World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:5, No:8, 2011

Stability of a special class of switched positive systems

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Open Science Index, Mathematical and Computational Sciences Vol:5, No:8, 2011 publications.waset.org/6709.pdf

Abstract—This paper is concerned with the existence of a linear copositive Lyapunov function(LCLF) for a special class of switched positive linear systems(SPLSs) composed of continuous- and discrete-time subsystems. Firstly, by using system matrices, we construct a special kind of matrices in appropriate manner. Secondly, our results reveal that the Hurwitz stability of these matrices is equivalent to the existence of a common LCLF for arbitrary finite sets composed of continuous- and discrete-time positive linear time-invariant(LTI) systems. Finally, a simple example is provided to illustrate the implication of our results.

Keywords—Linear copositive Lyapunov functions; Positive systems; Switched systems.

I. Introduction

GENERALLY speaking, a dynamical system is called *positive* if for any nonnegative initial condition, the corresponding solution of the system is also nonnegative (see [1], [2]). In real world, the positivity requirement is often introduced in the system model whenever the physical nature of the describing variables constrains them to take only positive (or at least nonnegative) values. As a result, positive linear systems naturally arise in fields such as bioengineering, economic modeling, behavioral science, and stochastic processes. This feature makes analysis and synthesis of positive systems a challenging and interesting job (see, for example, [3], [4], [5], [6] and references therein).

In this perspective, SPLSs are mathematical models which keep into account two different aspects: the fact that the system dynamics can be suitably described by means of a family of subsystems, each of them formalizing the system laws under specific operating conditions, among which the system commutes, and the nonnegativity constraint the physical variables are subject to. This is the case when trying to describe certain physiological and pharmacokinetic processes, like the insulin-sugar metabolism. Of course, the need for this class of systems in specific research contexts [4], [5] has stimulated an interest in issues related to them, in particular, stability issues [7], [8], [9], [10], [11], [12], [14] and references therein.

A key result in this connection is that stability of SPLSs under arbitrary switching laws is equivalent to the existence of a common Lyapunov function [7]. Generally speaking, three classes of Lyapunov function naturally suggest themselves for SPLSs: common quadratic Lyapunov functions, common diagonal Lyapunov functions, and common LCLFs. For continuous-time positive LTI systems, the authors of [8] and independently Dvid Angeli, posed a conjecture that the existence of common quadratic Lyapunov function can be determined by testing the Hurwitz-stability of an associated convex set of matrices. Gurvits, Shorten and Mason [9] proved that this conjecture is true for pairs of second order systems and is false in general. In the paper [10], a necessary and sufficient was derived for the existence of common diagonal Lyapunov function for pairs of positive LTI systems with irreducible system matrices. It is well known that traditional Lyapunov functions may give conservative stability conditions for SPLSs as they fail to take account that the trajectories are naturally constrained to the positive orthant. Therefore, it is natural to adopt common LCLFs which can guarantee the stability of SPLSs [11]. Work discussed in [12] provided a method for determining whether or not a given SPLS composed of two subsystems is stable. Such an approach is based upon determining verifiable conditions for a common LCLF. [13] considered some reduced cases when the system matrices are 2×2 dimensions and have the same block lower triangular forms, some results were presented by means of properties of geometry. For discrete-time SPLSs, switched copositive Lyapunov function method is proposed in [14]. It is emphasize that the existence of switched copositive Lyapunov function is just a sufficient condition of the stability for SPLSs. [15] presented a compact and easily verifiable equivalent conditions of the existence of a common LCLF on pairs of SPLSs by means of properties of geometry.

This paper will investigate the stability of a special class of SPLSs composed of a set of continuous-time subsystems and a set of discrete-time subsystems. Simple necessary and sufficient conditions for the existence of a common LCLF will be established. The remainder of this paper is structured as follows. In the next section, we give some notations and preliminary results which will be used in the sequel. Section III is dedicated to derive some checkable necessary and sufficient conditions for the existence of a common LCLF for arbitrary finite sets composed of continuous- and discrete-time positive LTI systems. Section IV provides a simple example to illustrate the main results of this paper, and some concluding remarks are presented in Section V.

II. Notation and Background

Throughout, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^n(\mathbb{R}^n_{0,+}, \mathbb{R}^n_+)$ stands for the $n-$dimensional real (nonnegative, positive) vector space and $\mathbb{R}^{n\times n}$ is the space of $n\times n$ matrices with real entries. For A in $\mathbb{R}^{n\times n}$, a_{kl} denotes the element in the (kl) position of A. $A \succeq 0(\preceq 0)$ means

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Open Science Index, Mathematical and Computational Sciences Vol:5, No:8, 2011 publications.waset.org/6709.pdf

that all elements of matrix A are nonnegative (nonpositive) and $A \succ 0(\prec 0)$ means that all elements of matrix A are positive (negative). The notion $A > 0(< 0)$ means that A is a symmetric positive (negative) definite matrix. Meanwhile, $A^T(A^{-1})$ represents the transpose (inverse) of matrix A. Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{N}_0 = \{0\} \bigcup \mathbb{N}$. $\lambda(A)$ represents the eigenvalue of A , and $\rho(A)$ denotes the spectral radius of A. Also, when referring to switched linear systems, stability shall be used to denote asymptotic stability under arbitrary switching signals.

