

# Adaptation of Iterative Methods to Solve Fuzzy Mathematical Programming Problems

Ricardo C. Silva, Luiza A. P. Cantão, and Akebo Yamakami

**Abstract**—Based on the fuzzy set theory this work develops two adaptations of iterative methods that solve mathematical programming problems with uncertainties in the objective function and in the set of constraints. The first one uses the approach proposed by Zimmermann to fuzzy linear programming problems as a basis and the second one obtains cut levels and later maximizes the membership function of fuzzy decision making using the bound search method. We outline similarities between the two iterative methods studied. Selected examples from the literature are presented to validate the efficiency of the methods addressed.

**Keywords**—Fuzzy Theory, Nonlinear Optimization, Fuzzy Mathematics Programming.

## I. INTRODUCTION

**O**PERATIONAL Research (OR) is an area of science that develops techniques to optimize systems efforts. Applications of OR are found in business, industrial, military and governmental areas, among others. The study of OR can be divided in several subareas and mathematical programming is one of them. Mathematical programming is used to solve problems that involve minimization (or maximization) of the objective function in a function domain that can be constrained or not.

Optimization models have used traditional mathematical programming, which attempts to represent the optimization operation of interest by constructing an exact mathematical model. The studies may overlook ambiguities, that exist in actual optimization operations. In recent years, Fuzzy Logic [1] has shown great potential for modelling systems which are non-linear, complex, ill-defined and not well understood. Fuzzy Logic has found numerous applications due to its ease of implementation, flexibility, tolerant nature to imprecise data, and ability to model non-linear behavior of arbitrary complexity because of its basis in terms of natural language.

A fuzzy set is defined by a membership function  $\mu_{\tilde{A}}(x)$ , that establishes to each  $x$  a pertinence degree to the set  $A$ ,

with  $\mu_{\tilde{A}} \in [0, 1]$ . One way to express a fuzzy set as:

$$\mu_{\tilde{A}} = \begin{cases} \frac{x - a_{inf}}{a_{mod} - a_{inf}} & x \in [a_{inf}, a_{mod}] \\ \frac{a_{sup} - x}{a_{sup} - a_{mod}} & x \in [a_{mod}, a_{sup}] \\ 0 & \text{otherwise,} \end{cases}$$

where  $a_{mod}$  is modal value,  $a_{inf}$  and  $a_{sup}$  are lower and upper bounds, respectively. Figure 1 shows the membership function described for equation above.

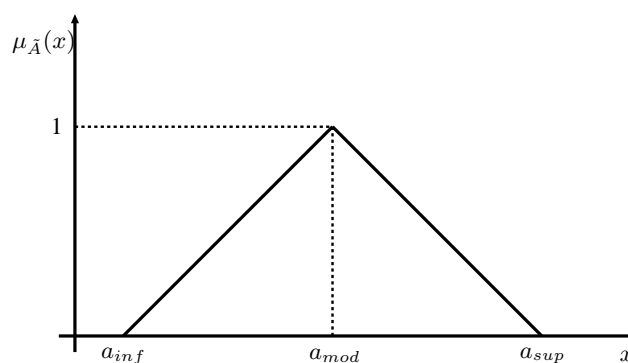


Fig. 1 Membership function  $\mu_{\tilde{A}}(x)$

Mathematical programming problems need a precise definition of both the constraints and the objective function to be optimized. Fuzzy sets help handle uncertainties when mathematical programming problems are formalized in the following form:

$$\begin{aligned} \min \quad & f(\tilde{\mathbf{a}}; \mathbf{x}) \\ \text{s.t.} \quad & g_k(\mathbf{x}) \lesssim b_k, \quad k = 1, \dots, m \\ & h_l(\mathbf{x}) \cong c_l, \quad l = 1, \dots, n \\ & \mathbf{x} \geq 0 \end{aligned} \quad (1)$$

where  $\tilde{\mathbf{a}}$  represents a fuzzy parameter and,  $\lesssim$  and  $\cong$  represent the uncertainties in the constraints.

This work is divided as follows. Section II deals with the constraint types and the increasing computational effort increase in some mathematical problems; section III introduces the adaptations of two versions of iterative methods to solve mathematical programming problems with fuzzy parameters in the objective function and uncertainties in set of constraints; section IV presents numerical simulations for selected problems and an analysis of the obtained results. Finally, concluding remarks are found in section V.

