

Non-Polynomial Spline Method for the Solution of Problems in Calculus of Variations

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Abstract—In this paper, a numerical solution based on non-polynomial cubic spline functions is used for finding the solution of boundary value problems which arise from the problems of calculus of variations. This approximation reduce the problems to an explicit system of algebraic equations. Some numerical examples are also given to illustrate the accuracy and applicability of the presented method.

Keywords—Calculus of variation; Non-polynomial spline functions; Numerical method

INTRODUCTION

THE calculus of variations and its extensions are devoted to finding the optimum function that gives the best value of the economic model and satisfies the constraints of a system. The need for an optimum function, rather than an optimal point, arises in numerous problems from a wide range of fields in engineering and physics, which include optimal control, transport phenomena, optics, elasticity, vibrations, statics and dynamics of solid bodies and navigation[1]. In computer vision the calculus of variations has been applied to such problems as estimating optical flow[2] and shape from shading [3]. Several numerical methods for approximating the solution of problems in the calculus of variations are known. Galerkin method is used for solving variational problems in [4]. The Ritz method [5], usually based on the subspaces of kinematically admissible complete functions, is the most commonly used approach in direct methods of solving variational problems. Chen and Hsiao [6] introduced the Walsh series method to variational problems. Due to the nature of the Walsh functions, the solution obtained was piecewise constant. Some orthogonal polynomials are applied on variational problems to find the continuous solutions for these problems [7-9]. A simple algorithm for solving variational problems via Bernstein orthonormal polynomials of degree six is proposed by Dixit et al. [10]. Razzaghi et al. [11] applied a direct method for solving variational problems using Legendre wavelets. He's variational iteration method has been employed for solving some problems in calculus of variations in [12].

Spline functions are special functions in the space of which approximate solutions of ordinary differential equations. In other words spline function is a piecewise polynomial,

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satisfying certain conditions of continuity of the function and its derivatives. The applications of spline as approximating interpolating and curve fitting functions have been very successful[13-16]. Quadratic and cubic polynomial and non-polynomial spline functions based methods have been presented to find approximate solutions to second order boundary value problems[17]. Khan [18] used parametric cubic spline function to develop a numerical method, which is fourth order for a specific choice of the parameter. The main purpose of the present paper is to use non-polynomial cubic spline method for numerical solution of boundary value problems which arise from problems of calculus of variations. The method consists of reducing the problem to a set of algebraic equations. The outline of the paper is as follows. First, in Section 2, we introduce the problems in calculus of variations and explain their relations with boundary value problems. Section 3 outlines non-polynomial cubic spline and basic equations that are necessary for the formulation of the discrete system. Also in this section, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering two numerical examples.

II. STATEMENT OF THE PROBLEM

The general form of a variational problem is finding extremum of the

$$J[u_1(t), u_2(t), \dots, u_n(t)] = \int_a^b G(t, u_1(t), u_2(t), \dots, u_n(t), u_1'(t), u_2'(t), \dots, u_n'(t)) dt. \quad (1)$$

To find the extreme value of J , the boundary conditions of the admissible curves are known in the following form:

$$u_i(a) = \gamma_i, \quad i = 1, 2, \dots, n, \quad (2)$$

$$u_i(b) = \delta_i, \quad i = 1, 2, \dots, n. \quad (3)$$

The necessary condition for $u_i(t)$, $i = 1, 2, \dots, n$, to extremize $J[u_1(t), u_2(t), \dots, u_n(t)]$ is to satisfy the Euler-Lagrange equations that is obtained by applying the well known procedure in the calculus of variation [5],

$$\frac{\partial G}{\partial u_i} - \frac{d}{dt} \left(\frac{\partial G}{\partial u_i'} \right) = 0, \quad i = 1, 2, \dots, n \quad (4)$$

subject to the boundary conditions given by Eqs. (2)-(3).

