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Anti-Homomorphism in Fuzzy Ideals

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Abstract—The anti-homomorphic image of fuzzy ideals, fuzzy ideals of near-rings and anti ideals are discussed in this note. A necessary and sufficient condition has been established for near-ring anti ideal to be characteristic.

Keywords—Fuzzy Ideals, Anti fuzzy subgroup, Anti fuzzy ideals, Anti homomorphism, Lower α level cut.

I. Introduction

In 1971, Rosenfeld [11] constituted the elementary concepts of fuzzy subgroupoid, fuzzy ideals and fuzzy subgroups. Biswas [3] introduced the notion of anti fuzzy subgroups. Fuzzy subnear-rings are introduced by Abou-Zaid [1]. He studied fuzzy left (resp. right) ideals of a near-ring and gave some properties of fuzzy prime ideals of a near-ring. In [7], it has been established that homomorphic image of a fuzzy left (resp. right) ideal which has "sup property" is a fuzzy left (resp. right) ideal. In the year 1998, Sung M.H. et al. [12] proved the same result using the level fuzzy subsets and obtained some properties based on near-ring homomorphism. Properties of anti-homomorphic images of near-rings are discussed in [5]. Homomorphic images and pre images of anti fuzzy ideals are investigated by K.H. Kim et al. [9]. The notion of anti homomorphic image and pre image of fuzzy and anti fuzzy ideals are investigated in this paper. Also, near-ring anti homomorphic image and pre image of ideals are obtained.

A. Preliminaries

In this section, review of fuzzy set theoretic concepts are given briefly (for details one can refer [4], [11] and [10]). A fuzzy set μ of a set N is a function $\mu : N \to [0,1]$.

μ will be called a fuzzy left ideal [11], if $\mu(xy) \geq \mu(y)$; a fuzzy right ideal, if $\mu(xy) \geq \mu(x)$; anti fuzzy left ideal [3] if $\mu(xy) \leq \mu(y)$; anti fuzzy right ideal, if $\mu(xy) \leq \mu(x)$;.

Let $f : N \to N'$ be a function and let μ and ν be fuzzy sets in N and N' respectively. Then $f(\mu)$ [11], the image of μ under f is a fuzzy set in N' defined by

$$f(\mu)(y) = \begin{cases} \sup\left(\mu(x) : x \in f^{-1}(y)\right) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{Otherwise} \end{cases}$$

for all $y \in N'$. $f^{-1}(\nu)$ [11], the preimage of ν under f is a fuzzy set in N given by

$$f^{-1}(\nu)(x) = \nu\left(f(x)\right)$$

for all $x \in N$.

Similar to an α level cut [4], we have lower level cut [9] as follows:

Let μ be a fuzzy set in a set N. For $\alpha \in [0,1]$, the lower α level cut of μ is denoted by ${}_{\alpha}N_{\mu}$ and is given by

$${}_{\alpha}N_{\mu} = \{n \in N : \mu(n) \leq \alpha\}.$$

Definition 1.1: [1], [7] Let N be a left near-ring and μ be a non empty fuzzy sub set of N. μ is said to be a fuzzy left N-ideal if

I-1. $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$,
I-2. $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in N$,
I-3. $\mu(y+x-y) \geq \mu(x)$ and
I-4. $\mu(xy) \geq \mu(y)$ where $x, y \in N$

are satisfied. If axioms (I-1), (I-2), (I-3) with

I-5. $\mu((x+z)y - xy) \geq \mu(z)$

holds, μ is a fuzzy right N-ideal.

From the definition of right near-ring [6], ideals can be defined as follows:

Definition 1.2: Let N be a right near-ring and μ be a non empty fuzzy sub set of N. μ is said to be a fuzzy left N-ideal if (I-1), (I-2), (I-3) and

I-6. $\mu(xy) \geq \mu(y)$ where $x, y \in N$

are satisfied. If (I-7) is postulated instead of (I-5) of fuzzy N-ideal of left near-ring, μ is a fuzzy right N-ideal where

I-7. $\mu(y(x+z) - yx) \geq \mu(z)$.

Definition 1.3: [9] Let N be a left near-ring and μ be a non empty fuzzy sub set of N. μ is said to be an anti fuzzy left N-ideal if

AI-1. $\mu(x-y) \leq max(\mu(x), \mu(y))$,
AI-2. $\mu(xy) \leq max(\mu(x), \mu(y))$,
AI-3. $\mu(y+x-y) \leq \mu(x)$ and
AI-4. $\mu(xy) \leq \mu(x)$ where $x, y \in N$.

If axioms (AI-1), (AI-2), (AI-3) with the following (AI-5) are satisfied then μ is an anti fuzzy right N-ideal;

AI-5. $\mu((x+z)y - xy) \leq \mu(z)$.