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## II. SET OF CONSTRAINTS

Mathematical programming problems can have three types constraint formulation: (i) equality constraints; (ii) inequality constraints; (iii) equality and inequality mixed constraints.

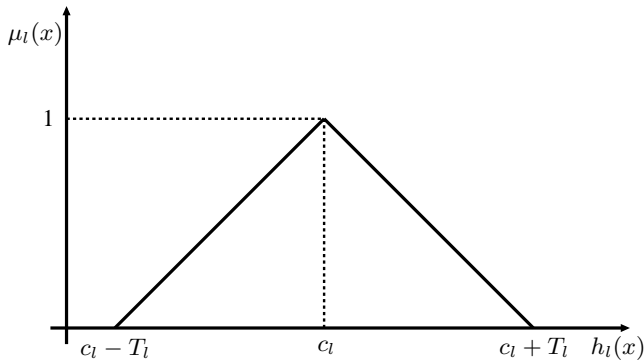


Fig. 2 Membership function of the equality constraint  $h_l(\mathbf{x}) = c_l$

Figure 2 shows the membership function of the equality constraints with uncertainties, while Figures 3 and 4 show the membership functions of the inequality constraints with uncertainties.

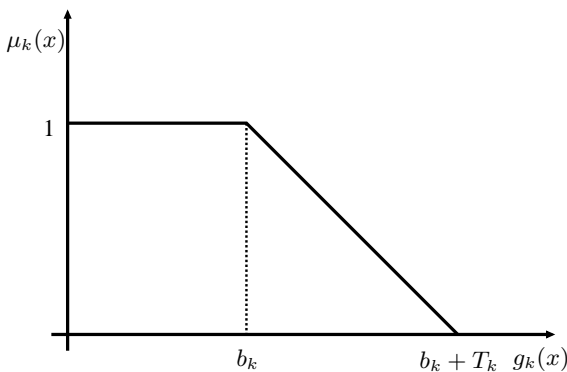


Fig. 3 Membership function of the inequality constraint  $g_k(\mathbf{x}) \lesssim b_k$ .

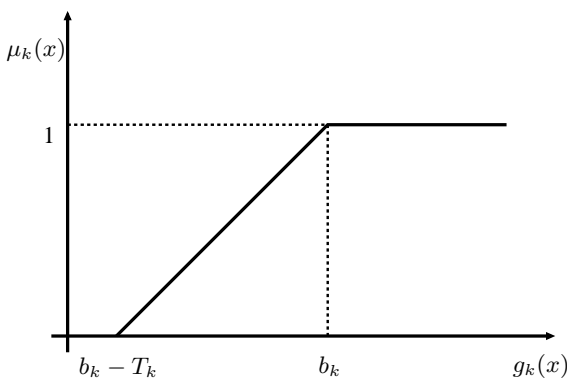


Fig. 4 Membership function of the inequality constraint  $g_k(\mathbf{x}) \gtrsim b_k$ .

In the literature, each equality constraint is partitioned into two inequality constraints, as presented in Figures 3 and 4. Thus, the computational effort increases with the number of equality constraints.

The adaptations presented in this work solve only mathematical programming problems with inequality constraints.

Thus, each equality constraint is generates originally two inequality constraints. Therefore, Figure 2 is divided in the Figures 3 and 4, where  $c_l = b_k$  and  $T_l = T_k$ . Hence, the number of constraints in Problem (1) is  $m + 2n$ , where  $m$  is the number of inequality constraints of original problem and  $2n$  represent the inequality constraints generated from the  $n$  equality constraints.

In [2], a transformation of an equality constraint in an only inequality constraint using the module function, without loss of generality.

## III. ITERATIVE METHODS

The methods presented in this section are proposed to solve mathematical programming problems with uncertainties both in objective function and in set of constraints. These methods are adaptations of the methods developed to solve problems with uncertainties in the set of constraints, described in [3]–[5], which adapt classic methods that solve classic mathematical programming problems.

In [6] it was developed an adaptation of classic methods to solve fuzzy linear programming problems, while in [3] and [4] this method was generalized to solve fuzzy nonlinear programming problems. Based on this generalization, we consider in subsection III-A some modifications to solve Problem (1).