In this paper, we consider the special form of the variational problem(1) as

$$J[u(t)] = \int_a^b G(t, u(t), u'(t)) dt, \quad (5)$$

with boundary conditions

$$u(a) = \gamma, \quad u(b) = \delta, \quad (6)$$

and

$$J[u_1(t), u_2(t)] = \int_a^b G(t, u_1(t), u_2(t), u_1'(t), u_2'(t)) dt \quad (7)$$

subject to boundary conditions

$$u_1(a) = \gamma_1, \quad u_1(b) = \delta_1, \quad (8)$$

$$u_2(a) = \gamma_2, \quad u_2(b) = \delta_2. \quad (9)$$

Thus, for solving the variational problems (5), we consider the second order differential equation

$$\frac{\partial G}{\partial u} - \frac{d}{dt} \left(\frac{\partial G}{\partial u'} \right) = 0, \quad (10)$$

with the boundary condition (6). And also, for solving the variational problems (7), we find the solution of the system of second-order differential equations

$$\frac{\partial G}{\partial u_i} - \frac{d}{dt} \left(\frac{\partial G}{\partial u_i'} \right) = 0, \quad i = 1, 2, \quad (11)$$

with the boundary conditions (8)-(9). Therefore, by applying non-polynomial cubic spline method for the Euler-Lagrange equations (10) and (11) we can obtain an approximate solution to the variational problems (5) and (7).

III. Non-polynomial Cubic spline method

Consider the partition $\Delta = \{t_0, t_1, t_2, \dots, t_n\}$ of $[a, b] \subset R$.

Let $S_k(\Delta)$ denote the set of piecewise polynomials of degree k on subinterval $I_i = [t_i, t_{i+1}]$ of partition Δ . In this work, we consider non-polynomial cubic spline method for finding approximate solution of variational problems.

Consider the grid points t_i on the interval $[a, b]$ as follows:

$$a = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = b \quad (12)$$

$$t_i = t_0 + ih, \quad i = 0, 1, 2, \dots, n, \quad (13)$$

$$h = \frac{b-a}{n}, \quad (14)$$

where n is a positive integer. Let $u(t)$ be the exact solution of the Eq.(10) and $S_i(t)$ be an approximation to

$u_i = u(t_i)$ obtained by the segment $P_i(t)$. Each non-polynomial spline segment $P_i(t)$ has the form:

$$P_i(t) = a_i \sin k(t-t_i) + b_i \cos k(t-t_i) + c_i(t-t_i) + d_i \quad (15)$$

$$i = 0, 1, 2, \dots, n-1,$$

where a_i, b_i, c_i and d_i are constants and k is the frequency of the trigonometric functions which will be used to raise the accuracy of the method and Eq. (15) reduce to cubic polynomial spline function in $[a, b]$ when $k \rightarrow 0$.

We consider the following relations:

$$P_i(t_{i+j}) = u_{i+j}, \quad j = \frac{1}{2}, \frac{3}{2},$$

$$P_i'(t_{i+j}) = D_{i+j}, \quad j = \frac{1}{2}, \frac{3}{2}, \quad (16)$$

$$P_i''(t_{i+j}) = M_{i+j}, \quad j = \frac{1}{2}, \frac{3}{2}.$$

We can obtain the values of a_i, b_i, c_i and d_i via a straightforward calculation as follows:

$$a_i = h^2 \frac{M_{i+\frac{1}{2}} \cos \theta - M_{i+\frac{3}{2}}}{\theta^2 \sin \theta}, \quad b_i = -h^2 \frac{M_{i+\frac{1}{2}}}{\theta^2}, \quad (17)$$

$$c_i = \frac{D_{i+1} + D_{i+\frac{1}{2}}}{2} - h \frac{(M_{i+\frac{1}{2}} \cos \theta - M_{i+\frac{3}{2}})(1 + \cos \theta)}{2\theta \sin \theta}$$

$$- h \frac{M_{i+\frac{1}{2}} \sin \frac{\theta}{2}}{2\theta}, \quad (18)$$

$$d_i = u_{i+\frac{1}{2}} + h^2 \frac{M_{i+\frac{1}{2}}}{\theta^2}, \quad (19)$$

where $\theta = kh$ and $i = 0, 1, \dots, n-1$. Using the continuity conditions $P_{i-1}^{(n)}(x_{i+\frac{1}{2}}) = P_i^{(n)}(x_{i+\frac{1}{2}})$, $n = 0, 1$, we get the

following relations for $i = 0, 1, \dots, n-1$:

