World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:5, No:4, 2011

Topological Properties of an Exponential Random Geometric Graph Process

Yilun Shang

Abstract—In this paper we consider a one-dimensional random geometric graph process with the inter-nodal gaps evolving according to an exponential AR(1) process. The transition probability matrix and stationary distribution are derived for the Markov chains concerning connectivity and the number of components. We analyze the algorithm for hitting time regarding disconnectivity. In addition to dynamical properties, we also study topological properties for static snapshots. We obtain the degree distributions as well as asymptotic precise bounds and strong law of large numbers for connectivity threshold distance and the largest nearest neighbor distance amongst others. Both exact results and limit theorems are provided in this paper.

Keywords—random geometric graph, autoregressive process, degree, connectivity, Markovian, wireless network.

I. INTRODUCTION

MANY randomly deployed networks, such as wireless sensor networks, are properly characterized by random geometric graphs (RGGs). Given a specified norm on the space under consideration, an RGG is usually obtained by placing a set of n vertices independently at random according to some spatial probability distribution and connecting two vertices by an edge if and only if their distance is less than a critical cutoff r. Topological properties of RGGs are comprehensively analyzed in e.g. [16], [22], [23]; also see [8] for a latter survey in the context of wireless networks. Although extensive simulations and empirical studies are performed in dynamical RGGs, analytical treatments of topological properties are merely done in static RGGs in the previous work. A recent paper [4] is a remarkable exception, in which the authors conduct the first analytical research on the connectivity of mobile RGG in the torus $[0, 1)^2$. In this paper, we will also present analytical results and consider an one-dimensional exponential RGG process $G(t, r, \Lambda)$ evolving with time, where vertices are randomly placed along a semi-infinite line. One-dimensional exponential RGGs are newly investigated by some authors[7], [10], [11], which offer a significant variant from the familiar uniformly $U[0, 1]$ distributed nodes, see e.g.[3], [6], [8], [21] and references therein.

In [12], the distributions of distances between successive vertices rather than those of vertices themselves are examined, and as it is stated in the same paper, this assumption is more natural since "*sensors are usually thrown one by one along a trajectory of a vehicle*." We will then follow suit, and assume exponential distributions for inter-nodal distances of the graph process $G(t, r, \Lambda)$. Every segment between two successive vertices is supposed to evolve following a stationary TEAR(1) process[13] with exponential marginal. This linear process has no zero-defect and thus surpasses the elementary AR(1) process involved in [11]. We believe such mobile scheme has broad potential applications due to the flexible double randomness mechanism (see Section 2). Since the evolution of connectivity and the number of components in $G(t, r, \Lambda)$ are both Markovian, we will address the transition probabilities and limiting distributions of these two process G_t and G'_t respectively by employing Markov chain theory[17], [19]. It is worth noting that there are several Markov chains coupled in our model stemming from the first order autoregressive properties endowed in the evolution of inter-nodal distances.

In addition to dynamical properties, we also establish static properties for fixed t. Vertices in $G(t, r, \Lambda)$, for any given t, form nearly a Poisson point process (more precisely, a continuous time pure birth Markov process). Connectivity of Poisson RGG is well-studied in the literature (e.g.[1], [5], [14], [15]), especially in the context of ad hoc networks. We will investigate some topological properties basically along the lines of [7]. We give new results as well as corroborate some known results (see Section 6.1) by different approach. We mention that, in our opinion, the aforementioned simple idea in [12] reflects a conception of one step "memory" essentially. We show (Theorem 8) that "1-step memory" + "growth" are not enough to produce power law distribution reminiscent of the architecture of Polya urn process, where typically infinite memory generates the power law [2].

In this paper both exact and asymptotic formula are provided. We remark here that exact solutions are important since the asymptotic results can not be applied to real network when not knowing the rate of convergence.

The rest of this paper is organized as follows. Section 2 gives definition of the exponential RGG process and some preliminaries. Section 3 and 4 deal with the transition probability matrix, stationary distribution of G_t and G'_t respectively. Section 5 includes the analysis for hitting time of G_t for disconnectivity. In Section 6, we present some topological properties for snapshot of $G(t, r, \Lambda)$. The degree distribution and strong laws of connectivity and the largest nearest neighbor distances are given among other things. Section 7 contains further discussion and some open problems.

Y. Shang is with the Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China e-mail: shylmath@hotmail.com, shyl@sjtu.edu.cn. This paper is an extended version of a work [20] presented in CyberC 2009.

II. MODEL AND PRELIMINARIES

The RGG process $G(t, r, \Lambda)$ is constructed as a discrete time process with n vertices deployed in one dimension on $[0, \infty)$.

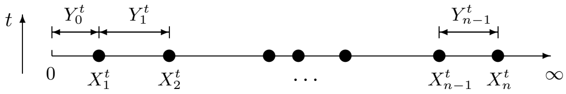

Fig. 1. One-dimensional exponential RGG process. Envision time evolving upward, and possibly n growing along x-axis.

Let $X_1^t, \cdots, X_n^t$ denote the vertices of the network at time t, for $t \geq 0$. Set $Y_l^t := X_{l+1}^t - X_l^t$, for $l = 1, 2, \cdots, n-1$ and $Y_0^t := X_1^t$, see Fig.1.