Another optimization method to solve fuzzy nonlinear programming problems has been described in [5]. Subsection III-B presents an approach to solve problems with fuzzy parameters in the objective function and uncertainties in set of constraints.

### A. Adaptation of Zimmermann's Method

A mathematical programming problem with a fuzzy objective function and fuzzy constraints can be described generically, in the from of Problem (1), as follows:

$$\begin{aligned} \min \quad & g_0(\tilde{\mathbf{a}}; \mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \lesssim b_i, \quad i = 1, \dots, m + 2n \\ & \mathbf{x} \in \Omega, \end{aligned} \quad (2)$$

where  $\tilde{a}$  represents a fuzzy parameter in the objective function and  $\lesssim$  denotes the uncertainties in the set of constraints. It can be noted that the formulation without  $\sim$  is the original classic mathematical programming model. The objective function with fuzzy parameters can be interpreted as:

$$\text{IDf}(g_0(\tilde{a}; x)) \lesssim \text{IDf}(\tilde{b}_0)$$

where  $\text{IDf}(\cdot)$  represent the defuzzification function, that uses Yager's first index, described in [7]–[9].

As suggested in [6], we transform each fuzzy inequality constraint into a classic inequality constraint. This transformation is done performed by introducing variable  $t_j$ ,  $j = 0, 1, \dots, m + 2n$ , that indicate the fuzzy constraints violation

level of the fuzzy constraints. Thus, Problem (2) becomes

$$\begin{aligned} \text{IDf}(g_0(\tilde{\mathbf{a}}; \mathbf{x})) &\leq \text{IDf}(\tilde{b}_0) + \text{IDf}(\tilde{t}_0) \\ g_i(\mathbf{x}) &\leq b_i + t_i, \quad i = 1, \dots, m + 2n \\ \mathbf{x} &\geq \mathbf{0} \\ 0 &\leq \text{IDf}(\tilde{t}_0) \leq \text{IDf}(\tilde{T}_0) \\ 0 &\leq t_i \leq T_i \end{aligned}$$

where  $T_j = [\text{IDf}(\tilde{T}_0), T_1, \dots, T_{m+2n}]$  is the maximum tolerable violation of each constraints.

The membership functions  $\mu_0(g_0(\tilde{\mathbf{a}}, \mathbf{x}))$  and  $\mu_i(g_i(\mathbf{x}))$ , with  $i = 0, 1, \dots, m + 2n$ , are described below:

$$\mu_0(g_0(\tilde{\mathbf{a}}, \mathbf{x})) = \begin{cases} 0, & \text{se } \text{IDf}(\tilde{t}_0) \geq \text{IDf}(\tilde{T}_0) \\ 1 - \frac{\tilde{t}_0}{\tilde{T}_0}, & \text{se } 0 < \text{IDf}(\tilde{t}_0) < \text{IDf}(\tilde{T}_0) \\ 1, & \text{se } \text{IDf}(\tilde{t}_0) \leq 0 \end{cases}$$

$$\mu_i(g_i(\mathbf{x})) = \begin{cases} 0, & \text{se } t_i \geq T_i \\ 1 - \frac{t_i}{T_i}, & \text{se } 0 < t_i < T_i \\ 1, & \text{se } t_i \leq 0 \end{cases}$$

where the satisfaction level for the decision maker can be defined by the degree of membership to each constraint  $j$ . However, we can define an aggregate function for each constraint as:

$$\bigcap_{0 \leq j \leq m+2n} \bigcap_{\mathbf{x}} \mu_j(g_j(\mathbf{x})) \in [0, 1]$$

where  $\bigcap$  denotes intersection.