$$\frac{h}{2} (D_i + D_{i-\frac{1}{2}}) = u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}} + M_{i+\frac{1}{2}} \left(\frac{1}{k^2} - \frac{h(\cos \frac{\theta}{2} + 1)}{2k \sin \theta} \right) +$$

$$M_{i-\frac{1}{2}} \left(\frac{h \cos \theta \cos \frac{\theta}{2} + h \cos \theta + h \sin \theta \sin \frac{\theta}{2}}{2k \sin \theta} - \frac{1}{k^2} \right), \quad (20)$$

and

$$\begin{aligned} & \frac{1}{2}(D_i + D_{i-\frac{1}{2}} - D_{i+1} - D_{i+\frac{1}{2}}) = \\ & M_{i+\frac{1}{2}} \left(\frac{\cos\theta}{k \sin\theta} - \frac{\sin\frac{\theta}{2}}{2k} - \frac{\cos\theta(1+\cos\frac{\theta}{2})}{2k \sin\theta} + \frac{1}{k \tan\theta} \right. \\ & \left. - \frac{1+\cos\frac{\theta}{2}}{2k \sin\theta} \right) + M_{i-\frac{1}{2}} \left(\frac{-\cos\theta}{k \tan\theta} - \frac{\sin\theta}{k} + \frac{\cos\theta(1+\cos\frac{\theta}{2})}{2k \sin\theta} \right. \\ & \left. + \frac{\sin\frac{\theta}{2}}{2k} \right) + M_{i+\frac{3}{2}} \left(-\frac{1}{k \sin\theta} + \frac{1+\cos\frac{\theta}{2}}{2k \sin\theta} \right). \end{aligned} \quad (21)$$

By reducing the indices of Eqs. (20) and (21) by one, we get the following equations:

$$\begin{aligned} & \frac{1}{2}(D_{i-1} + D_{i-\frac{3}{2}}) = \frac{u_{i-\frac{1}{2}} - u_{i-\frac{3}{2}}}{h} + M_{i-\frac{1}{2}} \left(\frac{1}{hk^2} - \frac{\cos\frac{\theta}{2} + 1}{2k \sin\theta} \right) \\ & + M_{i-\frac{3}{2}} \left(\frac{\cos\frac{\theta}{2} + \cos\theta}{2k \sin\theta} - \frac{1}{hk^2} \right), \end{aligned} \quad (22)$$

and also

$$\begin{aligned} & \frac{1}{2}(D_{i-1} + D_{i-\frac{3}{2}} - D_i - D_{i+\frac{1}{2}}) = \\ & M_{i-\frac{1}{2}} \left(\frac{\cos\theta}{k \sin\theta} - \frac{\sin\frac{\theta}{2}}{2k} - \frac{\cos\theta(1+\cos\frac{\theta}{2})}{2k \sin\theta} + \frac{1}{k \tan\theta} - \frac{1+\cos\frac{\theta}{2}}{2k \sin\theta} \right) \\ & + M_{i-\frac{3}{2}} \left(\frac{-\cos\theta}{k \tan\theta} - \frac{\sin\theta}{k} + \frac{\cos\theta(1+\cos\frac{\theta}{2})}{2k \sin\theta} + \frac{\sin\frac{\theta}{2}}{2k} \right) + \\ & M_{i+\frac{1}{2}} \left(\frac{-1}{k \sin\theta} + \frac{1+\cos\frac{\theta}{2}}{2k \sin\theta} \right). \end{aligned} \quad (23)$$

D_{i+j} , $j = -\frac{3}{2}, -1, 0, \frac{1}{2}$ are eliminated from Eq. (23) by using

Eq. (22). As a result we get the following scheme:

$$\begin{aligned} & u_{i+\frac{1}{2}} - 2u_{i-\frac{1}{2}} + u_{i-\frac{3}{2}} = \\ & h^2 [\alpha M_{i-\frac{3}{2}} + 2\beta M_{i-\frac{1}{2}} + \alpha M_{i+\frac{1}{2}}], \quad i = 2, 3, \dots, n-1 \end{aligned} \quad (24)$$

where

$$\alpha = \frac{1}{\theta \sin\theta} - \frac{1}{\theta^2}, \quad \beta = \frac{1}{\theta^2} - \frac{\cos\theta}{\theta \sin\theta} \quad (25)$$

In order to illustrate the performance of the non-polynomial cubic spline method, we present two examples.

