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Approximation Approach to Linear Filtering Problem with Correlated Noise

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Abstract—The (sub)-optimal soolution of linear filtering problem with correlated noises is considered. The special recursive form of the class of filters and criteria for selecting the best estimator are the essential elements of the design method. The properties of the proposed filter are studied. In particular, for Markovian observation noise, the approximate filter becomes an optimal Gevers-Kailath filter subject to a special choice of the parameter in the class of given linear recursive filters.

Keywords—Linear dynamical system, filtering, minimum mean square filter, correlated noise

I. Introduction

Consider a standard linear filtering problem

$$x_{k+1} = \Phi_k x_k + G_k w_k, \tag{1}$$

$$z_{k+1} = H_{k+1} x_{k+1} + v_{k+1}, k = 0, 1, 2, ... \tag{2}$$

here x_k is the n-dimensional system state at k instant, Φ_k is the (nxn) fundamental matrix, z_k is the p-dimensional observation vector, H_k is the (pxn) observation matrix, w_k, v_k are the model and observation noises. The statistical characteristics of the entering random variables are given as

$$E[x_0] = \bar{x_0}, E[x_0 x_0^T] = M_0, \tag{3}$$

$$E[w_k] = 0, E[w_k w_l^T] = Q_{kl}, \tag{4}$$

$$E[v_k] = 0, E[v_k v_l^T] = R_{kl}, E[w_k v_l] = K_{wv}(k), \tag{5}$$

$$E[(x_0 - \bar{x_0}) w_k^T] = K_{xw}(k), E[(x_0 - \bar{x_0}) v_k^T] = K_{xv}(k). \tag{6}$$

Denote by $\hat{x}_k$ a minimum mean square (MMS) estimator for the state x_k. The Kalman filter (KF) yields the MMS solution to this filtering problem with white and uncorrelated process and observation noises [10]. The extension of the KF to the systems with colored noises that are Markovian is studied on the basis of innovation process [6]. The filtering problems with correlated noises are widely encountered in engineering applications (data assimilation in meteorology and oceanography [4], GPS position time series [2], halftoning systems with blue noise [5], speech signal processing [13], navigation [14], guidance [3] .... For many practical applications, the assumption on Markovian noises is nevertheless not necessary. The filtering problem in the form (1)(2)(3)-(6) has been considered in [12]. Generally speaking, due to assumptions (3)-(6) the estimator $\hat{x}_{k+1}$, written in recursive form, depends on $\hat{x}_k$ and all the observations $\{z_1, ..., z_{k+1}\}$. This dependence makes implementation of the optimal filter extremely difficult for large k.

The present paper aims to overcome the mentioned above difficulty. The approach follows that reported in [8], with emphasis on the linear filtering problem considered in [12] : Given the system dynamics and observations contaminated by correlated noises, the task is to construct an algorithm providing an (sub)-optimal filtered estimate of high quality. Concretely, according to [8], the class of filters $\{\tilde{x}_k(n_k)\}$ with $n_k \leq k$ - some positive integer number, is introduced in a way such that $\tilde{x}_{k+1}(n_{k+1})$ depends on $\tilde{x}_k(n_k)$ and n_{k+1} latest observations. One important requirement to the algorithm will be that the produced estimate $\tilde{x}_k(n_{k+1})$ will be truly MMS if $\tilde{x}_k(n_k) = \hat{x}_k$ and $n_{k+1} = k + 1$. Such algorithm has a merit to be studied in more detail, noticing in practice the time correlation generally becomes weaker as the time difference increases. More importantly, for a particular case of the Markovian observation noise with memory m, the sub-optimal filter becomes truly MMS in the class of filters being linear functions of the last estimate $\hat{x}_k$ and $m + 1$ last observations. The case of Markovian noise sequence with memory $m = 1$ will be studied in detail in section 6.

The paper is organized as follows. In section 2, for the time-invariant system state, the main theoretical results on MMS filter optimal in a given class of linear filters are presented. These results will be extended to the general time-varying system state in section 3. The properties of the obtained filter are studied in section 4. Conditions for equivalence of two estimators obtained on the basis of the last estimator and two different numbers of latest observations are given in section 5. Application of the theoretical results to the design of the MMS filter subject to the Markovian noise sequence with memory $m = 1$ is considered in section 6. The conclusions are given in section 7.