Then, the degree of conjunctive satisfaction can be maximized over all constraints. Setting  $\tilde{S} = \min_{0 \leq j \leq m+2n} \mu_j(g_j)$ , the fuzzy mathematical model can be formulated as:

$$\begin{aligned} \max \quad & \tilde{S} \\ \text{s.t.} \quad & (a) \quad \text{IDf}(g_0(\tilde{\mathbf{a}}; \mathbf{x}) - \tilde{b}_0 - \tilde{T}_0 * (1 - \tilde{S})) \leq 0 \\ & (b) \quad g_i(\mathbf{x}) - b_i - T_i * (1 - \text{IDf}(\tilde{S})) \leq 0 \quad (3) \\ & (c) \quad \tilde{S} \in [0, 1], \mathbf{x} \in \Omega, \\ & (d) \quad \tilde{T}_0 \geq \tilde{0}, T_i \geq 0, \quad i = 1, \dots, m + 2n \end{aligned}$$

Note that the constraint (a) is the objective function of original problem and the constraints (b), (c) e (d) represent the set of constraints of the original problem. The objective  $\tilde{S}$  is represented as fuzzy number because the membership function of the original objective objective is a fuzzy value.

### B. Adaptation of Xu's Method

The Two-Phase Method transforms a fuzzy optimization problem into classical equivalent. This is done by changing Problem (2) into:

$$\begin{aligned} \min \quad & g_0(\tilde{\mathbf{a}}; \mathbf{x}) \\ \text{s.a} \quad & g_i(\mathbf{x}) \leq b_i + T_i, \quad i = 1, \dots, m + 2n \quad (4) \\ & \mathbf{x} \in \Omega \end{aligned}$$

However, using a function  $\mu_i(\mathbf{x}) : \mathbb{R}^n \rightarrow [0, 1]$ , we have distinct satisfaction levels inside of the unitary interval  $[0, 1]$ . The membership function for each inequality constraint can be expressed in two forms, depending on the inequality type, as follows:

1) Decreasing membership function, Figure 3:

$$\mu_i(g_i(\mathbf{x})) = \begin{cases} 0, & \text{se } g_i(\mathbf{x}) \geq b_i + T_i \\ \frac{(b_i + T_i) - g_i(\mathbf{x})}{T_i}, & \text{se } b_i < g_i(\mathbf{x}) < b_i + T_i \\ 1, & \text{se } g_i(\mathbf{x}) \leq b_i \end{cases}$$

2) Increasing membership function, Figure 4:

$$\mu_i(g_i(\mathbf{x})) = \begin{cases} 0, & \text{se } g_i(\mathbf{x}) \leq b_i - T_i \\ \frac{g_i(\mathbf{x}) - (b_i - T_i)}{T_i}, & \text{se } b_i - T_i < g_i(\mathbf{x}) < b_i \\ 1, & \text{se } g_i(\mathbf{x}) \geq b_i - T_i \end{cases}$$

On the other hand, we need to parameterize the set of constraints using the  $\alpha$ -cut levels defined in fuzzy logic, [1], such that

$$C_\alpha = \{x \mid x \in \mathbb{R}^n, \mu_c(x) \geq \alpha\}, \quad \forall \alpha \in [0, 1].$$

An intersection is applied to all constraints with the aggregate operator in the form

$$\mu_C(\mathbf{x}) = \min_{i=1}^{m+2n} \mu_i(\mathbf{x}), \quad \forall x \in \mathbb{R}^n,$$

where the aggregate operator is the minimum function.

Thus, the Problem (4) can be transformed into a fuzzy problem into a parametric classical problem.

$$\begin{aligned} \min \quad & g_0(\tilde{\mathbf{a}}; \mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq b_i + T_i(1 - \alpha), \\ & \mathbf{x} \in \Omega, \quad \alpha \in [0, 1], \quad i = 1, \dots, m + 2n \end{aligned} \quad (5)$$

However, some values of the objective function depend on the parameter  $\alpha$ .

The function  $g_0(\mathbf{x}^*(\alpha))$  shown in Figure 5, represents fuzzy solutions to Problem (5), for  $\alpha \in [0, 1]$ , which is a monotone increasing function.