Example 1. We first consider the following variational problem with the exact solution $u(t) = e^{3t}$ in [12]:

$$\min J = \int_0^1 (u(t) + u'(t) - 4e^{3t}) dt, \quad (26)$$

subject to boundary conditions

$$u(0) = 1, \quad u(1) = e^3. \quad (27)$$

Considering the Eq. (26), the Euler-Lagrange equation of this problem can be written in the following form:

$$u''(t) - u(t) - 8e^{3t} = 0. \quad (28)$$

The solution of the second-order differential equation (28) with boundary conditions (27) is approximated by the presented spline method. For our purpose, We consider the boundary value problem (28) in general form as follows:

$$u''(t) = g(t)u(t) + f(t), \quad (29)$$

Where $g(t) = 1$ and $f(t) = 8e^{3t}$. The exact solution of this problem is $u(t) = e^{3t}$. For a numerical solution of the boundary-value problem (29), the interval $[0,1]$ is divided into a set of grid points with step size h . Setting $t = t_{i+j}$, $j = -\frac{3}{2}, -1, \frac{1}{2}$, in Eq. (29), we obtain

$$u''_{i+j} = g(t_{i+j})u_{i+j} + f(t_{i+j}), \quad (30)$$

by using the assumption $P''(t_{i+j}) = M_{i+j}$ in (30) we have

$$M_{i+j} = g(t_{i+j})u_{i+j} + f(t_{i+j}), \quad j = -\frac{3}{2}, -1, \frac{1}{2}. \quad (31)$$

Replacing M_{i+j} as Eq. (31) in Eq.(24), we get

$$\begin{aligned} & u_{i+\frac{1}{2}} - 2u_{i-\frac{1}{2}} + u_{i-\frac{3}{2}} = h^2 [\alpha(g(t_{i-\frac{3}{2}})u_{i-\frac{3}{2}} + f(t_{i-\frac{3}{2}})) \\ & + 2\beta(g(t_{i-\frac{1}{2}})u_{i-\frac{1}{2}} + f(t_{i-\frac{1}{2}})) + \alpha(g(t_{i+\frac{1}{2}})u_{i+\frac{1}{2}} + f(t_{i+\frac{1}{2}}))], \\ & i = 2, 3, \dots, n-1 \end{aligned} \quad (32)$$

(24) or

with boundary conditions (40) and (41). In a similar manner and applying (24), we assume that functions $u_1(t)$ and

$$u_2(t) \text{ defined over the interval } [0, \frac{\pi}{2}] \text{ are approximated by}$$

$$P_{1,i}(t) = a_{1,i} \sin k(t-t_i) + b_{1,i} \cos k(t-t_i) + c_{1,i}(t-t_i) + d_{1,i},$$

$$i = 0, 1, \dots, n-1, \quad (43)$$

$$P_{2,i} = a_{2,i} \sin k(t-t_i) + b_{2,i} \cos k(t-t_i) + c_{2,i}(t-t_i) + d_{2,i},$$

$$i = 0, 1, \dots, n-1, \quad (44)$$

Where $a_{j,i}$, $b_{j,i}$, $c_{j,i}$ and $d_{j,i}$, $j = 1, 2$ are constants and k is the frequency of the trigonometric functions. Similarly, we can obtain the following results:

$$\begin{cases} u_{1,i+\frac{1}{2}} - 2u_{1,i-\frac{1}{2}} + u_{1,i-\frac{3}{2}} = \\ h^2[\alpha M_{1,i-\frac{3}{2}} + 2\beta M_{1,i-\frac{1}{2}} + \alpha M_{1,i+\frac{1}{2}}], \quad i = 2, 3, \dots, n-1, \\ u_{2,i+\frac{1}{2}} - 2u_{2,i-\frac{1}{2}} + u_{2,i-\frac{3}{2}} = \\ h^2[\alpha M_{2,i-\frac{3}{2}} + 2\beta M_{2,i-\frac{1}{2}} + \alpha M_{2,i+\frac{1}{2}}], \quad i = 2, 3, \dots, n-1, \end{cases} \quad (45)$$