We now recall some basic facts about positive systems. See [11] for a description of the basic theory and several applications of positive linear systems.

Definition 1: The continuous-time LTI system

$$\Sigma_{A^c} : \dot{x}(t) = A^c x(t), \quad x_0 = x(0)$$

is said to be positive if $x_0 \succeq 0$ implies that $x(t) \succeq 0$ for all $t \geq 0$.
The discrete-time LTI system

$$\Sigma_{A^d} : x(k+1) = A^d x(k), \quad x_0 = x(0)$$

is said to be positive if $x_0 \succeq 0$ implies that $x(k) \succeq 0$ for all $k \in \mathbb{N}$.

The following Lemma ensures positivity of the systems Σ_{A^c} and Σ_{A^d}.

Lemma 1: Continuous-time system Σ_{A^c} is positive if and only if off-diagonal entries of the matrix A^c are non-negative. Discrete-time system Σ_{A^d} is positive if and only if the matrix A^d satisfies $A \succeq 0$.

It is well known that a matrix whose off-diagonal entries are non-negative is said to be *Metzler*, then the positivity of the system Σ_{A^c} is equivalent to Metzler of A^c. A matrix A is said to be *Hurwitz* if and only if all its eigenvalues lie in the open left half of the complex plane. And a matrix is said to be *Schur* if and only if spectral radius less than 1. A classic result shows that the stability of positive system Σ_{A^c} and Σ_{A^d} is equivalent to Hurwitz of A^c and Schur of A^d, respectively.

The next results summarize some basic properties of for a Metzler matrix A^c to be Hurwitz and a matrix $A^d \succeq 0$ to be Schur, which will be used in the next section.

Lemma 2: [6] Let matrix $A \in \mathbb{R}^{n\times n}$ be Metzler. Then A is a Hurwitz matrix if and only if there exists a vector $v \succ 0$ in $\mathbb{R}^n$ with $Av \prec 0$.
Let $A \succeq 0$ in $\mathbb{R}^{n\times n}$. Then A is a Schur matrix if and only if there exists a vector $v \succ 0$ in $\mathbb{R}^n$ with $(A - I)v \prec 0$.

Lemma 3: Let matrix A in $\mathbb{R}^{n\times n}$ be Schur, then $A - I$ is a Hurwitz matrix.
The following result can be derived from [19].

Lemma 4: Consider Metzler matrices A, B with $A \succeq B$, if A is Hurwitz, then B is also Hurwitz.
Consider matrices A, B with $A \succeq B \succeq 0$, if A is Schur, then B is also Schur.

Lemma 5: Let Metzler matrix $A \in \mathbb{R}^{n\times n}$ be Hurwitz, then $\det A \neq 0$.
Let $A \succeq 0$ in $\mathbb{R}^{n\times n}$ be Schur, then $\det(A - I) \neq 0$.

A convex cone in $\mathbb{R}^n$ is a set $\Omega \in \mathbb{R}^n$ such that, for any $x, y \in \Omega$ and any $\alpha \geq 0, \beta \geq 0$, $\alpha x + \beta y \in \Omega$. The convex cone is said to be open (closed) if it is open (closed) with respect to the usual Euclidean topology on $\mathbb{R}^n$. For an open convex cone Ω, $\bar{\Omega}$ denotes the closure of Ω.

Lemma 6: [16] Let Ω_1, Ω_2 be open convex cones in $\mathbb{R}^n$. Suppose that $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \{0\}$. Then there is a vector $v \in \mathbb{R}^n$ such that

$$v^T x < 0 \quad \textit{for all} \quad x \in \Omega_1$$

and

$$v^T x > 0 \quad \textit{for all} \quad x \in \Omega_2.$$

III. Main Results

This section studies the existence of a LCLF for a special class of SPLS which is composed of a set of stable continuous-time subsystems

$$\dot{x}(t) = A^c(t)x(t), \; x(0) = x_0 \succeq 0, t \geq 0 \tag{1}$$

and a set of stable discrete-time subsystems

$$x(k+1) = A^d(k)x(k), \; x(0) = x_0 \succeq 0, k \in \mathbb{N}_0, \tag{2}$$

where, in continuous-time case, $x(t) \in \mathbb{R}^n$ are the states, $A^c(t) \in \{A^c_1, \cdots, A^c_{m^c}\}$ with Metzler matrices A^c_i in $\mathbb{R}^{n\times n}$, $i \in \mathcal{I}^c = \{1, \cdots, m^c\}$ is an index set, and m^c denotes the number of continuous-time subsystems, in discrete-time case, $x(k) \in \mathbb{R}^n$ are the states, $A^d(k) \in \left\{A^d_{m^c+1}, \cdots, A^d_{m^c+m^d}\right\}$ with $A^d_i \succeq 0$ in $\mathbb{R}^{n\times n}$, $i \in \mathcal{I}^d = \{m^c+1, \cdots, m^c+m^d\}$ is an index set, and m^d denotes the number of discrete-time subsystems. Also, use $\mathcal{I}$ to denote the index set $\{1, \cdots, m^c + m^d\}$.