II. Preliminaries results : time-invariant system state

For simplifying the presentation, first consider the filtering problem (1),(3)-(5) under assumptions

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$$\Phi_k = I, G_k = 0, \text{i.e.}, x_{k+1} = x_k = x. \quad (7)$$

The system state is then time-invariant. Throughout this paper, I denotes a unit matrix of the appropriate dimension. Introduce the notations

$$z_k^i = (z_i^T, ..., z_k^T)^T, z_k^k = z_k, k \geq i, \quad (8)$$
$$H_k^i = (H_i^T, ..., H_k^T)^T, H_k^k = H_k, k \geq i, \quad (9)$$
$$v_k^i = (v_i^T, ..., v_k^T)^T, v_k^k = v_k, k \geq i, \quad (10)$$
$$V_k^i = E(v_k^i v_k^{i,T}). \quad (11)$$

Let $\tilde{x}_k(n_k)$ be a sequence of estimators for x given by $\{z_1, ..., z_k\}$ such that each next estimator $\tilde{x}_{k+1}(n_{k+1})$ is a linear function of $\tilde{x}_k(n_k)$ and n_{k+1} last observations. According to notations in [8], we have

$$\tilde{x}_{k+1}(n_{k+1}) = \delta_k \xi[\tilde{x}_k(n_k)), z_{k+1}^{k+2-n_{k+1}}] + \gamma_k = [\delta_{k,1}, \delta_{k,2}][\xi_{k,1}^T, \xi_{k,2}^T]^T + \gamma_k,$$
$$\xi_{k,1} := \tilde{x}_k(n_k), \xi_{k,2} := z_{k+1}^{k+2-n_{k+1}}.$$

For simplicity and without generality, assume that the sequence $\{\tilde{x}_k(n_k)\}$ is unbiased. Then one can set $\gamma_k = 0$. In what follows we will use the following notation for the sequence $\{\tilde{x}_k(n_k)\}$

$$\tilde{x}_{k+1}(n_{k+1}) = A_k \tilde{x}_k(n_k) + B_k z_{k+1}^{k+2-n_{k+1}}. \quad (12)$$

where A_k, B_k are matrices of appropriate dimensions. Denote by $X_{k+1}(n_{k+1})$ the class of all unbiased estimators having the structure (12), where $\tilde{x}_k(n_k)$ is unbiased estimator too.

Definition 1. We shall call $\tilde{x}_{k+1} := \tilde{x}_{k+1}(n_{k+1})$ an optimal MMS estimator in the class $X_{k+1}(n_{k+1})$ if it satisfies

$$\text{(i) } E[\tilde{x}_{k+1}] = E(x);$$
$$\text{(ii) } \tilde{x}_{k+1} = \arg\min_{x' \in X_{k+1}^u} J(x'),$$
$$J(x') = \text{tr}[E(x'-x)(x'-x)^T]$$

where $X_{k+1}^u = \{x' \in X_{k+1}(n_{k+1}) : E(x') = E(x)\}$, tr (.) denotes the trace operator.

In the present paper, for simplicity, we assume the existence of all figured inverse matrices.

Lemma 1. Let $\tilde{x}_k(n_k) = \hat{x}$, P_k be its error covariance matrix (ECM). Then $\tilde{x}_{k+1}$ is defined by

$$\tilde{x}_{k+1}(n_{k+1}) = \hat{x} + K_{k+1}[z_{k+1}^*(n_{k+1}) - H_{k+1}^* \hat{x}], \quad (13)$$

