On Best Estimation for Parameter Weibull Distribution

Hadeel Salim Alkutubi

***Abstract*—**The objective of this study is to introduce estimators to the parameters and survival function for Weibull distribution using three different methods, Maximum Likelihood estimation, Standard Bayes estimation and Modified Bayes estimation. We will then compared the three methods using simulation study to find the best one base on MPE and MSE.

***Keywords*—**Maximum Likelihood estimation , Bayes estimation, Jeffery prior information, Simulation study

I. Introduction

BAYESIAN statistics is the only statistical theory that combines modeling inherent uncertainty and statistical uncertainty . The theorem of Bayes provides a solution on how to learn from data. Related to survival function and by using Bayes estimator, [4] estimated the shape and scale parameters of the Weibull distribution by assuming a weighted squared error loss function. They minimized the corresponding expected loss with respect to a given posterior distribution. Sinha & Sloan [6], obtained Bayes estimator of three parameters Weibull distribution and compared the posterior standard deviation estimates with the corresponding asymptotic standard deviation of their maximum likelihood counterparts and numerical examples are given. In 2002, Klaus Felsenstein [5] developed Bayesian procedures for vague data. These data were assumed to be vague in the sense that the likelihood is a mixture of the model distribution with error distribution. In this case the standard updating procedure of the model prior would fail. Al-Bayyati [1] studied the problem of estimating parameters of Weibull distribution and reliability function in situation where there is no information on the parameters. He proposed a method based on the primary information with weighted Bayes. An extension of Jeffery prior information with square error loss function in exponential distribution was studied by Al-Kutubi [2]. In this paper, Al-kutubi [3] was proposed an extension of Jeffery prior information with a new loss function and then compeer it with standard Bayes to find the best. In this paper we will introduce estimators to the parameters and survival function for Weibull distribution using three different methods, Maximum Likelihood estimation, Standard Bayes estimation and Modified Bayes estimation. We will then compare the three methods using simulation study to find the best one base on MPE and MSE.

Hadeel S. H. Alkutubi is with College of Nursing, Kufa University, Iraq.
e-mail: hadeelkutubi@live.com

II. Materials and Methods

Let $t_1, t_2, \dots, t_n$ be the life time of a random sample of size n with distribution function and probability density function .In the Weibull case , we assumed that the probability density function of the life time is given by

$$f(t,\theta,c)=\frac{c}{\theta}\left(\frac{t_i}{\theta}\right)^{c-1}\exp\left(\frac{-t}{\theta}\right)^{c}$$

To find Maximum likelihood estimator , we will calculate the likelihood function such that

$$L(t,\theta,c)=\Pi f(t,\theta,c)$$

$$=\frac{c^n}{\theta^{cn}}(\Pi t_i^{\,c-1})\exp\left(-\frac{\sum_{i=1}^{n}t_i^{\,c}}{\theta^c}\right)$$

$$Ln=(\,L(t,\theta,c)=n\,ln-cn\,ln\theta+\sum_{i=1}^{n}\ln t_i^{\,c-1}$$

Using score vector , such that

$$V(\theta)=\frac{\partial\ln(L(t,\theta,c)}{\partial\theta}=\frac{-cn}{\theta}+\frac{\sum t_i^{\,c}\,c\theta^{c-1}}{\theta^{2c}}$$

Let $V(\theta)=0$, then the Maximum likelihood estimator is

$$\widetilde{\theta_1}=\frac{\sum t_i}{\sqrt[c]{cn}}$$

To obtain Bayes estimator, the following steps are needed [3].