To discuss the stability of the overall switched positive systems, we shall make some illustration for this type of switched systems. For simplicity, suppose that the sampling period of all the discrete-time subsystems is τ. Since the states of the discrete-time subsystems can be viewed as piecewise constant vectors between sampling points, we can consider the value of the system states in the continuous-time domain. For example, if continuous-time subsystem (1) is activated on $[t_0, t_1]$ and then discrete-time subsystem (2) is activated for k steps and subsystem $\Sigma_{A^c_2}$ on $[t_1 + k\tau, t_2]$, the time domain thus can be divided into

$$[t_0, t_2] = [t_0, t_1] \cup [t_1, t_1 + k\tau] \cup [t_1 + k\tau, t_2].$$

This argument is repeated and ends in turning out the running status of the whole systems.

For later discussion, we should give some relevant notation. Use Ψ_{n,m^c+m^d} to denote the set of all mapping $\varphi : \{1, \cdots, n\} \rightarrow \mathcal{I}$. Define the matrix $A_\varphi = A_\varphi\left(A^c_1, \cdots, A^c_{m^c}, A^d_{m^c+1} - I, \cdots, A^d_{m^c+m^d} - I\right)$ by

$$\begin{aligned} & A_\varphi\left(A^c_1, \cdots, A^c_{m^c}, A^d_{m^c+1} - I, \cdots, A^d_{m^c+m^d} - I\right) \\ & = \left(A^{(1)}_{\varphi(1)}, \cdots, A^{(n)}_{\varphi(n)}\right). \end{aligned} \tag{3}$$

Obviously, for $1 \leq j \leq n$, A_φ is a matrix in $\mathbb{R}^{n\times n}$, whose the jth column $A^{(j)}_{\varphi(j)}$ is the jth column of one of the $A^c_1, \cdots, A^c_{m^c}, A^d_{m^c+1} - I, \cdots, A^d_{m^c+m^d} - I$. Meanwhile, let

$$\begin{aligned} & \mathscr{A}(A^c_1, \cdots, A^c_{m^c}, A^d_{m^c+1} - I, \cdots, A^d_{m^c+m^d} - I) \\ & = \left\{A_\varphi | \varphi \in \Psi_{n,m^c+m^d}\right\} \end{aligned} \tag{4}$$

represent all matrices formed in such way. For simplicity, use $\mathscr{A}$ to denote the set (4). It is easy to see that

$$\left\{A_1^c, \cdots, A_{m^c}^c, A_{m^c+1}^d - I, \cdots, A_{m^c+m^d}^d - I\right\} \subset \mathscr{A}.$$

It is well known that a switched system composed of stable subsystems could be unstable if the switching is not done appropriately. When the stability of SPLSs is considered, it is natural to adopt common LCLFs since the existence of such a function have less conservative than traditional Lyapunov functions. The common LCLF approach relies on the following fact.

Definition 2: For the finite sets composed of continuous-time positive LTI systems (1) and discrete-time positive LTI systems (2), the function $V(x) = v^T x$ is said to be a common LCLF if and only if there exists a vector $v \succ 0$ in $\mathbb{R}^n$ such that $A_i^{cT} v \prec 0, \forall i \in \mathcal{I}^c$ and $(A_i^d - I)^T v \prec 0, \forall i \in \mathcal{I}^d$.

Before presenting our main results, we need the following technical lemma.

Lemma 7: Let Metzler matrices $A_1^c, \cdots, A_{m^c}^c$ in $\mathbb{R}^{n\times n}$ be Hurwitz and $A_{m^c+1}^d, \cdots, A_{m^c+m^d}^d \succeq 0$ in $\mathbb{R}^{n\times n}$ be Schur. For $v \in \mathbb{R}^n$, denote

$$\Omega_{A_i^c} = \{v \succ 0 | A_i^{cT} v \prec 0, i \in \mathcal{I}^c\} \tag{5}$$

and

$$\Omega_{A_i^d} = \{v \succ 0 | (A_i^d - I)^T v \prec 0, i \in \mathcal{I}^d\} \tag{6}$$

such that

$$\bigcap_{i=1}^{m^c} \bar{\Omega}_{A_i^c} \bigcap_{i=m^c+1}^{m^c+m^d} \bar{\Omega}_{A_i^d} = \{0\}. \tag{7}$$

Then there exist $m^c + m^d$ positive definite diagonal matrices D_i in $\mathbb{R}^{n\times n}$ such that

$$\sum_{i=1}^{m^c} A_i^c D_i + \sum_{i=m^c+1}^{m^c+m^d} (A_i^d - I) D_i = 0. \tag{8}$$

Proof: Conversely, we show that if there exist no m^c+m^d positive definite diagonal matrices D_i in $\mathbb{R}^{n\times n}$ such that (8) holds, then at least one nonzero vector $v \succeq 0$ in $\mathbb{R}^n$ belongs to the set $\bigcap_{i=1}^{m^c} \bar{\Omega}_{A_i^c} \bigcap_{i=m^c+1}^{m^c+m^d} \bar{\Omega}_{A_i^d}$.