For $0 \leq p < 1$, we assume that $\{Y_l^t\}$ evolves following:

$$Y_l^{t+1} = \begin{cases} Y_l^t + \varepsilon_l^t & w.p. \quad p \\ \varepsilon_l^t & w.p. \quad 1-p \end{cases} \tag{1}$$

where the innovation sequences $\{\varepsilon_l^t\}_{t \geq 0}$ consist of i.i.d. nonnegative random variables. The behavior of this autoregressive process $\{Y_l^t\}_{t \geq 0}$ is characterized by runs of rising values (with geometrically distributed run length) when choosing $Y_l^t + \varepsilon_l^t$, followed by a sharp fall when choosing ε_l^t without inclusion of the previous values. Furthermore, we assume that Y_l^t, $l = 0, 1, \cdots, n-1$ are independent for any t.

In particular, we set $\varepsilon_l^t := (1-p)Z_l^t$, where $Z_l^t \sim Exp(\lambda_l)$ is an exponential random variable with mean $\lambda_l^{-1} > 0$. Let $\Lambda := \{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}\}$. In this case, as is shown in [13], the above TEAR(1) process $\{Y_l^t\}_{t>0}$ would be a stationary sequence of marginally exponentially distributed random variables with parameter λ_l, assuming that the initial internodal gaps Y_l^0 are exponentially distributed with parameter λ_l. That means $Y_l^t \sim Exp(\lambda_l)$. In this case, the auto correlation function of $\{Y_l^t\}$ is $\mathrm{Corr}(Y_l^t, Y_l^{t+j}) = p^j$, being nonnegative. [9] showed that (1) is stationary for each $0 \leq p < 1$ iff Y_l^t is geometrically infinitely divisible. For further extension and discussion of (1) we refer the reader to [18].

Vertices in snapshot of $G(t, r, \Lambda)$ constitute a counting process with inter-nodal distances having distribution $Exp(\lambda_l)$, while in standard exponential RGG, the corresponding distributions are relevant to n the total number of vertices (see [7] Lemma 1); hence relying on the global information. Besides, notice that the cutoff $r = r(n, t)$ may depend on n and t. However, we restrict ourselves to fixed r in order to keep calculations clear though some results may be generalized without much effort. The popular assumption $\lim_{n\to\infty} r(n) = 0$ is not necessary here in virtue of unbounded support.

III. Stationary distribution of G_t

Let us denote by $\mathcal{C}_t$ and $\mathcal{D}_t$ the events that $G(t, r, \Lambda)$ is connected and disconnected at time t, respectively. Define G_t as a discrete time stochastic process describing connectivity of the graph process $G(t, r, \Lambda)$. Therefore $\mathcal{C}_t = \{G_t = \text{“conneted”}\}$ and $\mathcal{D}_t = \{G_t = \text{“disconneted”}\}$. It's easy to see that G_t is a homogeneous Markov chain, assuming the cutoff r is independent of t. We abbreviate as usual the states as 1=“connected”($\mathcal{C}$) and 2=“disconnected”($\mathcal{D}$). Our main results in this section then read as follows:

Theorem 1. *G_t is a time-reversible, homogeneous finite Markov chain, with one step transition probability matrix*

$$P(n) = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix},$$

where

$$p_{11} = \prod_{l=1}^{n-1} \left(1 - \frac{(1-p)e^{-\lambda_l r}\left(1 - e^{-\frac{\lambda_l r}{1-p}}\right)}{1 - e^{-\lambda_l r}} \right), \tag{2}$$

$$\begin{aligned} p_{21} &= \frac{1}{1 - \prod_{l=1}^{n-1}(1 - e^{-\lambda_l r})} \\ &\quad \cdot \Bigg(\sum_{\emptyset \neq A \subseteq [n-1]} (1-p) \prod_{l \in A} e^{-\lambda_l r}\left(1 - e^{-\frac{\lambda_l r}{1-p}}\right) \\ &\quad \cdot \prod_{l \in [n-1] \setminus A} \Big(1 - e^{-\lambda_l r} - (1-p)e^{-\lambda_l r} \\ &\quad \cdot \left(1 - e^{-\frac{\lambda_l r}{1-p}}\right)\Big) \Bigg), \end{aligned} \tag{3}$$

$p_{12} = 1 - p_{11}$ *and* $p_{22} = 1 - p_{21}$.

Theorem 2. *G_t has a unique stationary distribution $\pi(n) = (\pi_1(n), \pi_2(n))$, where*

$$\begin{cases} \pi_1(n) = & \frac{(1-p_{22})^2}{p_{11}(1-p_{22})^2 + p_{21}p_{12}(2-p_{22})}, \\ \pi_2(n) = & \frac{(1-p_{11})^2}{p_{22}(1-p_{11})^2 + p_{12}p_{21}(2-p_{11})} \end{cases} \tag{4}$$

Theorem 3. *Suppose $\lambda_l \equiv \lambda$, for $l = 0, 1, \cdots, n-1$. Let $P(\infty)$ be the transition probability matrix of G_t as n tends to infinity, and $\pi(\infty)$ the (unique) stationary distribution corresponding to $P(\infty)$. Then $\pi(\infty) = (0, 1)$ and*

$$\lim_{n\to\infty} \pi(n)P(n) = \pi(\infty)P(\infty).$$

Theorem 3 implies that we can swap the order of obtaining stationary distribution and taking limit w.r.t. n.

