^* = z^2/4$, then it gives the following result in the notation obtained in [2]:

$$\mu_k' = \frac{\left[\omega\Gamma_{z^*}(b;c;\frac{\lambda+k}{\mu};p)\right]}{\left[\omega\Gamma_{z^*}(b;c;\frac{\lambda}{\mu};p)\right]}$$

Theorem -2 easily follows from the definition (29) on using the special cases of the S-function discussed in the preceding section of the paper.

Expected Value: Let $\Psi(x)$ be a function of a continuous real random variable x with density $f(x)$, then the expected value of $\Psi(x)$ is defined as [21]

$$E(\Psi) = \int_0^\infty \Psi(x) f(x)\,dx$$

The existence of the expected value $E(\Psi)$ depends upon the behavior of $f(x)$ and $\Psi(x)$. It is interesting to observe that for a positive real random variable x with density $f(x)$, where $f(x) = 0$ for $x<0$, the expected value of x^{s-1} is the Mellin transform of $f(x)$.

Remark 2: It may be mentioned here that for k=1,the result (59) yields the result for the mean $E_x = \int_0^\infty x\, f(x)dx$, and for $k = s$-1 , it yields the result for the expected value of x^{s-1} and the density function $f(x)$ with x>0 ,defined by (22).

The hazard rate function (failure rate) is defined as

$$h(x) = \frac{f(x)}{S^{(*)}(x)}$$

where $S^{(*)}(x)$ is the survivor function of x

$$S^{(*)}(x) = 1 - F(x) > 0 \text{ for } x>0$$

and $F(x)$ being the cumulative density function (c.d.f) namely

$$F(x) = \int_0^x f(u)du$$

This function $S^{(*)}(x)$ has its origin in reliability theory.

Theorem 3: For the density function defined by (22), we have:

$$F(x) = \frac{S_0^{x^\mu}\left[\begin{matrix} a,a_2,...,a_p;b_1,...,b_q;\frac{\lambda}{\mu},\nu,p \\ \omega,\frac{\rho}{\mu}\end{matrix}(z,x)\right]}{S^{a,a_2,...,a_p,b_1,...,b_q;\frac{\lambda}{\mu},\nu,p}_{\omega,\frac{\rho}{\mu}}(z)}$$

$$S^{(*)}(x) = \frac{S_{x^\mu}^{\infty}\left[\begin{matrix} a,a_2,...,a_p;b_1,...,b_q;\frac{\lambda}{\mu},\nu,p \\ \omega,\frac{\rho}{\mu}\end{matrix}(z,x)\right]}{S^{a,a_2,...,a_p,b_1,...,b_q;\frac{\lambda}{\mu},\nu,p}_{\omega,\frac{\rho}{\mu}}(z)}$$

and

$$h(t) = \frac{\left(\frac{2p}{\pi}\right)^{1/2} \mu t^{\lambda-1} K_\nu(pt^\mu)\ {}_pR_q(a,a_2,...,a_p;b_1,...,b_q;\omega;-\frac{z}{t^\rho})}{S_{x^\mu}^{\infty}\left[\begin{matrix} a,a_2,...,a_p;b_1,...,b_q;\frac{\lambda}{\mu},\nu,p \\ \omega,\frac{\rho}{\mu}\end{matrix}(z,x)\right]}$$

where the incomplete S-functions $S_0^x(.)$ and S_x^∞ are defined in (15) and (17) respectively.

Proof: The Theorem -3 follows from the following equations:

$$F(x) = \frac{\left(\frac{2p}{\pi}\right)^{\frac{1}{2}} \mu \int_0^x t^{\lambda-1} K_\nu(pt^\mu)\ {}_pR_q(a,a_2,...,a_p;b_1,...,b_q;\omega;-\frac{z}{t^\rho})dt}{\left[S^{a,a_2,...,a_p;b_1,...,b_q;\frac{\lambda}{\mu},\nu.p}_{\omega,\rho/\mu}(z)\right]}$$

$S(x)$ =1− $F(x)$ =

$$\frac{\left[S^{a,a_2,...,a_p,b_1,...,b_q;\frac{\lambda}{\mu},\nu.p}_{\omega,\rho/\mu}(z,x)\right] - S_0^{x^\mu}\left[\begin{matrix}a,a_2,...,a_p,b_1,...,b_q;\frac{\lambda}{\mu},\nu.p \\ \omega,\rho/\mu\end{matrix}(z,x)\right]}{\left[S^{a,a_2,...,a_p;b_1,...,b_q;\frac{\lambda}{\mu},\nu.p}_{\omega,\rho/\mu}(z,x)\right]}$$

The Mean Residue Life Function: The mean residue life function is defined as

$$K(x) = \frac{1}{S^*(x)}\int_x^\infty (t-x)f(t)dt.$$

Separating the variables, we obtain

$$\int_x^\infty tf(t)dt = \theta \int_x^\infty t^\lambda K_\nu(pt^\mu)\ {}_pR_q(a,a_2,...,a_p;b_1,...,b_q;\omega;-z/t^\rho)dt \tag{30}$$

$$= \frac{S_{x^{\mu}}^{\infty}\left[\begin{matrix} a,a_2,...,a_p;b_1,...,b_q;\frac{\lambda+1}{\mu},\nu.p \\ \omega,\rho/\mu \end{matrix} (z,x)\right]}{S_{\omega,\rho/\mu}^{a,a_2,...,a_p;b_1,...,b_q;\frac{\lambda}{\mu},\nu.p}(z)} \quad (31)$$

As, $\frac{x}{S^*(x)}\int_x^{\infty} f(t)\,dt = x$ so that from (30) and (31), we have

$$\text{K (x)} = \frac{S_{x^{\mu}\ \omega,\rho/\mu}^{\infty\ a,a_2,...,a_p;b_1,...,b_q,c;\frac{\lambda+1}{\mu},\nu.p}\ \mu(z,x)}{S_{\omega,\rho/\mu}^{a,a_2,...,a_p;b_1,...,b_q;\frac{\lambda}{\mu},\nu.p}(z)} - x$$

The Moment Generating Function:

The moment generating function of a continuous and random variable x, denoted by $M_X(t)$, is defined by

$$E(e^{tX}) = M_X(t) = \int_0^{\infty} e^{tx} f(x)dx$$

with certain restrictions on the parameters in the density function $f(x)$. It can be easily seen that for the density defined by (27), namely

$$f(x) = \Lambda_6 x^{\lambda-1}\exp(-px)\ {}_1R_1(b;c;\omega;-z/x)$$

the moment generating function can be derived in a simplified form $E(e^{tx}) = \frac{S_{\omega,1}^{b,c;\lambda+\frac{1}{2},\frac{1}{2}p-t}(z)}{S_{\omega,1}^{b,c;\lambda+\frac{1}{2},\frac{1}{2}p}(z)}$ as given in the paper by [2].

VI. CONCLUSION

For various suitable choices of the parameters involved in our results in the preceding sections, we can easily deduce several special cases which were considered in some of the earlier works cited in the abstract .The details involved in these specializations are being left as an exercise for the interested reader.

In conclusion, it is expected that the research workers in the fields of applied statistics, generalized special functions and reliability theory may find this work useful in their applications.

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