First of all, suppose that there exist no $m^c + m^d$ positive definite diagonal matrices D_i in $\mathbb{R}^{n\times n}$ such that (8) holds. Then for some vector $w \succ 0$ in $\mathbb{R}^n$, which leads to

$$\sum_{i=1}^{m^c} A_i^c D_i w + \sum_{i=m^c+1}^{m^c+m^d} (A_i^d - I) D_i w \neq 0. \tag{9}$$

Furthermore, for all $i \in \mathcal{I}$, set $D_i w = w_i$ in $\mathbb{R}^n$, then (9) can be rewritten in the form

$$\sum_{i=1}^{m^c} A_i^c w_i + \sum_{i=m^c+1}^{m^c+m^d} (A_i^d - I) w_i \neq 0. \tag{10}$$

As Metzler A_i^c are Hurwitz and $A_i^d \succeq 0$ are Schur, it follows from Lemma 2 that

$$\left\{\sum_{i=1}^{m^c} A_i^c w_i + \sum_{i=m^c+1}^{m^c+m^d} (A_i^d - I) w_i | w_i \succ 0\right\} \cap \mathbb{R}_+^n = \emptyset. \tag{11}$$

For simplicity, use Ω to denote set $\left\{\sum_{i=1}^{m^c} A_i^c w_i + \sum_{i=m^c+1}^{m^c+m^d} (A_i^d - I) w_i | w_i \succ 0\right\}$. From (11), it is easy to show that $\bar{\Omega} \cap \mathbb{R}_{0,+}^n = \{0\}$. By Lemma 6, there exists a vector $v \in \mathbb{R}^n$ such that

$$v^T x < 0 \quad \text{for all} \quad x \in \Omega$$

and

$$v^T x > 0 \quad \text{for all} \quad x \in \mathbb{R}_+^n.$$

It is easy to show from the later inequality that $v \succeq 0$. In this case, for $w_i \succ 0$, due to $x \in \Omega$, then the front inequality becomes

$$\sum_{i=1}^{m^c} v^T A_i^c w_i + \sum_{i=m^c+1}^{m^c+m^d} v^T (A_i^d - I) w_i < 0. \tag{12}$$

As $v \succeq 0$(nonzero) and $w_i \succ 0$, (12) indicates that $A_i^{cT} v \preceq 0, \forall i \in \mathcal{I}^c$ and $(A_i^d - I)^T v \preceq 0, \forall i \in \mathcal{I}^d$ always hold. Finally, we find a nonzero $v \succeq 0$ in $\mathbb{R}^{n\times n}$ such that $v \in \bigcap_{i=1}^{m^c} \bar{\Omega}_{A_i^c} \bigcap_{i=m^c+1}^{m^c+m^d} \bar{\Omega}_{A_i^d}$. This completes the proof. ■

Based on the preliminary results above, we now present the following Theorem for SPLSs composed of continuous-time subsystem (1) and discrete-time subsystem (2).

Theorem 1: Let Metzler matrices $A_1^c, \cdots, A_{m^c}^c$ in $\mathbb{R}^{n\times n}$ be Hurwitz and $A_{m^c+1}^d, \cdots, A_{m^c+m^d}^d \succeq 0$ in $\mathbb{R}^{n\times n}$ be Schur. Then the following statements are equivalent. (i)

(i) For any $A \in \mathscr{A}$, A is a Hurwitz matrix.
(ii) The finite sets composed of positive LTI systems $\Sigma_{A_1^c}, \cdots, \Sigma_{A_{m^c}^c}, \Sigma_{A_{m^c+1}^d}, \cdots, \Sigma_{A_{m^c+m^d}^d}$ have a common LCLF.
(iii) The finite sets composed of continuous-time positive LTI systems $\Sigma_{\bar{A}_1^c}, \cdots, \Sigma_{\bar{A}_{m^*}^c}$ have a common LCLF, where $\bar{A}_k^c \in \mathscr{A}$ in $\mathbb{R}^{n\times n}$ with $k = 1, \cdots, m^*$ and $m^c + m^d \leq m^* \leq (m^c + m^d)^n$.

Proof: This proof will be accomplished by showing ((i)⇔((ii) and ((i)⇔((iii), respectively.