Proof of Theorem 1. The probability density function of ε_l^t can be shown to be given by $f_l(s) = \frac{\lambda_l}{1-p} e^{-\lambda_l s/(1-p)} 1_{[s>0]}$. Also, the conditional density function for Y_l^t in the connected network is $g_{Y_l|\mathcal{C}}(y) = \frac{\lambda_l e^{-\lambda_l y}}{1 - e^{-\lambda_l r}} 1_{[0<y<r]}$, since the connectivity of network means $Y_l^t < r$ for all l. By independence property, we have $p_{11} = P(\mathcal{C}_{t+1}|\mathcal{C}_t) = \prod_{l=1}^{n-1} P(Y_l^{t+1} < r | Y_l^t < r)$. Our aim now turns to evaluate the probability $P(Y_l^{t+1} < r | Y_l^t < r)$. Let $V_l^t \sim Bin(p)$ independently, then the scheme (1) becomes

$$Y_l^{t+1} = \varepsilon_l^t + V_l^t Y_l^t. \tag{5}$$

Let $\widetilde{Y}_l^{t+1}$ denote Y_l^{t+1} conditional on $\{Y_l^t < r\}$. For a nonnegative random variable X with density function $f(x)$, Laplace-Stieltjes transform is defined by $\mathcal{L}(X)(s) =$

$\mathcal{L}(f)(s)=\int_0^\infty f(x)e^{-sx}\mathrm{d}x$. We have by (5),

$$\begin{aligned}\mathcal{L}(\widetilde{Y}_l^{t+1})(s) &= \mathcal{L}(\varepsilon_l^t)(s)\cdot\mathcal{L}(V_l^t\widetilde{Y}_l^t)(s)\\ &= \int_0^\infty e^{-su}\frac{\lambda_l}{1-p}e^{-\frac{\lambda_l u}{1-p}}\mathrm{d}u\\ &\quad\cdot\int_0^r e^{-sy}\Big((1-p)\delta(y)+\frac{p\lambda_l e^{-\lambda_l y}}{1-e^{-\lambda_l r}}\Big)\mathrm{d}y\\ &= \frac{\lambda_l}{\lambda_l+s(1-p)}\\ &\quad\cdot\Big((1-p)+\frac{p\lambda_l(1-e^{-(\lambda_l+s)r})}{(s+\lambda_l)(1-e^{-\lambda_l r})}\Big)\end{aligned}$$

where $\delta(y)$ is the Dirac-delta function. Inverting the above to get

$$\begin{aligned}\mathcal{L}^{-1}(\mathcal{L}(\widetilde{Y}_l^{t+1}))(y) &= \lambda_l e^{-\frac{\lambda_l y}{1-p}}1_{[y>0]}+\frac{2\lambda_l e^{-\frac{\lambda_l(2-p)y}{2(1-p)}}}{1-e^{-\lambda_l r}}\\ &\quad\cdot\mathrm{sh}\Big(\frac{\lambda_l py}{2(1-p)}\Big)1_{[y>0]}\\ &\quad-\frac{2\lambda_l e^{-\lambda_l\left(r+\frac{(2-p)(y-r)}{2(1-p)}\right)}}{1-e^{-\lambda_l r}}\\ &\quad\cdot\mathrm{sh}\Big(\frac{\lambda_l p(y-r)}{2(1-p)}\Big)1_{[y>r]}.\end{aligned}$$

Hence

$$\begin{aligned}P(Y_l^{t+1}<r|Y_l^t<r) &= \int_0^r\mathcal{L}^{-1}(\mathcal{L}(\widetilde{Y}_l^{t+1}))(y)\mathrm{d}y\\ &= 1-\frac{(1-p)e^{-\lambda_l r}}{1-e^{-\lambda_l r}}\\ &\quad\cdot\big(1-e^{-\frac{\lambda_l r}{1-p}}\big) \qquad (6)\end{aligned}$$

which gives (2).

Let $\emptyset\neq A\subseteq[n-1]$. Denote the event $E_A:=\{Y_l^t>r,\forall l\in A;Y_l^t<r,\forall l\in[n-1]\backslash A\}$, then

$$\begin{aligned}P(\mathcal{C}_{t+1}|E_A) &= \prod_{l\in A}P(Y_l^{t+1}<r|Y_l^t>r)\\ &\quad\cdot\prod_{l\in[n-1]\backslash A}P(Y_l^{t+1}<r|Y_l^t<r)\\ &= \prod_{l\in[n-1]\backslash A}\Big(1-\frac{(1-p)e^{-\lambda_l r}\big(1-e^{-\frac{\lambda_l r}{1-p}}\big)}{1-e^{-\lambda_l r}}\Big)\\ &\quad\cdot\prod_{l\in A}(1-p)\big(1-e^{-\frac{\lambda_l r}{1-p}}\big),\end{aligned}$$

where we used the expression $P(Y_l^{t+1}<r|Y_l^t>r)=(1-p)\big(1-e^{-\frac{\lambda_l r}{1-p}}\big)$. Since $P(E_A)=\prod_{l\in A}e^{-\lambda_l r}\prod_{l\in[n-1]\backslash A}(1-e^{-\lambda_l r})$ and $P(\mathcal{D}_t)=1-\prod_{l=1}^{n-1}(1-e^{-\lambda_l r})$, (3) follows by noting that

$$\begin{aligned}p_{21} &= P(\mathcal{C}_{t+1}|\mathcal{D}_t)\\ &= \sum_{\emptyset\neq A\subseteq[n-1]}P(\mathcal{C}_{t+1}|E_A)\cdot P(E_A)/P(\mathcal{D}_t).\end{aligned}$$

G_t is time-reversible by standard results of Markov chain[17]. □

Proof of Theorem 2. Since G_t is an irreducible finite Markov chain, $\mathcal{C}$ and $\mathcal{D}$ are both positive recurrent. Also since they are both non-periodical, $\mathcal{C}$ and $\mathcal{D}$ are ergodic state. Set $T_{ij}:=\min\{k:k\geq 1,G_k=j,G_0=i\}$, for $i,j\in\{1,2\}$. If the righthand side of the above definition is $\emptyset$, set $T_{ij}=\infty$. The first hitting probability is then given by $f_{ij}^{(k)}=P(T_{ij}=k|G_0=i)$.