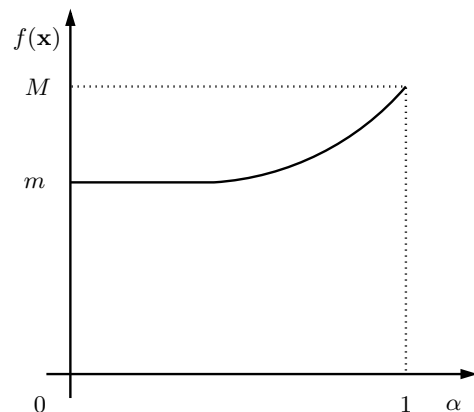


Fig. 5 Fuzzy solution

Generally speaking, the fuzzy decision  $D$  characterizes by its membership function  $\mu_D$  may be viewed as the intersection of the fuzzy constraints and fuzzy goal, of according to Bellman and Zadeh [10].

$$\mu_D = \mu_C \cap \mu_{\tilde{G}},$$

where  $\mu_D, \mu_{\tilde{G}}, \mu_C : \mathbb{F}(\mathbb{R}^n) \rightarrow [0, 1]$ . The fuzzy constraints  $C$  and the fuzzy goal  $G$  in (5) are defined as fuzzy sets in the space of alternatives, characterized by their membership functions  $\mu_C$  and  $\mu_D$ , respectively.

The optimal decision is to select the best alternative from those contained in the fuzzy decision space, which maximizes the membership function of the fuzzy decision, i.e.

$$\mu_D(\mathbf{x}^*) = \max_{x \in \mathbb{R}^n} \mu_D(\mathbf{x}). \quad (6)$$

In order to illustrate the above principle, let us imagine one fuzzy goal  $G$  with one fuzzy constraint  $C$ . The membership functions  $\mu_G, \mu_C$  and their intersection  $\mu_D$  are plotted for this case in Figure 6. This figure also shows that the point  $A$  represent the optimal decision which has the maximum degree of membership in the fuzzy decision set.

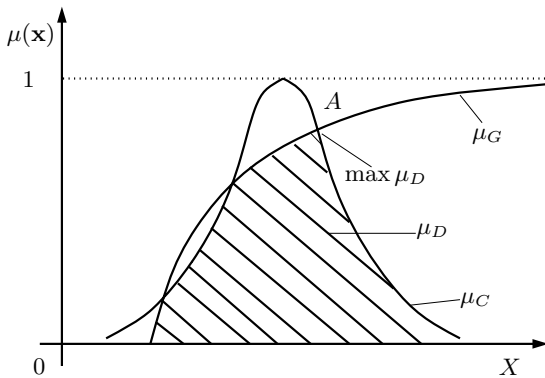


Fig. 6 Fuzzy decision making

Equation (6) can obtain a optimum level  $\alpha^*$  and the optimum point  $x^*$  such that

$$\mu_{\tilde{G}}(\mathbf{x}^*) = \max_{x \in C_{\alpha^*}} \mu_{\tilde{G}}(\mathbf{x}), \quad (7)$$

where  $C_{\alpha^*}$  is the  $\alpha^*$ -cut level of the fuzzy constraints set  $C$ .

The fuzzy solution inside of a limited interval is given by the upper and lower values, described in Figure 5 and obtained by function  $\mu_G(\mathbf{x})$  as follows

$$\begin{aligned} \tilde{m} &= g_0(\tilde{\mathbf{a}}; \mathbf{x}^*(0)) = \min_{\mathbf{x} \in C_0} g_0(\tilde{\mathbf{a}}; \mathbf{x}) \\ \tilde{M} &= g_0(\tilde{\mathbf{a}}; \mathbf{x}^*(1)) = \min_{\mathbf{x} \in C_1} g_0(\tilde{\mathbf{a}}; \mathbf{x}), \end{aligned}$$

where  $C_0, C_1$  are the cut levels of  $\alpha = 1$  and 0 of the fuzzy constraints set  $C$ . The method presented in [11] relates to the iterative methods presented here. In that work the value  $\tilde{m}$  and  $\tilde{M}$  are respectively equal to  $\tilde{b}_0$  and  $\tilde{T}_0$ .