((i)⇔((ii) Sufficiency: Conversely, we wish to show that if the statement ((ii) is not true, then there exists at least one matrix $A \in \mathscr{A}$ not to be Hurwitz.

To this end, let $\Omega_{A_i^c}, \Omega_{A_i^d}$ be defined as (5) and (6). Complete the proof in two steps.

Step 1. We prove that, under a stronger assumption

$$\bigcap_{i=1}^{m^c} \bar{\Omega}_{A_i^c} \bigcap_{i=m^c+1}^{m^c+m^d} \bar{\Omega}_{A_i^d} = \{0\}. \tag{13}$$

than the false of statement ((ii), at least one matrix $A \in \mathscr{A}(A_1, \cdots, A_m)$ is not Hurwitz. With this in mind, by Lemma 7, it follows that there exist m^c+m^d positive definite diagonal matrices $D_i \in \mathbb{R}^{n\times n}$, denoted as $D_i = \mathrm{diag}\{d_i^{(j)}\}$ for all $i \in \mathcal{I}, 1 \leq j \leq n$, such that

$$\sum_{i=1}^{m^c} A_i^c D_i + \sum_{i=m^c+1}^{m^c+m^d} (A_i^d - I) D_i = 0. \tag{14}$$

Moreover, (14) indicates that the determinant

$$\det\left(\sum_{i=1}^{m^c} A_i^c D_i + \sum_{i=m^c+1}^{m^c+m^d} (A_i^d - I) D_i\right) = 0. \tag{15}$$

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According to the basic properties of determinant, taking into account the construction of the matrices A_φ given by (3), by elementary calculation, the left-side determinant of (15) becomes

$$\begin{aligned}&\det\left(\sum_{i=1}^{m^c} A_i^c D_i + \sum_{i=m^c+1}^{m^c+m^d}(A_i^d - I)D_i\right)\\&= \sum_{\varphi\in\Psi_{n,m}} \det A_\varphi \prod_{j=1}^{n} d_{\varphi(j)}^{(j)},\end{aligned} \tag{16}$$

where the notation of $d_{\varphi(j)}^{(j)}$ corresponds the mapping $\varphi(j) \in \Psi_{n,m}$.

Now, for $\varphi \in \Psi_{n,m}$, consider the determinant $\det A_\varphi$. If all matrices belonging to the set $\mathscr{A}$ are Hurwitz, then $\det A_\varphi \neq 0$ results form Lemma 5. Hence, it follows from $d_{\varphi(j)}^{(j)} > 0$ and (16) that

$$\det\left(\sum_{i=1}^{m^c} A_i^c D_i + \sum_{i=m^c+1}^{m^c+m^d}(A_i^d - I)D_i\right) \neq 0, \tag{17}$$

which would contradict (15). We thus conclude that, if (13) is true, then at least one $A \in \mathscr{A}(A_1, \cdots, A_m)$ not to be Hurwitz.

Step 2. We wish to improve this result above to the case of

$$\bigcap_{i=1}^{m^c} \Omega_{A_i^c} \bigcap_{i=m^c+1}^{m^c+m^d} \Omega_{A_i^d} = \{0\}, \tag{18}$$

i.e., show that, if (18) holds, there is at least one $A \in \mathscr{A}$ not to be Hurwitz. To this end, we distinguish two cases.

Case 1. $\bigcap_{i=1}^{m^c} \bar{\Omega}_{A_i^c} \bigcap_{i=m^c+1}^{m^c+m^d} \bar{\Omega}_{A_i^d} = \{0\}$.

In this case, by hypothesis (18), based on the above discussion, the result is straightforward.

Case 2. $\bigcap_{i=1}^{m^c} \bar{\Omega}_{A_i^c} \bigcap_{i=m^c+1}^{m^c+m^d} \bar{\Omega}_{A_i^d} \neq \{0\}$.

This implies that there exist some nonzero $v \succeq 0$ belonging to the set $\bigcap_{i=1}^{m^c} \bar{\Omega}_{A_i^c} \bigcap_{i=m^c+1}^{m^c+m^d} \bar{\Omega}_{A_i^d}$. In this case, select a vector $\delta = (\delta_1, \cdots, \delta_{m^c}, \delta_{m^c+1}, \cdots, \delta_{m^c+m^d})^T \succ 0$. For all $i \in \mathcal{I}$, define $A_i^c(\delta_i) = A_i^c + \delta_i \mathbf{1}_{n\times n}$ and $A_i^d(\delta_i) = A_i^d + \delta_i \mathbf{1}_{n\times n}$ with the $n \times n$ matrix $\mathbf{1}_{n\times n}$ consisting of all ones. In addition, set

$$\Omega_{A_i^c(\delta_i)} = \{v \succ 0 | (A_i^c + \delta_i \mathbf{1}_{n\times n})^T v \prec 0\}, i \in \mathcal{I}^c$$

and

$$\Omega_{A_i^d(\delta_i)} = \{v \succ 0 | (A_i^d + \delta_i \mathbf{1}_{n\times n} - I)^T v \prec 0\}, i \in \mathcal{I}^d.$$

It is easy to verify that $\Omega_{A_i^c(\delta_i)}$ and $\Omega_{A_i^d(\delta_i)}$ are open convex cones.