A number of r items put to test and the life times of this random samples are recorded with the probability density function $f(t,\theta,c)$ .The life time probability density function $f(t,\theta,c)$ are regarded as a conditional probability density function $f(t|\theta,c)$ where the marginal probability density function of θ is given by $g(\theta)$, the Jeffery prior information.

$$f(t,\theta,c)=\frac{c}{\theta}\left(\frac{t}{\theta}\right)^{c-1}e^{\left(-\frac{t}{\theta}\right)^{c}}$$

$$\ln f(t,\theta,c)=\ln c-\ln\theta+(c-1)\ln(t)-(c-1)\ln\theta-\left(\frac{t}{\theta}\right)^{c}$$

$$\frac{\partial\ln f(t,\theta,c)}{\partial\theta^2}=\frac{c}{\theta^2}-2c\left(\frac{t}{\theta}\right)^{c-1}\left(\frac{t}{\theta^3}\right)-c(c-1)\left(\frac{t}{\theta}\right)^{c-2}\left(\frac{t}{\theta}\right)^{2}$$

=M

We find Jeffery prior by taking $g(\theta) \propto \sqrt{I(\theta)}$, where fisher information $I(\theta)$ is given by

$$I(\theta) = -n\, E\left(\frac{\partial^2 \ln f(t,\theta)}{\partial \theta^2}\right)$$

$$E\left(\frac{\partial^2 \ln f(t,\theta,c)}{\partial \theta^2}\right) = E(M) = \int_0^\infty M\, \frac{c}{\theta}\left(\frac{t}{\theta}\right)^{c-1} e^{-\left(\frac{t}{\theta}\right)^c} dt = \frac{-c^2}{\theta^2}$$

Then $I(\theta) = \frac{nc^2}{\theta^2}$, so the Jeffery prior information is given by

$$g(\theta) = k\sqrt{I(\theta)} = k\sqrt{\frac{nc^2}{\theta^2}} = k\frac{c\sqrt{n}}{\theta}$$

The joint probability density function is:

$$H(t_1,\dots,t_n;\theta,c) = L(t_1,\dots,t_n|\theta,c)\, g(\theta)$$
$$= \frac{kc^2\sqrt{n}}{\theta^{n+1}}\left(\frac{\sum t_i}{\theta}\right)^{c-1} exp\left(-\left(\frac{\sum t_i}{\theta}\right)^c\right)$$

The marginal probability density function of $(t_i, \dots t_n, \theta, c)$ is given by:

$$P(t_1,\dots,t_n) = \int_0^\infty H(t_1,\dots,t_n;\theta,c)\, d\theta$$
$$= \int_0^\infty \frac{kc^2\sqrt{n}}{\theta^{n+1}}\left(\frac{\sum t_i}{\theta}\right)^{c-1} \exp\left(-\left(\frac{\sum t_i}{\theta}\right)\right) d\theta$$
$$= kc^2\sqrt{n}\int_0^\infty \frac{1}{\theta^{n+1}}\left(\frac{\sum t_i}{\theta}\right)^{c-1} exp\left(-\left(\frac{\sum t_i}{\theta}\right)^c\right) d\theta$$
$$= \frac{kc\sqrt{n}}{(\sum t_i)^n}\left(\frac{n-1}{c}\right)!$$

Then we can get posterior distribution such that

$$\pi(\theta,c|t_1,\dots,t_n) = \frac{H(t_1,\dots,t_n;\theta,c)}{P(t_1,\dots,t_n)}$$
$$= \frac{c(\sum t_i)^n\left(\frac{\sum t_i}{\theta}\right)^{c-1} \exp\left(-\left(\frac{\sum t_i}{\theta}\right)^c\right)}{\theta^{n+1}\left(\frac{n-1}{c}\right)!}$$

By using squared error loss function

$\ell(\hat{\theta}-\theta) = c(\hat{\theta}-\theta)^2$, we can obtain the Risk function, such that

$$R(\hat{\theta},\theta) = EL(\hat{\theta},\theta) = \int_0^\infty c(\hat{\theta}-\theta)^2 \pi(\theta|t_1,\dots t_n) d\theta$$

$$= c\hat{\theta}^2 - \frac{2c\hat{\theta}\sum t_i}{\left(\frac{n-1}{c}\right)} + \frac{c(\sum t_i)^2}{\left(\frac{n-1}{c}\right)\left(\frac{n-2}{c}\right)}$$

$$\frac{\partial R(\hat{\theta},\theta)}{\partial \hat{\theta}} = 2c\hat{\theta} - \frac{2c(\sum t_i)}{\left(\frac{n-1}{c}\right)}$$