By a standard result from [19], an irreducible ergodic Markov chain has unique stationary distribution $\pi(n)$, and $\pi_i(n)$ is given by $\pi_i(n)=1/\sum_{k=1}^\infty kf_{ii}^{(k)}$, for i=1,2 in the present case. Thereby, (4) follows easily from the facts $f_{11}^{(1)}=p_{11}$, $f_{11}^{(k)}=p_{21}p_{22}^{k-2}p_{12}$, for $k\geq 2$; and $f_{22}^{(1)}=p_{22}$, $f_{22}^{(k)}=p_{12}p_{11}^{k-2}p_{21}$, for $k\geq 2$. □

Proof of Theorem 3. When $\lambda_l\equiv\lambda$, the righthand side of expression (6) belongs to interval $(0,1)$. Hence p_{11} tends to 0 as $n\to\infty$ in view of (2). Since $(1-p)e^{-\lambda r}\big(1-e^{-\frac{\lambda r}{1-p}}\big)+\big(1-e^{-\lambda r}-(1-p)e^{-\lambda r}\big(1-e^{-\frac{\lambda r}{1-p}}\big)\big)=1-e^{-\lambda r}<1$, p_{21} tends to 0 as $n\to\infty$ by the binomial theorem and (3). Then we have $P(\infty)=\begin{pmatrix}0&1\\0&1\end{pmatrix}$. In this case, $\mathcal{C}$ is a transient state and $\mathcal{D}$ is an absorbing and positive recurrent state. By a standard result (see e.g. [19]), the stationary distribution corresponding to $P(\infty)$ exists and is unique. Direct calculation gives $\pi(\infty)=(0,1)$. It is straightforward to verify that $\pi(n)\to\pi(\infty)$ as n tends to infinity. The theorem is thus concluded by exploiting the relation $\pi P=\pi$. □

IV. Transition probability matrix of G'_t

In this section we show a refinement stochastic process G'_t from G_t. To be precise, let $\{G'_t=i\}$ denote the event that $G(t,r,\Lambda)$ has i components at time t, for $1\leq i\leq n$. Therefore, G'_t is a homogeneous Markov chain with state space $[n]$. It's clear that $\{G'_t=1\}=\mathcal{C}_t$.

Let the transition probabilities of G'_t be $p'_{ij}:=P(G'_{t+1}=j|G'_t=i)$. Set $A,B\subseteq[n-1]$ with $|A|=i-1$ and $|B|=j-1$, $1\leq i,j\leq n$. Denote the event $E_A:=\{Y_l^t>r,\forall l\in A;Y_l^t<r,\forall l\in[n-1]\backslash A\}$ and similarly for E_B. We obtain by the total probability formula,

$$\begin{aligned}p'_{ij} &= \sum_{\substack{A,B\subseteq[n-1]\\|A|=i=1,|B|=j-1}}P(E_B|E_A)\cdot P(E_A)/P(G'_t=i),\\ &\qquad 1\leq i,j\leq n. \qquad (7)\end{aligned}$$

We have derived $P(E_A)$ in the proof of Theorem 1, and $P(G'_t=i)=\sum_{A\subseteq[n-1],|A|=i-1}P(E_A)$. To evaluate (7), we still need the probability $P(E_B|E_A)$, but it is also at hand already:

$$\begin{aligned}P(E_B|E_A) &= \prod_{l\in A\cap B}P(Y_l^{t+1}>r|Y_l^t>r)\\ &\quad\cdot\prod_{l\in A\backslash B}P(Y_l^{t+1}<r|Y_l^t>r)\\ &\quad\cdot\prod_{l\in B\backslash A}P(Y_l^{t+1}>r|Y_l^t<r)\\ &\quad\cdot\prod_{l\in[n-1]\backslash A\cup B}P(Y_l^{t+1}<r|Y_l^t<r).\end{aligned}$$

The second and fourth terms in the above expression have been obtained in the proof of Theorem 1, and clearly $P(Y_l^{t+1} > r|Y_l^t > r) = 1 - P(Y_l^{t+1} < r|Y_l^t > r), P(Y_l^{t+1} > r|Y_l^t < r) = 1 - P(Y_l^{t+1} < r|Y_l^t < r)$. Now we arrive at the main result.

Theorem 4. *The transition probability matrix of G'_t is $P' = (p'_{ij})_{n\times n}$, which is given by (7).*

Of course, we have $p'_{11} = p_{11}$ and $\sum_{j=2}^n p'_{1j} = p_{12}$. Since G'_t is an irreducible ergodic chain, it has a unique stationary distribution which may be deduced analogously as in Section 3.