$$z_{k+1}^*(n_{k+1}) = z_{k+1} - \Sigma_{21}\Sigma_{11}^{-1} z_k^{k+2-n_{k+1}}, \quad (14)$$
$$H_{k+1}^* = H_{k+1} - \Sigma_{21}\Sigma_{11}^{-1} H_k^{k+2-n_{k+1}}, \quad (15)$$
$$K_{k+1} = (P_k H_{k+1}^T - N_{k+1})\tilde{\Sigma}_{22}^{-1}, \quad (16)$$
$$\tilde{\Sigma}_{22} = [\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}]^{-1}, \quad (17)$$
$$\Sigma_{22} = R_{k+1} + H_{k+1} P_k H_{k+1}^T - H_{k+1} N_{k+1} - (H_{k+1} N_{k+1})^T, \quad (18)$$
$$\Sigma_{11} = V_k^{k+2-n_{k+1}} - H_k^{k+2-n_{k+1}} E_k^{k+2-n_{k+1}}, \quad (19)$$
$$\Sigma_{21} = \Sigma_{12}^T, \quad (20)$$
$$\Sigma_{12} = \bar{K}_k^{k+2-n_{k+1}} - H_k^{k+2-n_{k+1}} N_{k+1}, \quad (21)$$
$$N_{k+1} = E[(\hat{x}_k - x) v_{k+1}^T], \quad (22)$$
$$E_k^{k+2-n_{k+1}} = E[(\hat{x}_k - x) v_k^{k+1-n_{k+1},T}] = P_k H_k^{k+2-n_{k+1},T}, \quad (23)$$
$$\bar{K}_k^{k+2-n_{k+1}} = E[v_k^{k+2-n_{k+1}} v_{k+1}^T], \quad (24)$$
$$P_k = E[(\hat{x}_k - x)(\hat{x}_k - x)^T], \quad (25)$$
$$P_{k+1}(n_{k+1}) = E[(\tilde{x}_{k+1} - x)(\tilde{x}_{k+1} - x)^T] = P_k - K_{k+1}(P_k H_{k+1}^T - N_{k+1})^T. \quad (26)$$

Proof: The proof is similar to that presented in [8]: From the requirement (i) on unbiasedness of $\tilde{x}_{k+1}$ we have $A_k = I - B_k H_{k+1}^{k+2-n_{k+1}}$. Substituting A_k into (12) and taking the gradient of $J(.)$ with respect to B_k leads to the equation for finding B_k. Thus,

$$A_k = I - B_k H_{k+1}^{k+2-n_{k+1}}, B_k = -\Sigma_3 \Sigma_1^{-1}. \quad (27)$$

where Σ_1, Σ_3 are defined in (29). We have the ECM P_{k+1},

$$P_{k+1}(n_{k+1}) = B_k \Sigma_1 B_k^T + B_k \Sigma_2 + \Sigma_3 B_k^T + \Sigma_4, \quad (28)$$

$$\Sigma_1 = H_{k+1}^{k+2-n_{k+1}} P_k (H_{k+1}^{k+2-n_{k+1}})^T + V_{k+1}^{k+2-n_{k+1}} + \Delta\Sigma_1,$$
$$\Delta\Sigma_1 = -H_{k+1}^{k+2-n_{k+1}} E_{k+1}^{k+2-n_{k+1}} - (H_{k+1}^{k+2-n_{k+1}} E_{k+1}^{k+2-n_{k+1}})^T,$$

$$\Sigma_2 = -H_{k+1}^{k+2-n_{k+1}} P_k + (E_{k+1}^{k+2-n_{k+1}})^T,$$
$$\Sigma_3 = \Sigma_2^T, \Sigma_4 = P_k, \quad (29)$$

$E_{k+1}^{k+2-n_{k+1}}$ is defined by (23). Using A_k from (27) the estimator $\tilde{x}_{k+1}$ can be rewritten as

$$\tilde{x}_{k+1} = \hat{x}_k + B_k[z_{k+1}^{k+2-n_{k+1}} - H_{k+1}^{k+2-n_{k+1}} \hat{x}_k]. \quad (30)$$

Compute the matrix B_k. Since $z_k^1 = H_k^1 x + v_k^1$, the MMS estimate $\hat{x}_k = T_k z_k^1, T_k = P_k H_k^1 V_k^{1,-1}$, $P_k = (H_k^1 V_k^{1,-1} H_k^{1,T})^{-1}$, $V_k^1 = E(v_k^1 v_k^{1,T})$ is an unbiased estimate, $T_k H_k^1 = I$. Hence

$$E_{k+1}^{k+2-n_{k+1}} = E\{[T_k(H_k^1 x + v_k^1) - x](v_{k+1}^{k+2-n_{k+1}})^T\} =$$

$$= T_k E\{v_k^1 (v_{k+1}^{k+2-n_{k+1}})^T\}. \quad (31)$$

Let

$$V_k^1 = [V_k^T(1), ..., V_k^T(k)]^T, V_k(i) = E[v_i(v_i^1)^T],$$
$$V_k^{1,-1} = [\tilde{V}_k(1), ..., \tilde{V}_k(k)].$$