Definition 1.4: Let N be a right near-ring and μ be a non empty fuzzy sub set of N. μ is said to be an anti fuzzy left N-ideal when AI-1 to AI-3 along with

AI-6. $\mu(xy) \leq \mu(y)$ where $x, y \in N$

are postulated. If axioms (AI-1), (AI-2), (AI-3) with (AI-7) are satisfied then μ is an anti fuzzy right N-ideal;

AI-7. $\mu(y(x+z) - yx) \leq \mu(z)$.

Recall that, a function $f : N \to N'$ of near-rings is called an ***anti-homomorphism*** [5] when

1. $f(n+m) = f(m) + f(n)$
2. $f(nm) = f(m)f(n)$, for all $n, m \in N$.

A surjective anti-homomorphism is called an **anti-epimorphism** ($\cong$).

II. Main Results

A. Fuzzy Ideals

It is to be noted that the anti homomorphic image pre-image of a fuzzy groupoid is again a fuzzy groupoid. Where as,

Result 2.1: An anti homomorphic pre-image of a right (left) ideal is a left (right) ideal.

Proof: Let ν be a fuzzy left ideal. Then

$$\begin{aligned}\mu(xy) &= \nu(f(xy)) \\ &= \nu(f(y)f(x)) \\ &\geq \nu(f(x)) \\ &= \mu(x),\end{aligned}$$

a right ideal. When ν is a fuzzy right ideal,

$$\begin{aligned}\mu(xy) &= \nu(f(xy)) \\ &= \nu(f(y)f(x)) \\ &\geq \nu(f(y)) \\ &= \mu(y),\end{aligned}$$

a left ideal. □

Result 2.2: Anti-homomorphic image of a fuzzy left (right) ideal, with spermium property, is a fuzzy right (left) ideal.

Proof: Let μ be a fuzzy left ideal with sup property. Given $f(x), f(y)$ in $f(N)$, let $x_0 \in f^{-1}[f(x)], y_0 \in f^{-1}[f(y)]$ be such that

$$\mu(x_0) = \sup_{t \in f^{-1}[f(x)]} \mu(t), \quad \mu(y_0) = \sup_{t \in f^{-1}[f(y)]} \mu(t)$$

respectively. Then

$$\begin{aligned}\nu[f(x)f(y)] &= \nu[f(yx)] \\ &= \sup_{t \in f^{-1}[f(yx)]} \mu(t) \\ &\geq \mu(y_0 x_0) \\ &\geq \mu(x_0) \\ &= \sup_{t \in f^{-1}[f(x)]} \mu(t) \\ &= \nu[f(x)],\end{aligned}$$

implies ν is a fuzzy right ideal. Similarly, when μ is a fuzzy right ideal with above property, we have ν is a fuzzy left ideal. □

In sequel to the above results, the following also can be established for the case of anti fuzzy left (right) ideals.

Result 2.3: An anti homomorphic pre-image of an anti fuzzy right (left) ideal is an anti fuzzy left (right) ideal.

Result 2.4: An anti homomorphic image of an anti fuzzy right (left) ideal with sup property, is an anti fuzzy left (right) ideal.

B. Fuzzy Ideals in Near-rings

Result 2.5: ([5] Theorem 2.2) Anti homomorphic image of a right near-ring (left near-ring) is a left near-ring (right near-ring).

Result 2.6: Let $f : N \to N'$ be an anti-epimorphism of near-rings. If ν is a fuzzy (left/right) ideal in the right (left) near-ring N', then μ, which is $f^{-1}(\nu)$ is a fuzzy (left/right) ideal in the left (right) near-ring N.

Proof:
Let ν be a fuzzy left ideal of right near-ring N'. The proof of conditions (I-1), (I-2) and (I-3) of definition 1.1 are similar to that of proof of [7] Theorem 2.12. For any $x, y \in N$, we have

$$\begin{aligned}\mu(xy) &= \nu(f(xy)) \\ &= \nu(f(y)f(x)) \\ &\geq v(f(y)) \\ &= \mu(y).\end{aligned}$$

Thus μ is a fuzzy left ideal of the left near-ring N. When ν is a right ideal of right near-ring N', for any $x, y, z \in N$ we have,

$$\begin{aligned}\mu((x+z)y - xy) &= \nu(f((x+z)y - xy)) \\ &= \nu(f(y)f(x+z) - f(xy)) \\ &= \nu(f(y)(f(x)+f(z)) - f(y)f(x)) \\ &\geq \nu(f(z)) \\ &= \mu(z),\end{aligned}$$

μ is a fuzzy right ideal of left near-ring N.