V. Hitting time

Suppose $\mathcal{C}_t$ holds at time t, and we will consider the Markov chain G_t. Denote $T := \min\{k : k \geq 1, \mathcal{D}_{t+k} \text{ holds}\}$, then T is the hitting time for disconnectivity. We could obtain the expectation of T using the transition probabilities derived in Section 3 by a routine approach[19]. In this section, we will instead depict an algorithm for getting the distribution of T directly.

The event $\{T > k\}$ is equivalent to $\{Y_l^{t+1} < r, Y_l^{t+2} < r, \cdots, Y_l^{t+k} < r, \forall\ 1 \leq l \leq n-1\}$. In view of (5), we can interpret the above as follows

$$\begin{aligned} Y_l^{t+1} &= \varepsilon_l^t + V_l^t Y_l^t < r, \\ Y_l^{t+2} &= \varepsilon_l^{t+1} + V_l^{t+1}\varepsilon_l^t + V_l^{t+1}V_l^t Y_l^t < r, \\ &\cdots \\ Y_l^{t+k} &= \varepsilon_l^{t+k-1} + V_l^{t+k-1}\varepsilon_l^{t+k-2} + \cdots \\ &\quad + V_l^{t+k-1}\cdots V_l^{t+1}\varepsilon_l^t \\ &\quad + V_l^{t+k-1}\cdots V_l^t Y_l^t < r. \end{aligned}$$

Set $U_l^{t+j} := V_l^{t+j}\varepsilon_l^{t+j-1} + \cdots + V_l^{t+j}\cdots V_l^{t+1}\varepsilon_l^t + V_l^{t+j}\cdots V_l^t Y_l^t$, for $1 \leq j \leq k-1$ and $U_l^t := V_l^t Y_l^t$. Therefore, condition on $Y_l^t, V_l^t, \cdots, V_l^{t+k-1}$, the probability that the above k inequalities holds simultaneously is shown to be given by

$$\begin{aligned} &P_l^k(Y_l^t, \{V_l^t, \cdots, V_l^{t+k-1}\}) \\ = &\int_0^{r-U_l^t} f_l(\varepsilon_l^t)\mathrm{d}\varepsilon_l^t \cdots \\ &\cdot \int_0^{r-U_l^{t+k-1}} f_l(\varepsilon_l^{t+k-1})\mathrm{d}\varepsilon_l^{t+k-1}, \end{aligned} \tag{8}$$

where $f_l(\cdot)$ is given in the proof of Theorem 1. Denote the last $i+1$ integrals of (8) by $I_{l,k-i}$, $0 \leq i \leq k-1$. For $i = 0$,

$$I_{l,k} = \int_0^{r-U_l^{t+k-1}} \frac{\lambda_l}{1-p} e^{-\frac{\lambda_l s}{1-p}}\mathrm{d}s = 1 - e^{-\frac{\lambda_l\left(r-U_l^{t+k-1}\right)}{1-p}}.$$

For $i = 1$,

$$\begin{aligned} I_{l,k-1} &= \int_0^{r-U_l^{t+k-2}} \frac{\lambda_l}{1-p} e^{-\frac{\lambda_l \varepsilon_l^{t+k-2}}{1-p}} I_{l,k}\mathrm{d}\varepsilon_l^{t+k-2} \\ &= 1 - e^{-\frac{\lambda_l\left(r-U_l^{t+k-2}\right)}{1-p}} \\ &\quad - \frac{\lambda_l(r-U_l^{t+k-2})}{1-p} e^{-\frac{\lambda_l\left(r-U_l^{t+k-2}\right)}{1-p}} 1_{[V_l^{t+k-1}=1]} \\ &\quad - \left(1 - e^{-\frac{\lambda_l\left(r-U_l^{t+k-2}\right)}{1-p}}\right) e^{-\frac{\lambda_l r}{1-p}} 1_{[V_l^{t+k-1}=0]}. \end{aligned}$$

In general, for $0 \leq i \leq k-1$,

$$I_{l,k-i} = \int_0^{r-U_l^{t+k-i-1}} \frac{\lambda_l}{1-p} e^{-\frac{\lambda_l \varepsilon_l^{t+k-i-1}}{1-p}} I_{l,k-i+1}\mathrm{d}\varepsilon_l^{t+k-i-1}.$$

We can proceed using this recursive formula by induction and integration by parts. Notice that $P_l^k(Y_l^t, \{V_l^t, \cdots, V_l^{t+k-1}\}) = I_{l,1}$ from (8).

Consequently, given $Y_l^t < r$, the probability that $Y_l^{t+1} < r$, $Y_l^{t+2} < r, \cdots, Y_l^{t+k} < r$ hold all together is seen to be given by

$$\begin{aligned} \widetilde{P}_l^k : &= \frac{\lambda_l}{1-e^{-\lambda_l r}} \sum_{i=0}^k p^i(1-p)^{k-i} \\ &\quad \cdot \sum_{\substack{k-\text{vector } \xi \\ \text{consisting of } k\ 1's, k-i\ 0's}} \int_0^r P_l^k(y,\xi)e^{-\lambda_l y}\mathrm{d}y. \end{aligned}$$

Now we state our result as follows, whose proof is now straightforward.