Then we have $V_k(i)\tilde{V}_k(j) = I\delta_{ij}$ where δ_{ij} is the Kronecker symbol. But $E[v_k^{k+2-n_{k+1}} v_k^{1,T}]$ $=[V_k^T(k+2-n_{k+1}), ..., V_k^T(k)]^T$ hence

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$$E_k^{k+2-n_{k+1}} = T_k E[v_k^1 (v_k^{k+2-n_{k+1}})^T] = ([V_k^T(k+2-n_{k+1}),...,V_k^T(k)]^T[\tilde{V}_k(1),...,\tilde{V}_k(k)] H_k^1 P_k)^T = [(O,I)\left\{\begin{array}{c} H_{k+1-n_{k+1}}^1 \\ H_k^{k+2-n_{k+1}} \end{array}\right\} P_k]^T = P_k H_k^{k+2-n_{k+1},T}. \quad (32)$$

Taking into account (29)(31)(32) one can write

$$B_k = -\Sigma_3\Sigma_1^{-1} = [E_{k+1}^{k+2-n_{k+1}} - P_k H_{k+1}^{k+2-n_{k+1},T}]\Sigma_1^{-1} = -[(E_k^{k+2-n_{k+1}}, N_{k+1}) - (P_k H_k^{k+2-n_{k+1},T}, P_k H_{k+1}^T)]\Sigma_1^{-1} = (0, P_k H_{k+1}^T - N_{k+1})\Sigma_1^{-1}$$

or

$$B_k = (P_k H_{k+1}^T - N_{k+1})\tilde{\Sigma}_{22}(-\Sigma_{21}\Sigma_{11}^{-1}, I), \quad (33)$$

here N_{k+1} is defined as in (22) and for

$$\Sigma_1 = \left\{\begin{array}{cc} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{array}\right\}, \Sigma_1^{-1} = \left\{\begin{array}{cc} \tilde{\Sigma}_{11} & \tilde{\Sigma}_{12} \\ \tilde{\Sigma}_{21} & \tilde{\Sigma}_{22} \end{array}\right\}, \quad (34)$$

the Lemma on Inversion of block matrix [10] yields

$$\tilde{\Sigma}_{22} = [\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}]^{-1}$$

which shows (17). Substituting (33) into (30) yields (13) with K_{k+1} defined in (16). The formula (26) is obtained by using (27),(28) and (33).

To show (18)-(20), noticing from (29) that Σ_1 can be written as the ECM of the following random vector

$$\Sigma_1 = E(\xi\xi^T), \xi := H_{k+1}^{k+2-n_{k+1}}(\hat{x}_k - x) + v_{k+1}^{k+2-n_{k+1}}.$$

Represent $\xi = (\xi_1^T, \xi_2^T)^T$, from (34) one sees that

$$\Sigma_{11} = V_k^{k+2-n_{k+1}} + H_k^{k+2-n_{k+1}} P_k (H_k^{k+2-n_{k+1}})^T - H_k^{k+2-n_{k+1}} E_k^{k+2-n_{k+1}} - (H_k^{k+2-n_{k+1}} E_k^{k+2-n_{k+1}})^T$$
$$\Sigma_{22} = R_{k+1} + H_{k+1}P_k H_{k+1}^T - H_{k+1}N_{k+1} - (H_{k+1}N_{k+1})^T,$$
$$\Sigma_{12} = \Sigma_{21}^T = \bar{K}_k^{k+2-n_{k+1}} + H_k^{k+2-n_{k+1}} P_k H_{k+1}^T - H_{k+1}N_{k+1} - (H_{k+1}N_{k+1})^T,$$

$\bar{K}_k^{k+2-n_{k+1}}$ is defined by (24). These formulas imply (18)-(20) noticing from (32) that Σ_{11}, Σ_{12} can be simplified. ■

Comment 1. As shown by [8], when Σ_1 in (27) is singular, the matrix B_k is defined by $B_k = -\Sigma_3\Sigma_1^+$. The solution $\tilde{x}_{k+1}$ then exists and is unique (almost surely). The uniqueness of $\tilde{x}_{k+1}$ for non-singular Σ_1 follows automatically.