Let $\frac{\partial R(\hat{\theta},\theta)}{\partial \hat{\theta}} = 0$, then Bayes estimator is

$$\hat{\theta}_2 = \frac{\sum t_i}{\left(\frac{n-1}{c}\right)}$$

To find survival function, we will solve the following equation

$$\hat{S}_2(t) = \int_0^\infty \exp\left(-\frac{t_i}{\theta}\right)^c \pi(\theta,c|t_1,\dots,t_n) d\theta$$

$$= \int_0^\infty \exp\left[-\left(\frac{t_i}{\theta}\right)^c\right]\left\{\frac{c[(\sum t_i)^n]\left[\left(\frac{\sum t_i}{\theta}\right)^{c-1}\right] exp\left[-\left(\frac{\sum t_i}{\theta}\right)^c\right]}{\theta^{n+1}\left(\frac{n-1}{c}\right)!}\right\} d\theta$$

Then the survival function is

$$\hat{S}_1(t) = \left(\frac{\sum t_i}{t_i + \sum t_i}\right)^{n+c-1}$$

By using new loss function, we can get new estimator, such that

$$R(\hat{\theta},\theta) = EL(\hat{\theta},\theta) = \int_0^\infty \sqrt{\theta}(\hat{\theta}-\theta)^2 \pi(\theta|t_1,\dots t_n) d\theta$$

$$= \int_0^\infty \left(\hat{\theta}^2\theta^{\frac{1}{2}} - 2\hat{\theta}\theta^{\frac{3}{2}} + \theta^{\frac{5}{2}}\right)\left(\frac{c(\sum t_i)^n\left(\frac{\sum t_i}{\theta}\right)^{c-1} \exp\left[-\left(\frac{\sum t_i}{\theta}\right)^c\right]}{\theta^{n+1}\left(\frac{n-1}{c}\right)!}\right) d\theta$$

Solving equation in above to get the second Bayes estimator

$$\hat{\theta}_3 = \frac{\sum t_i}{\left(\frac{n-\frac{3}{2}}{c}\right)}$$

III. Results

In simulation study, we have chosen $n = 35, 75, 100$ to represent small, moderate and large sample size, several values of parameter θ=0.5, 1, 1.5, 2 with four values of constants. The number of replication used was R=1000. The simulation program was written by using Matlab program. After the parameter was estimated , mean square error (MSE) and mean percentage error (MPE) were calculated to compare the methods of estimation, where

$$MSE(\hat{\theta}) = \frac{\sum_{i=1}^{1000}(\hat{\theta}-\theta)^2}{R} \quad and \quad MPE(\hat{\theta}) = \frac{\sum_{i=1}^{1000}\frac{|\hat{\theta}-\theta|}{\theta}}{R}$$

TABLE I THE ORDERING OF THE ESTIMATORS WITH RESPECT TO MSE

Size	θ	C	Theta Hat		
			$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$
35	0.5	0.4	0.200	0.135	0.211
		0.8	0.201	0.133	0.211
	1	0.4	0.222	0.209	0.229
		0.8	0.222	0.209	0.229
	1.5	0.4	0.234	0.231	0.233
		0.8	0.233	0.232	0.233
75	0.5	0.4	0.125	0.121	0.129
		0.8	0.125	0.122	0.128
	1	0.4	0.121	0.120	0.122
		0.8	0.123	0.120	0.123
	1.5	0.4	0.127	0.125	0.130
		0.8	0.128	0.126	0.133
100	0.5	0.4	0.091	0.090	0.094
		0.8	0.091	0.090	0.094
	1	0.4	0.088	0.086	0.087
		0.8	0.088	0.086	0.087
	1.5	0.4	0.088	0.085	0.089
		0.8	0.088	0.087	0.090