Theorem 5. *Suppose the hitting time T of G_t is defined as above, then the distribution $P(T \leq k) = 1 - \prod_{l=1}^{n-1}\widetilde{P}_l^k$ and it's expectation $ET = \sum_{k=0}^\infty \prod_{l=1}^{n-1}\widetilde{P}_l^k$. The complexity to compute ET is $O(n)$.*

In principle, by the truncation of k, we may approximate ET discretionarily close.

VI. Snapshots of $G(t, r, \Lambda)$

For fixed t, we denote by $G(r, \Lambda)$ the static case which can be regarded as a snapshot of the dynamical process $G(t, r, \Lambda)$. Also, we omit the superscript t typically, e.g. Y_l, etc.

A. Cluster structure

Let $P_n(\mathcal{C})$ denote the probability that $G(r, \Lambda)$ is connected. We have the following result regarding connectivity.

Theorem 6. *We have*

$$P_n(\mathcal{C}) = \prod_{l=1}^{n-1}(1 - e^{-\lambda_l r}).$$

Moreover, suppose there exists $M > 0$ such that $\lambda_l < M$, for all l, then $P_n(\mathcal{C}) \to 0$ as $n \to \infty$.

Proof. Since Y_l, $1 \leq l \leq n-1$ are independent random variables, $P_n(\mathcal{C}) = \prod_{l=1}^{n-1} P(Y_l < r) = \prod_{l=1}^{n-1}(1 - e^{-\lambda_l r})$. When λ_l is bounded by M, we observe that $\ln P_n(\mathcal{C})$ tends to 0, as $n \to \infty$. □

Let $\psi_n(k)$ denote the probability that $G(r, \Lambda)$ consists of k components and $P_n^m(k)$ the probability that there are k components in $G(r, \Lambda)$, each of which having size m (i.e. m vertices).

Theorem 7. *Suppose there exists $M > 0$ such that $\lambda_l < M$, for all l. Then, for any fixed k, $\psi_n(k) \to 0$ as $n \to \infty$; and for any fixed k, m, $P_n^m(k) \to 0$ as $n \to \infty$.*

Proof. Mimicking the proof of Theorem 3 & 4 in [7] gives the result. □

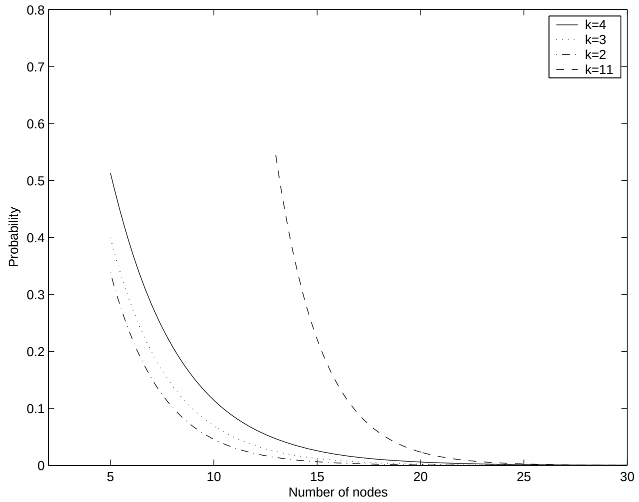

Fig. 2. Probability that $G(r,\Lambda)$ contains k components for different values of k.

In Figure 2, we plot $\psi_n(k)$ as function of n number of vertices for different k. We take $\lambda_i = 1$ for $1 \le i \le 10$, and $\lambda_i = 2$ for $i > 10$. Observe that the convergence to the asymptotic value 0 is very fast.

We may thus conclude that this static network is almost surely divided into an infinite number of finite clusters. This observation was first made by Dousse et.al.[5].

B. Degree distribution

Let $G(r,\lambda)$ denote the graph $G(r,\Lambda)$ when $\Lambda = \{\lambda,\cdots,\lambda\}$.

Theorem 8. *In the graph $G(r,\lambda)$, the degree distribution can be divided into three classes: the degree distribution of X_1 and X_n is $Poi(\lambda r)$; and for $k+1 \le i \le n-k$, that of X_i is $\{e^{-2\lambda r}\frac{(2\lambda r)^k}{k!}\}_{k\in\mathbb{N}}$. For $2 \le i \le k$, the degree distribution of X_i and X_{n+1-i} is $\{e^{-2\lambda r}\frac{(\lambda r)^k}{k!}\sum_{j=0}^{i-1}\binom{k}{j}\}_{k\in\mathbb{N}}$, which varies between Poisson distributions on the border and in the middle.*

Proof. Let $\{Y_i\}$, $\{Y_i'\}$ be independent $Exp(\lambda)$. Denote the degree of vertex X_i as d_i. We get

$$\begin{aligned} P(d_n \ge k) &= P(d_1 \ge k) \\ &= P(Y_1 + \cdots + Y_k \le r) \\ &= e^{-\lambda r}\Big(\frac{(\lambda r)^k}{k!} + \frac{(\lambda r)^{k+1}}{(k+1)!} + \cdots\Big), \end{aligned}$$