In a the fuzzy optimization problem we can establish the fuzzy goal as follows:

$$\mu_{\tilde{G}}(\mathbf{x}) = \frac{\tilde{m}}{g_0(\tilde{\mathbf{a}}; \mathbf{x})}. \quad (8)$$

As expected, this fuzzy goal shows that when  $f$  reaches its infimum  $m$  the full membership ( $\mu_G = 1$ ) is obtained; as  $f$  increases  $\mu_G$  approaches the non-membership ( $\mu_G = 0$ ) boundary. Clearly, the upper and lower limits of the fuzzy goal are given by

$$\begin{aligned} \mu_{\tilde{G}}^u &= 1 \\ \mu_{\tilde{G}}^l &= \text{Df} \left( \frac{\tilde{m}}{\tilde{M}} \right). \end{aligned} \quad (9)$$

In addition, substituting Equation (8) into Equation (7), we have

$$\mu_{\tilde{G}}(\mathbf{x}^*) = \frac{1}{\tilde{m}} \min_{\mathbf{x} \in C_{\alpha^*}} g_0(\tilde{\mathbf{a}}; \mathbf{x}) \quad (10)$$

This approach allow is to optimize a fuzzy problem by means of the bound search method on the set of feasible solutions.

#### IV. NUMERICAL EXPERIMENTS

Subsection IV-A shows the formulation of the problems. The problems we use to evaluate the iterative methods are hypothetic formulations. Nevertheless, they are efficient to validate the study realized.

The computational results and a comparative analysis of the classic methods and the iterative methods responses are presented in section IV-B.

The tests were all performed on a Sun Blade 250 with two 1.28GHZ Ultra Sparc-IIIi processor, 4GB RAM running Solaris 9 operational system.

##### A. Formulation of the Problem

In this paper, we present some theoretical problems found in the literature with a view to validate the proposed algorithms. We simulate three nonlinear programming problems.

Uncertainties were inserted into the parameters of the objective functions in the form of a 10% variation in the modal value, e.g. the number  $\tilde{2}$  can vary up to 0.2 units positively or negatively. The optimal solutions to the problems without uncertainties are presented in the columns  $\bar{\mathbf{x}}^t$  and  $f(\mathbf{x})$  of tables I. The problems with inequality constraints, described in Table I, were copied from [12].

##### B. Results and Analysis

In this subsection we show the results obtained for the problems in section IV-A by the iterative methods introduced in section III. Table II depicts the optimal solutions of the problems in two forms: (i) totally satisfied constraints; and (ii) totally violated constraints. Table II shows the results for Problem (5) imposing  $\alpha = 1$ , for case (i), and  $\alpha = 0$ , for case (ii).

By examining the results presented in Table II, we can calculate the minimum satisfaction level to each problem in Table I. The main analysis is in choosing the relation between objective function value and satisfaction level. This choice depends on the decision maker because he has a previous knowledge of the main objective to be reached.

Table III shows the results obtained by the adaptation of Zimmermann's method, adaptation of Xu's method for the problem PG1. For this problem, the adaptation of Xu's method obtained better responses in every front, i.e., lower defuzzification value and lower convergence time, while the satisfaction level is a admissible value. The delayed procedure more was the adaptation of Zimmermann's method.

In Table IV we explore the results for the problem PG2. Note that the adaptations of Zimmermann's and Xu's methods present similar responses in terms of defuzzification value of objective function and satisfaction level, but processing time of Xu's method was higher.

The results for the problem PG3 are presented in Table V. For this example, the three methods achieved satisfaction levels higher than 90%. The adaptation of Zimmermann's method obtained better responses in every front, i.e., lower defuzzification value, higher satisfaction level and lower convergence time.

## V. CONCLUSION

We adapt Zimmermann's and Xu's methods to solve mathematical programming problems with uncertainties in objective function and in set of constraints. We present two iterative methods that transform each equality constraint into two inequality constraints. The introduced algorithms are similar in the sense that their convergence points are very close to each other.

The two iterative methods that use derived from the objective function presented good responses to hypothetic problems. The obtained results were better than the classic results found in the literature. However, they presented a satisfaction level lower than 100%, i.e., the optimum solution violates one or more problem constraints.