TABLE II THE ORDERING OF THE ESTIMATORS WITH RESPECT TO MPE

Size	θ	C	Theta Hat		
			$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$
35	0.5	0.4	0.020	0.013	0.021
		0.8	0.020	0.013	0.021
	1	0.4	0.022	0.020	0.022
		0.8	0.022	0.020	0.022
	1.5	0.4	0.023	0.023	0.024
		0.8	0.023	0.023	0.024
75	0.5	0.4	0.012	0.012	0.013
		0.8	0.012	0.012	0.014
	1	0.4	0.013	0.012	0.015
		0.8	0.013	0.012	0.015
	1.5	0.4	0.012	0.012	0.013
		0.8	0.012	0.012	0.013
100	0.5	0.4	0.008	0.007	0.009
		0.8	0.008	0.007	0.009
	1	0.4	0.006	0.005	0.007
		0.8	0.006	0.005	0.007
	1.5	0.4	0.007	0.005	0.006
		0.8	0.007	0.005	0.006

IV. DISCUSSION

The results of the simulation study are summarized and tabulated in Table 1 and Table 2 for the MSE and the MPE of the three estimators for all sample sizes and θ , C values respectively. The order of the estimator is from the best (smallest MSE) to the worst (largest MSE). It is obvious from these tables, Bayes estimator with new loss function with Jeffery prior information, $\hat{\theta}_3$ is the best estimator. In most of the cases, it is apparent that maximum likelihood estimator, $\hat{\theta}_1$ is the next best estimator. Standard Bayes estimator with Jeffery prior information $\hat{\theta}_2$ in most case has the largest MSE and MPE.

V. CONCLUSION

The new estimator with Jeffery prior information $\hat{\theta}_3$ is the best estimator when compared to standard Bayes and Maximum likelihood estimator. We can easily conclude that MSE and MPE of Bayes estimators decrease with an increased of sample size.

REFERENCES

[1] AlBayyati H.N. "Comparing methods of estimating Weibull failure models using simulation ". Ph.D. thesis, College of Administration and Economics, Baghdad University, Iraq.2002.

[2] AlKutubi H.S. " On Comparison estimation procedures for parameter and survival function exponential distribution using simulation" .Ph.D. thesis, College of Ibn Al-Hatham, Baghdad University, Iraq.2005.

[3] Alkutubi H. S. "Bayes estimator for exponential distribution with extension of Jeffery prior information" Malaysia journal of mathematical science. Malaysia. 3(2): 297-313.2009.

[4] Ellis W. C. and Rao T. V. "Minimum expected loss estimators of the shape and scale parameters of Weibull distribution" *IEE Transaction on Reliability*, Vol. 35, No. 2.1986.

[5] Felsenstein K. "Bayesian inference for questionable data" *Austrian Journal of Statistic*, Vol. 31, No. 2. 2002.

[6] Sinha S. K. and Sloan J. A. "Bayes estimation of the parameters and reliability function of the 3-parameters Weibull distribution" *IEEE Transaction on reliability* , Vol. 37, No. 4.1988.

Hadeel S. H. Alkutubi, born in Basreh , Iraq in 1973, she is Assistant Professor in College of Nursing, Kufa University, Iraq. Her general Specialization is Mathematics and specific specialization is Statistic, Survival Analysis. Her interesting Research are :Statistical analysis for health, medicine, and agriculture studies also estimation methods for parameter and survival function of failure distribution.

She has some researches in international and local Journals. All of them about Bayes Estimation, And Biostatistics. Alkutubi has Honoring from the ministry of higher education and scientific research in Iraq for publishing nine research papers in foreign journals and conferences.

She worked as associate Dean of the Faculty of Mathematics and Computer Science. Kufa University, Iraq in 2010. Also she is a member in many committees in Kufa University, like scientific committee, examination committee. She worked as managing editor of Kufa Journal,. University of Kufa in Iraq and - Managing editor of Tikrit Journal for Pharmaceutical Sciences. University of Tikrit., Tikret, Iraq.