Now, we claim that the intersection of the closure

$$\bigcap_{i=1}^{m^c} \bar{\Omega}_{A_i^c(\delta_i)} \bigcap_{i=m^c+1}^{m^c+m^d} \bar{\Omega}_{A_i^d(\delta_i)} = \{0\}$$

is always true. If not, i.e., there is a nonzero vector $v^* \succeq 0$ such that

$$\bigcap_{i=1}^{m^c} \bar{\Omega}_{A_i^c(\delta_i)} \bigcap_{i=m^c+1}^{m^c+m^d} \bar{\Omega}_{A_i^d(\delta_i)} \neq \{0\}.$$

This yeilds, for this v^*,

$$\begin{aligned}v^* &\in \bigcap_{i=1}^{m^c} \bar{\Omega}_{A_i^c(\delta_i)} \bigcap_{i=m^c+1}^{m^c+m^d} \bar{\Omega}_{A_i^d(\delta_i)}\\ &= \bigcap_{i=1}^{m^c} \{v \succeq 0 | (A_i^c + \delta_i \mathbf{1}_{n\times n})^T v \preceq 0\}\}\\ &\quad \bigcap_{i=m^c+1}^{m^c+m^d} \{v \succeq 0 | (A_i^d + \delta_i \mathbf{1}_{n\times n} - I)^T v \preceq 0\}\}\\ &= \bigcap_{i=1}^{m^c} \{v \succeq 0 | A_i^{cT} v \preceq -\delta_i \mathbf{1}_{n\times n} v\}\}\\ &\quad \bigcap_{i=m^c+1}^{m^c+m^d} \{v \succeq 0 | (A_i^d - I)^T v \preceq -\delta_i \mathbf{1}_{n\times n} v\}\}.\end{aligned} \tag{19}$$

As $\delta \succ 0$, we thus find from (19) that

$$\begin{aligned}v^* &\in \bigcap_{i=1}^{m^c} \bar{\Omega}_{A_i^c(\delta_i)} \bigcap_{i=m^c+1}^{m^c+m^d} \bar{\Omega}_{A_i^d(\delta_i)}\\ &= \bigcap_{i=1}^{m^c} \Omega_{A_i^c(\delta_i)} \bigcap_{i=m^c+1}^{m^c+m^d} \Omega_{A_i^d(\delta_i)} \neq \{0\},\end{aligned}$$

which would contradicts the assumption (18).

Now, choosing δ small enough to ensure that all Metzler matrices $A_i^c(\delta_i)$ are Hurwitz and $A_i^d(\delta_i) \succeq$ are Schur, in addition, similar to (4), construct the set

$$\begin{aligned}\mathscr{A}(\delta_i) = \mathscr{A}\,\big(&A_1^c(\delta_1), \cdots, A_{m^c}^c(\delta_{m^c}), A_{m^c+1}^d(\delta_m^c + 1) - I,\\ &\cdots, A_{m^c+m^d}^d(\delta_{m^c+m^d}) - I\big).\end{aligned}$$

With the above analysis, we thus conclude that there is at least one $A \in \mathscr{A}(\delta_i)$ not to be a Hurwitz or Schur. Finally, let $\delta \to 0$, a limiting argument ensures that this result will also be the case of $\mathscr{A}$.

Necessity: Suppose that the statement ((ii) holds. By Definition 2, there exists a vector $v \succ 0$ in $\mathbb{R}^n$ such that $A_i^{cT} v \prec 0, \forall i \in \mathcal{I}^c$ and $(A_i^d - I)^T v \prec 0, \forall i \in \mathcal{I}^d$.

On the one hand, for the case of $A_i^{cT} v \prec 0, \forall i \in \mathcal{I}^c$, it immediately follows that $A_i^{c(j)T} v \prec 0$ with $A_i^{c(j)}$ is the jth column of one of $A_1^c, \cdots, A_{m^c}^c$ for all $i \in \mathcal{I}^c, 1 \le j \le n$.

On the other hand, for the case of $(A_i^d - I)^T v \prec 0, \forall i \in \mathcal{I}^d$. Let $e^{(j)} = (0, \cdots, 1, \cdots, 0)^T$ denote the unit vector whose jth element is 1, we have $(A_i^{d(j)} - e^{(j)})^T v \prec 0$ with $A_i^{d(j)}$ is the jth column of one of $A_{m^c+1}^d, \cdots, A_{m^c+m^d}^d$ for all $i \in \mathcal{I}^d, 1 \le j \le n$.