where α and β are defined in (25). Now, consider the system (42) and substitute $t = t_i$, thus we can write:

$$u_{1,i}'' = u_{2,i}, \quad u_{2,i}'' = u_{1,i}, \quad (46)$$

consequently, we have:

$$M_{1,i} = u_{2,i}, \quad M_{2,i} = u_{1,i}. \quad (47)$$

By using relations (45) -(47), we get:

$$\begin{cases} u_{1,i+\frac{1}{2}} - 2u_{1,i-\frac{1}{2}} + u_{1,i-\frac{3}{2}} = \\ h^2[\alpha u_{2,i-\frac{3}{2}} + 2\beta u_{2,i-\frac{1}{2}} + \alpha u_{2,i+\frac{1}{2}}], \quad i = 2, 3, \dots, n-1, \\ u_{2,i+\frac{1}{2}} - 2u_{2,i-\frac{1}{2}} + u_{2,i-\frac{3}{2}} = \\ h^2[\alpha u_{1,i-\frac{3}{2}} + 2\beta u_{1,i-\frac{1}{2}} + \alpha u_{1,i+\frac{1}{2}}], \quad i = 2, 3, \dots, n-1, \end{cases} \quad (48)$$

The system (48) contains $2(n-2)$ equations with $2n$ unknown coefficients $u_{j,i-\frac{1}{2}}$, $j = 1, 2$, $i = 1, \dots, n$. To

obtain unique solution, four more equations are needed. These equations are found by using method of undetermined coefficients and are given below:

$$\begin{cases} 2u_{1,0} - 3u_{1,\frac{1}{2}} + u_{1,\frac{3}{2}} = \frac{h^2}{24}(15u_{2,\frac{1}{2}} + 3u_{2,\frac{3}{2}}), \quad i = 1, \\ u_{1,n-\frac{3}{2}} - 3u_{1,n-\frac{1}{2}} + 2u_{1,n} = \frac{h^2}{24}(3u_{2,n-\frac{3}{2}} + 15u_{2,n-\frac{1}{2}}), \quad i = n, \end{cases} \quad (49)$$

and

$$\begin{cases} 2u_{2,0} - 3u_{2,\frac{1}{2}} + u_{2,\frac{3}{2}} = \frac{h^2}{24}(15u_{1,\frac{1}{2}} + 3u_{1,\frac{3}{2}}), \quad i = 1, \\ u_{2,n-\frac{3}{2}} - 3u_{2,n-\frac{1}{2}} + 2u_{2,n} = \frac{h^2}{24}(3u_{1,n-\frac{3}{2}} + 15u_{1,n-\frac{1}{2}}), \quad i = n. \end{cases} \quad (50)$$

The Eqs. (48)-(50) produce a linear system that contains $2n$ equations with $2n$ unknown coefficients. Solving this linear system, we can obtain the approximate solution of the system of second-order boundary value problems(42).

Suppose $\|E_{u_1}(h)\|_\infty$ and $\|E_{u_2}(h)\|_\infty$ be the maximum absolute errors. We solved Example 2 for different values of n . The maximum of absolute errors on the uniform grid points (37) are tabulated in Table II .

TABLE II
RESULTS FOR EXAMPLE 2

n	h	$\ E_{u_1}(h)\ _\infty$	$\ E_{u_2}(h)\ _\infty$
4	0.392699	1.50959×10^{-5}	1.50959×10^{-5}
8	0.196350	1.56763×10^{-6}	1.56763×10^{-6}
16	0.098175	1.06058×10^{-7}	1.06058×10^{-7}
32	0.049087	6.75003×10^{-9}	6.75003×10^{-9}
64	0.024544	4.23519×10^{-10}	4.23519×10^{-10}
128	0.012272	2.65017×10^{-11}	2.65017×10^{-11}

IV. CONCLUSION

In this paper non-polynomial cubic spline method employed for finding the extremum of a functional over the specified domain. The main purpose is to find the solution of boundary value problems which arise from the variational problems. The non-polynomial cubic spline method reduce the computation of boundary value problems to some algebraic equations. The proposed scheme is simple and computationally attractive. Applications are demonstrated through illustrative examples

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