## ACKNOWLEDGMENT

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TABLE I  
 PROBLEMS WITH INEQUALITY CONSTRAINTS

Prob.	$f(\bar{a}; x)$	Variation Fuzzy	$x_{initial}$	Constraints	Violation	Classic solution	
						$\bar{x}^T$	$f(x)$
PG1	$(x_1 - \bar{2})^2 + (x_2 - \bar{1})^2$	10%	$[0.5, 0.5]^T$	$g_1(x) = x_1^2 - x_2 \leq 0$ $g_2(x) = x_2^2 - x_1 \leq 0$	$T_1 = 0.35$ $T_2 = 0.35$	$[1, 1]^T$	1
PG2	$x_1^2 + x_2^2 - 4x_1 + 4$	10%	$[1.0, 2.5]^T$	$g_1(x) = x_2 - x_1 - 2 \leq 0$ $g_2(x) = x_1^2 - x_2 + 1 \leq 0$ $g_3(x) = -x_1 \leq 0$ $g_4(x) = -x_2 \leq 0$	$T_1 = 1.0$ $T_2 = 0.5$ $T_3 = 0.0$ $T_4 = 0.0$	$[0.5536, 1.306]^T$	3.79894
PG3	$\bar{9}x_1^2 + x_2^2 + \bar{9}x_3^2$	10%	$[1.0, 1.0, 1.0]^T$	$g_1(x) = 1 - x_1x_2 \leq 0$ $g_2(x) = -x_2 \leq 0$ $g_3(x) = x_3 \leq 0$	$T_1 = 0.5$ $T_2 = 0.0$ $T_3 = 0.0$	$[0.5774, 1.732, -0.2 \times 10^{-5}]^T$	6

TABLE II  
 MAXIMUM AND MINIMUM LEVELS OF TOLERANCE FOR THE SET OF CONSTRAINTS

Prob.	Constraints	Optimal of $f(\bar{a}; x^*)$			Temp
		$x^*$	$f(\bar{x}; x^*)$	$F(f(\bar{a}; x^*))$	
PG1	Totally Satisfied	$[1.0005, 1.0004]^T$	$[0.80655, 0.99905, 1.2115]$	1.0041	2s
	Totally Violated	$[1.2074, 1.2073]^T$	$[0.50443, 0.67117, 0.86292]$	0.67739	1s
PG2	Totally Satisfied	$[0.55304, 1.3052]^T$	$[3.1761, 3.7973, 4.4185]$	3.7973	4s
	Totally Violated	$[0.57687, 1.2322]^T$	$[2.9128, 3.5436, 4.1743]$	3.5436	15s
PG3	Totally Satisfied	$[0.57802, 1.7287, -0.8 \times 10^{-12}]^T$	$[5.6945, 5.9952, 6.2959]$	5.9952	8s
	Totally Violated	$[0.54934, 1.6369, -0.1 \times 10^{-11}]^T$	$[5.1237, 5.3953, 5.6669]$	5.3953	11s

TABLE III  
 RESULT TO THE PROBLEMS

Prob.	Algorithm	Optimal of $f(\bar{a}; x^*)$				Temp
		$x^*$	$f(\bar{x}; x^*)$	IDf( $f(\bar{a}; x^*)$ )	$\mu$	
PG1	Subsection III-A	$[1.0535, 1.0534]^T$	$[0.71657, 0.89871, 1.1059]$	0.90494	0.77643	39s
	Subsection III-B	$[1.058, 1.0578]^T$	$[0.70907, 0.89076, 1.0974]$	0.89698	0.75749	4s

TABLE IV  
 RESULT TO THE PROBLEMS

Prob.	Algorithm	Optimal of $f(\bar{a}; x^*)$				Temp
		$x^*$	$f(\bar{x}; x^*)$	IDf( $f(\bar{a}; x^*)$ )	$\mu$	
PG2	Subsection III-A	$[0.55094, 1.2965]^T$	$[3.1603, 3.7807, 4.4011]$	3.7807	0.93768	2s
	Subsection III-B	$[0.55473, 1.3015]^T$	$[3.1607, 3.7826, 4.4045]$	3.7826	0.9361	11s

TABLE V  
 RESULT TO THE PROBLEMS

Prob.	Algorithm	Optimal of $f(\bar{a}; x^*)$				Temp
		$x^*$	$f(\bar{x}; x^*)$	IDf( $f(\bar{a}; x^*)$ )	$\mu$	
PG3	Subsection III-A	$[0.57749, 1.6907, 0.9 \times 10^{11}]^T$	$[5.5598, 5.8599, 6.1601]$	5.8599	0.92252	3s
	Subsection III-B	$[0.5764, 1.7191, -0.2 \times 10^{18}]^T$	$[5.6464, 5.9454, 6.2444]$	5.9454	0.91687	5s