Based on the above discussion, taking (3) and (4) into account, it turns out from Lemma 3 that $A^T v \prec 0$ for all $A \in \mathscr{A}$. Moreover, as Metzler matrices $A_1^c, \cdots, A_{m^c}^c$ in $\mathbb{R}^{n\times n}$ are Hurwitz and $A_{m^c+1}^d, \cdots, A_{m^c+m^d}^d \succeq 0$ in $\mathbb{R}^{n\times n}$ are Schur, we thus conclude from Lemma 2 that all matrices belonging to the set $\mathscr{A}$ must be Hurwitz. Hence, the statement ((i) holds.

((i)⇔((iii) Note that the finite sets $\mathscr{A}$ consists entirely of Hurwitz matrices and $m^c + m^d \le m^* \le (m^c + m^d)^n$, then the rest of the proof follows the same lines as the proof of ((i)⇔((ii). ■

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Remark 1: We should stress out that, in statement ((iii), the range of m^* is limited to the interval $[m^c + m^d, (m^c + m^d)^n]$, which means that we can arbitrarily choose matrices in such range to generate some SPLSs and under Theorem 1 these SPLSs are obvious uniformly asymptotically stable. In addition, note that the construction of the set $\mathscr{A}$ given in (4), then it easy to see that the set $\mathscr{A}$ has $(m^c + m^d)^n$ elements. Hence, the equivalence between ((i) and ((ii) in Theorem 1 indicates that the existence of a common LCLF for the finite sets composed of positive LTI systems (1) and (2) is equivalent to the Schur stability of $(m^c + m^d)^n$ matrices in $\mathscr{A}$. This will also be the equivalence between ((i) and ((iii). Moreover, note that the range of m^* and the construction of the set $\mathscr{A}$, the statement ((iii) immediately reduces the statement ((ii) if we choose $m^* = m^c + m^d$.

Remark 2: Obviously, if all matrices $A \in \mathscr{A}$ are Hurwitz, i.e., the finite system sets composed of (1) and (2) have a common LCLF. Then each of continuous-time systems $\Sigma_{A_i^c}, i \in \mathcal{I}^c$ and discrete-time systems $\Sigma_{A_i^d}, i \in \mathcal{I}^d$ is stable. In addition, for any $A \in \mathscr{A}$, the associated continuous-time positive system Σ_A is also stable. See Corollary 1.

Remark 3: Observe that, for the SLPS composed of (1) and (2). If select $m^d = 0$ or $m^c = 0$, the system reduces continuous- or discrete-time SLPSs, respectively. Corollary 2 considers such two reduced cases.

Remark 4: If we note the fact that, for 2×2 Hurwitz matrices with negative diagonal entries, this is equivalent to the determinants being positive. Then the following reduced results in Corollary 3-5 follow immediately from Theorem 1.

Corollary 1: If the finite sets composed of positive LTI systems $\Sigma_{A_1^c}, \cdots, \Sigma_{A_{m^c}^c}, \Sigma_{A_{m^c+1}^d}, \cdots, \Sigma_{A_{m^c+m^d}^d}$ have a common LCLF. Then each of the following statements is true. (i)

(i) The continuous-time LTI positive system $x(k+1) = Ax(k), \quad x_0 = x(0) \succeq 0$ is stable with any matrices A in $\mathscr{A}(A_1, \cdots, A_m)$.
(ii) Each continuous-time positive LTI systems $\Sigma_{A_i^c}, i \in \mathcal{I}^c$ is stable.
(iii) Each discrete-time positive LTI systems $\Sigma_{A_i^d}, i \in \mathcal{I}^d$ is stable.

Corollary 2: Consider SPLS composed of (1) and (2) with Metzler Hurwitz $A_i^c, i \in \mathcal{I}^c$ and Schur $A_i^d \succeq 0, i \in \mathcal{I}^d$, then the following statements hold. (i)

(i) If $m^d = 0$ and there exists a matrix A_j^c such that $A_j^c \succeq A_i^c, j \neq i, \forall i$, then continuous-time LTI positive systems $\Sigma_{A_1^c}, \cdots, \Sigma_{A_{m^c}^c}$ share a CLCLF.
(ii) If $m^c = 0$ and there exists a matrix A_j^d such that $A_j^d \succeq A_i^d, j \neq i, \forall i$, then discrete-time LTI positive systems $\Sigma_{A_1^d}, \cdots, \Sigma_{A_{m^c}^d}$ share a CLCLF.