Let ν be a right ideal of left near-ring N'. For any $x, y, z \in N$, we have

$$\begin{aligned}\mu(y(x+z) - yx) &= \nu(f((x+z)y - xy)) \\ &= \nu(f(x+z)f(y) - f(yx)) \\ &= \nu((f(x)+f(z))f(y) - f(x)f(y)) \\ &\geq \nu(f(z)) \\ &= \mu(z).\end{aligned}$$

Thus μ is a fuzzy left ideal of right near-ring N. Let ν is a fuzzy left ideal of left near-ring N'. Then

$$\begin{aligned}\mu(xy) &= \nu(f(xy)) \\ &= \nu(f(y)f(x)) \\ &\geq \nu(f(x)) \\ &= \mu(x)\end{aligned}$$

for all $x, y \in N$, implies μ is a left ideal of right near ring N. □

Result 2.7: Let $f : N \to N'$ be an anti-epimorphism of near-rings. If μ is a fuzzy (left/right) ideal in the left (right) near-ring N with sup property, then $\nu = f(\mu)$ is a fuzzy (left/right) ideal in the right (left) near-ring N'.

Proof: Let μ be a fuzzy left ideal of the left near-ring N with sup property and ν be the image of μ under f. Let $x_0 \in f^{-1}[f(x)]$, $y_0 \in f^{-1}[f(y)]$ such that

$$\mu(x_0) = \sup_{t \in f^{-1}[f(x)]} \mu(t), \;\; \mu(y_0) = \sup_{t \in f^{-1}[f(y)]} \mu(t).$$

$$\begin{aligned} \nu[f(x)f(y)] &= \sup_{t \in f^{-1}[f(yx)]} \mu(t) \\ &\geq \mu(y_0 x_0) \\ &\geq \mu(x_0) \\ &= \sup_{t \in f^{-1}[f(x)]} \mu(t) \\ &= \nu[f(x)]. \end{aligned}$$

That is ν is a fuzzy left ideal of right near-ring N'.

If μ is a fuzzy right ideal of left near-ring. For any $f(z) \in f(N)$, let $z_0 \in f^{-1}[f(z)]$ such that

$$\mu(z_0) = \sup_{t \in f^{-1}[f(z)]} \mu(z).$$

Now,

$$\begin{aligned} & \nu[f\{(x+z)y - (xy)\}] \\ = \;& \nu[f(y)f(x+z) - f(xy)] \\ = \;& \nu[f(y)\{f(x) + f(z)\} - f(y)f(x)] \\ = \;& \sup_{t \in f^{-1}\{f[f(y)\{f(x)+f(z)\} - f(y)f(x)]\}} \mu(t) \\ \geq \;& \mu[y_0\{x_0 + z_0\} - y_0 x_0] \\ \geq \;& \mu[z_0] \\ = \;& \sup_{t \in f^{-1}[f(z)]} \mu(t) \\ = \;& \nu[f(z)]. \end{aligned}$$

That is, ν is a fuzzy right ideal of right near-ring.

The image and pre-image of the fuzzy ideal of a fuzzy right near-ring N can be proved to be the fuzzy ideal of a left near-ring N'. □

C. Anti Fuzzy Ideals in Near-rings

Definition 2.8: A left N-ideal A of a near-ring is said to be characteristic [2], [8] if

$$f(A) = A \;\; \forall f \in Aut(N)$$

where $Aut(N)$ is set of all automorphism of N.

Anti fuzzy left N-ideal of μ of a near-ring N is said to be anti fuzzy characteristic if

$$\mu f_{(x)} = \mu(x) \quad \forall x \in N, f \in Aut(N).$$

Lemma 2.9: Let μ be an anti fuzzy left N-ideal of a near-ring N and let $x \in N$. Then $\mu(x) = s$ if and only if $x \in {}_sN_\mu$ and $x \notin {}_tN_\mu$ $\forall s > t$.

Proof is obvious.

The proof of the following theorem is analogous to the proof of theorem 3.9 [2], [8].

Theorem 2.10: Let μ be an anti fuzzy N-ideal of a near-ring N. Then each lower α level left N-ideal of μ is characteristic iff μ is an anti fuzzy characteristic of N.

K. H. Kim et al. [9] proved the following theorems:

Result 2.11 ([9], Theorem 3.19 (1)): Let $f : N \to N'$ be an epimorphism of near-rings. Let ν be an anti-fuzzy left N'-ideal and μ be the pre-image of ν under f. Then μ is an anti-fuzzy left N-ideal.

Result 2.12 ([9], Theorem 3.19 (2)): Let $f : N \to N'$ be a surjective homomorphism of near-rings. If μ is an anti fuzzy left N'-ideal then $f^{-1}(\mu)$ is an anti fuzzy left N-ideal.

Result 2.13: Let $f : N \to N'$ be an anti-epimorphism of near-rings. Let ν be an anti-fuzzy (left/right) ideal of right (left) near-ring N' then μ, the pre-image of ν under f, is an anti-fuzzy (left/right) ideal of left (right) near-ring N.

The proof of AI-4 and AI-5 are similar to the proof of result 2.6.

Result 2.14: Let $f : N \to N'$ be an anti epimorphism. If μ is an anti fuzzy (left/right) ideal of left (right) near-ring N with sup property, $f(\mu)$ is an anti fuzzy (left/right) ideal of right (left) near-ring N'.

The proof of AI-6 and AI-7 are similar to that of result 2.7.

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