where we used an equivalent definition of gamma distribution. Hence,

$$P(d_n = k) = P(d_1 = k) = e^{-\lambda r}\frac{(\lambda r)^k}{k!}.$$

Next, for $2 \le i \le k$,

$$\begin{aligned} P(d_{n+1-i} = k) &= P(d_i = k) \\ &= \sum_{j=0}^{i-1} P(Y_1 + \cdots + Y_j \le r, \\ &\quad Y_1 + \cdots + Y_{j+1} > r) \\ &\quad \cdot P(Y_1' + \cdots + Y_{k-j}' \le r, \\ &\quad Y_1' + \cdots + Y_{k-j+1}' > r) \\ &= \sum_{j=0}^{i-1} \int_0^r \lambda e^{-\lambda x}\frac{(\lambda x)^{j-1}}{(j-1)!} \\ &\quad \cdot \int_{r-x}^{\infty} \lambda e^{-\lambda y}\mathrm{d}y\mathrm{d}x \\ &\quad \cdot \int_0^r \lambda e^{-\lambda x}\frac{(\lambda x)^{k-j-1}}{(k-j-1)!} \\ &\quad \cdot \int_{r-x}^{\infty} \lambda e^{-\lambda y}\mathrm{d}y\mathrm{d}x \\ &= e^{-2\lambda r}\frac{(\lambda r)^k}{k!}\sum_{j=0}^{i-1}\binom{k}{j}. \end{aligned}$$

Finally, for $k+1 \le i \le n-k$,

$$\begin{aligned} P(d_i = k) &= \sum_{j=0}^{k} P(Y_1 + \cdots + Y_j \le r, \\ &\quad Y_1 + \cdots + Y_{j+1} > r) \\ &\quad \cdot P(Y_1' + \cdots + Y_{k-j}' \le r, \\ &\quad Y_1' + \cdots + Y_{k-j+1}' > r) \\ &= e^{-2\lambda r}\frac{(2\lambda r)^k}{k!} \end{aligned}$$

which concludes the proof. □

C. Strong law results

Define the connectivity distance $c_n := \inf\{r > 0 : G(r,\lambda) \text{ is connected}\}$; and the largest nearest neighbor distance $b_n := \max_{1\le i\le n}\min_{1\le j\le n, j\ne i}\{|X_i - X_j|\}$. We derive asymptotic tight bounds for c_n and strong law of large numbers for b_n, as n tends to infinity.

Theorem 9. *In the graph $G(r,\lambda)$, we have*
(i)

$$\limsup_{n\to\infty}\frac{\lambda c_n}{2\ln n} \le 1 \quad \text{and} \quad \liminf_{n\to\infty}\frac{\lambda c_n}{\ln n} \ge 1 \qquad a.s.$$

(ii)

$$\lim_{n\to\infty}\frac{\lambda b_n}{\ln n} = 1 \qquad a.s.$$

Proof. (i) Observe that $P(c_n \ge x) \le \sum_{l=1}^{n-1} e^{-\lambda_l x} = (n-1)e^{-\lambda x}$. Let $\varepsilon > 0$. Take $x = x_n = (2+\varepsilon)\ln n/\lambda$ in the above expression and sum in n, then we get

$$\sum_{n=1}^{\infty} P(c_n \ge x_n) \le \sum_{n=1}^{\infty} n^{-(1+\varepsilon)} < \infty.$$

By Borel-Cantelli lemma, $P(c_n \geq x \text{ i.o.}) = 0$. Hence, $\limsup_{n\to\infty} \frac{\lambda c_n}{2\ln n} \leq 1$ almost surely.

On the other hand, $P(c_n \leq y) = \prod_{l=1}^{n-1}(1-e^{-\lambda_l y}) = (1-e^{-\lambda y})^{n-1}$. Take $y = y_n = (1-\varepsilon)\ln n/\lambda$, then

$$\sum_{n=1}^{\infty} P(c_n \leq y_n) \leq \sum_{n=1}^{\infty} \left(1-n^{-(1-\varepsilon)}\right)^{n-1} \sim \sum_{n=1}^{\infty} e^{-n^{\varepsilon}} < \infty.$$

We conclude that $\liminf_{n\to\infty} \frac{\lambda c_n}{\ln n} \geq 1$ a.s. by using Borel-Cantelli lemma again.

(ii) By the independence of $\{Y_l\}$, we obtain

$$\begin{aligned} P(b_n \geq x) &= P\big(\cup_{i=2}^{n-1}\{\{Y_{i-1}\geq x\}\cap\{Y_i \geq x\}\} \\ &\quad \cup\{Y_1 \geq x\}\cup\{Y_{n-1}\geq x\}\big) \\ &\leq \sum_{i=2}^{n-1} P(Y_{i-1}\geq x)\cdot P(Y_i \geq x) \\ &\quad +P(Y_1\geq x)+P(Y_{n-1}\geq x) \\ &= (n-2)e^{-2\lambda x}+2e^{-\lambda x}. \end{aligned}$$

Take $x = x_n = (2+\varepsilon)\ln n/(2\lambda)$, then we get

$$\sum_{n=1}^{\infty} P(b_n \geq x_n) \leq \sum_{n=1}^{\infty} \left(n^{-(1+\varepsilon)} + 2n^{-(1+\frac{\varepsilon}{2})}\right) < \infty.$$

By Borel-Cantelli lemma, $\limsup_{n\to\infty} \frac{\lambda b_n}{\ln n} \leq 1$ almost surely.