Proof: By Lemma 4, these results follows from Theorem 1 immediately. ■

Corollary 3: Let A^c be Metzler and Hurwitz matrix, $A^d \succeq 0$ be schur matrix, and $A^c, A^d \in \mathbb{R}^{2\times 2}$. Then positive LTI systems $\Sigma_{A^c}, \Sigma_{A^d}$ have a common LCLF if and only if the following conditions are satisfied.

$$\begin{vmatrix} 1 - a_{11}^d & a_{12}^c \\ a_{21}^d & -a_{22}^c \end{vmatrix} > 0, \quad \begin{vmatrix} -a_{11}^c & a_{12}^d \\ a_{21}^c & 1 - a_{22}^d \end{vmatrix} > 0.$$

Corollary 4: [15] Let $A_1^d, A_2^d \succeq 0$ in $\mathbb{R}^{2\times 2}$ be schur matrices. Then the discrete-time positive LTI systems $\Sigma_{A_1^d}, \Sigma_{A_2^d}$ have a common LCLF if and only if the following conditions are satisfied.

$$\begin{vmatrix} 1 - a_{111}^d & a_{212}^d \\ a_{121}^d & 1 - a_{222}^d \end{vmatrix} > 0, \quad \begin{vmatrix} 1 - a_{211}^d & a_{112}^d \\ a_{221}^d & 1 - a_{122}^d \end{vmatrix} > 0.$$

Corollary 5: [13] Let $A_1^c, A_2^c \succeq 0$ in $\mathbb{R}^{2\times 2}$ Metzler and Hurwitz matrices. Then the continuous-time positive LTI systems $\Sigma_{A_1^c}, \Sigma_{A_2^c}$ have a common LCLF if and only if the following conditions are satisfied.

$$\begin{vmatrix} -a_{111}^c & a_{212}^c \\ a_{121}^c & -a_{222}^c \end{vmatrix} > 0, \quad \begin{vmatrix} -a_{211}^c & a_{112}^c \\ a_{221}^c & -a_{122}^c \end{vmatrix} > 0.$$

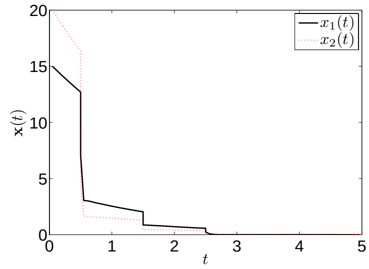

Fig. 1. The state variables of system (20).

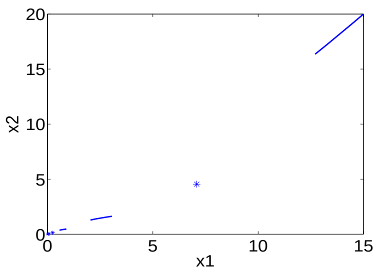

Fig. 2. The trajectory of system (20).

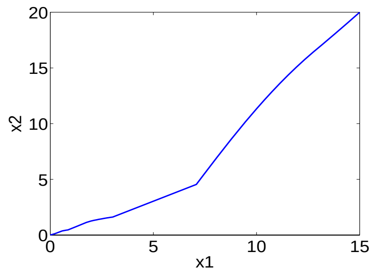

Fig. 3. The trajectory of system (20).

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IV. Example

As a simple example for Theorem 1, consider the SPLS composed of Σ_{A^c} and Σ_{A^d} with

$$A^c = \begin{pmatrix} -0.6 & 0.1 \\ 0.2 & -0.7 \end{pmatrix}, \quad A^d = \begin{pmatrix} 0.3 & 0.2 \\ 0.1 & 0.2 \end{pmatrix}. \qquad (20)$$

From (3) and (4), we have

$$\mathscr{A} = \left\{ \begin{bmatrix} -0.6 & 0.1 \\ 0.2 & -0.7 \end{bmatrix}, \begin{bmatrix} -0.7 & 0.2 \\ 0.1 & -0.8 \end{bmatrix}, \begin{bmatrix} -0.6 & 0.2 \\ 0.2 & -0.8 \end{bmatrix}, \begin{bmatrix} -0.7 & 0.1 \\ 0.1 & -0.7 \end{bmatrix} \right\}.$$

Then it is easy to check that all matrices in $\mathscr{A}$ are Hurwitz, which thus means that Σ_{A^c} and Σ_{A^d} have a common LCLF. In other words, the SPLS (20) is uniformly asymptotically stable. See Figure 1, Figure 2 and Figure 3. Figure 1 shows the positivity of SPLS (20). Figure 2 and Figure 3 show the uniform asymptotical stability of the SPLS, where the switching signal is generated randomly, the initial state is $[15\ 20]^T$. The mark '*' in the Figure 2 indicates the state change when the discrete-time subsystem Σ_{A^d} is activated. Figure 3 connects all the sampling points of the subsystem Σ_{A^d} into a continuous trajectory.

V. Conclusions

In this paper, we have investigated the existence of a LCLF for a special class of SPLSs composed of continuous- and discrete-time subsystems. Some necessary and sufficient conditions have been presented for the existence of a common LCLF. According to these conditions, we can easily verify the existence of a common LCLF for some given sets composed of continuous- and discrete-time positive LTI systems by algebraic approach. The example given in Section IV shows this advantage.

Acknowledgment

The authors would like to thank the associate editor and the anonymous reviewers for their detailed comments and suggestions.

This work was supported by the Foundation of National Nature Science of China(Grant No.11071178) and the Fostering Plan for Young and Middle Age Leading Research of UESTC(Grant No.Y02018023601033).

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