On the other hand,

$$\begin{aligned} P(b_n \leq y) &= P\big(\cap_{i=2}^{n-1}\{\{Y_{i-1}\leq y\}\cup\{Y_i \leq y\}\} \\ &\quad \cap\{Y_1 \leq y\}\cap\{Y_{n-1}\leq y\}\big) \\ &\leq \prod_{i=1}^{\lfloor \frac{n}{2}\rfloor} P(Y_{2i-1}\leq y)\cdot P(Y_{2i}\leq y) \\ &\sim (1-e^{-\lambda y})^n. \end{aligned}$$

Argue as the same case in (i), we can get $\liminf_{n\to\infty} \frac{\lambda b_n}{\ln n} \geq 1$ a.s.. This completes the proof. □

VII. Further discussion

Notice that every time-reversible finite Markov chain can be viewed as a random walk on undirected graphs[19], we may further analyze the mixing rate, cover time, spectral gap and so on. The interrelations of these Markov chains coupled in the main graph process $G(t, r, \Lambda)$ are of interest.

As for the idea of considering spacings, it may be extended to high dimensions in the following way. Deploy X_1 according to a probability density f, then place X_2 with the same probability density substituting the location of X_1 for the coordinate origin, and so forth. We deem the growing scheme would be an important alternative from the typical binomial or Poisson cases[16].

Other meaningful aspects include examination of "multiple spacings", reinforcing 1-step memory to finite steps memory even to infinite, which could be possible to result in power law degree distributions. Since we only treat the limit regime for constant λ_l, how to deal with λ_l approaching infinity is our future research. We believe the methods developed in this work would contribute to further in-depth research.

Acknowledgment

The author thanks the anonymous referees for their comments which have helped towards the improvement of the exposition of this paper.

References

[1] Y. C. Cheng and T. Robertazzi, Critical connectivity phenomena in multihop radio models. *IEEE Trans. on Commun.*, 37(1989) 770–777.
[2] F. Chung, S. Handjani and D. Jungreis, Generalizations of Polya's urn problem. *Annals of Combinatorics*, 7(2003) 141–153.
[3] S. Csörgő and W.-B. Wu, On the clustering of independent uniform random variables. *Random Structures and Algorithms*, 25(2004) 396–420.
[4] J. Díaz, D. Mitsche and X. Pérez-Giménez, On the connectivity of dynamic random geometric graphs. *Proc. of the 19th Annual ACM-SIAM Symposium on Discrete Algorithms*, San Francisco, 2008, 601–610.
[5] O. Dousse, P. Thiran and M. Hasler, Connectivity in ad hoc and hybrid networks. *Proc. of IEEE Infocom*, New York, 2002, 1079–1088.
[6] E. Godehardt and J. Jaworski, On the conncetivity of a random interval graph. *Random Structures and Algorithms*, 9 (1996) 137–161.
[7] B. Gupta, S. K. Iyer and D. Manjunath, Topological properties of the one dimensional exponential random geometric graph. *Random Structures and Algorithms*, 32(2008) 181–204.
[8] S. K. Iyer and D. Manjunath, Topological properties of random wireless networks. *Sādhanā*, 31(2006) 117–139.
[9] K. K. Jose and R. N. Pillai, Geometric infinite divisiblility and its applications in autoregressive time series modeling. In: V. Thankaraj (Ed.)*Stochastic Process and its Applications*, Wiley Eastern, New Delhi, 1995.
[10] N. Karamchandani, D. Manjunath and S. K. Iyer, On the clustering properties of exponential random networks. *IEEE Proc. of 6th WoWMoM*, 2005, 177–182.
[11] N. Karamchandani, D. Manjunath, D. Yogeshwaran and S. K. Iyer, Evolving random geometric graph models for mobile wireless networks. *IEEE Proc. of the 4th WiOpt*, Boston, 2006, 1–7.
[12] V. Kurlin, L. Mihaylova and S. Maskell, How many randomly distributed wireless sensors are enough to make a 1-dimensional network connected with a given probability? *Technical Report*, arXiv:0710.1001v1[cs.IT].
[13] A. J. Lawrance and P. A. W. Lewis, A new autoregressive time series model in exponential variables (NEAR(1)). *Advances in Applied Probability*, 13(1981) 826–845.
[14] D. Miorandi and E. Altman, Connectivity in one-dimensional ad hoc networks: A queueing theoretical approach. *Wireless Networks*, 12(2006) 573–587.
[15] S. Muthukrishnan and G. Pandurangan, The bin-covering technique for thresholding random geometric graph properties. *Proc. of 16th Annual ACM-SIAM Symposium on Discrete Algorithms*, Vancouver, 2005, 989–998.
[16] M. D. Penrose, *Random Geometric Graphs*, Oxford University Press, 2003.
[17] S. M. Ross, *Introduction to Probability Models*. Academic Press, 2006.
[18] V. Seetha Lekshmi and K. K. Jose, Autoregressive processes with Pakes and geometric Pakes generalized Linnik marginals. *Statist. Probab. Lett.*, 76(2006) 318–326.
[19] E. Seneta, *Non-negative Matrices and Markov Chains*. Springer-Verlag, 1981.
[20] Y. Shang, Exponential random geometric graph process models for mobile wireless networks. *International Conference on Cyber-Enabled Distributed Computing and Knowledge Discovery*, Zhangjiajie, 2009, 56-61.
[21] Y. Shang, Connectivity in a random interval graph with access points. *Information Processing Letters*, 109(2009), 446–449.
[22] Y. Shang, On the degree sequence of random geometric digraphs. *Applied Mathematical Sciences*, 4(2010) 2001–2012.
[23] Y. Shang, Laws of large numbers of subgraphs in directed random geometric networks. *International Electronic Journal of Pure and Applied Mathematics*, 2(2010) 69–79.