Comment 2. The estimate $\tilde{x}_{k+1}$ can be obtained in the following way [9]: Interpreting $z^* := \hat{x}_k$ as the "observation" available before arriving z_{k+1},

$$z^* = x + \epsilon_k, E(\epsilon_k) = 0, E(\epsilon_k\epsilon_k^T) = P_k, z^* = \hat{x}_k \quad (35)$$

and introducing $\tilde{z} = (z^{*,T}, z_{k+2-n_{k+1}}^T, ..., z_{k+1}^T)^T$, one has the following system of observations

$$\tilde{z} = \tilde{H}x + \tilde{v}, \quad (36)$$

$$\tilde{z} = (z^{*,T}, z_{k+2-n_{k+1}}^T, ..., z_{k+1}^T)^T,$$
$$\tilde{H} = (I, H_{k+2-n_{k+1}}^T, ..., H_{k+1}^T)^T,$$
$$\tilde{v} = (\epsilon_k^T, v_{k+2-n_{k+1}}^T, ..., v_{k+1}^T)^T, \tilde{V} = E(\tilde{v}\tilde{v}^T).$$

Then with probability 1 the estimator $\tilde{x}_{k+1}$ in Lemma 1 is equal to

$$\tilde{x}_{k+1} = (\tilde{H}^T\tilde{V}^{-1}\tilde{H})^{-1}\tilde{H}^T\tilde{V}^{-1}\tilde{z}. \quad (37)$$

Really, the estimator (37) is a linear function of $\hat{x}_k$ and $z_{k+1}^{k+2-n_{k+1}}$. From Theorem 6.1.11 of [1] it is the BLUE (unbiased and of minimum variance). Thus (37) must be also a MMS estimator by Definition 1.

Theorem 1. Let $\{\tilde{x}_k(n_k)\}$ be a sequence of unbiased estimators for the unknown vector x such that each estimator is obtained on the basis of the previous one and the n_k latest observations. Let these estimators be MMS according to Definition 1 subject to $n_{k+1} \leq n_k + 1$. Then

$$\tilde{x}_{k+1}(n_{k+1}) = \tilde{x}_k(n_k) + K_{k+1}[z_{k+1}^* - H_{k+1}^*\tilde{x}_k(n_k)]$$

where $z_{k+1}^*, H_{k+1}^*, K_{k+1} = B_k$ are determined by the formulas similar to (14),(15),(27),(29), only now we have $\tilde{x}_k(n_k)$, $P_k(n_k)$ instead of $\hat{x}$, P_k,

$$P_k(n_k) = E\{[\tilde{x}_k(n_k) - x][\tilde{x}_k(n_k) - x]^T\}.$$

The proof of Theorem 1 is analogous to the proof of Lemma 1, noticing from Comment 2,

$$\tilde{x}_k(n_k) = T_k(n_k)z_k(n_k),$$
$$T_k(n_k) = P_k(n_k)H_k^T(n_k)V_k^{-1}(n_k),$$
$$P_k(n_k) := [H_k^T(n_k)V_k^{-1}(n_k)H_k(n_k)]^{-1},$$
$$z_k(n_k) = H_k(n_k)x + v_k(n_k), V_k(n_k) = E\{v_k(n_k)v_k^T(n_k)\},$$
$$z_k(n_k) = \left[\begin{array}{c} \tilde{x}_{k-1}(n_{k-1}) \\ z_k^{k+1-n_k} \end{array}\right], H_k(n_k) = \left[\begin{array}{c} I \\ H_k^{k+1-n_k} \end{array}\right],$$
$$v_k(n_k) = \left[\begin{array}{c} \tilde{\epsilon}_{k-1} \\ \tilde{v}_k^{k+1-n_k} \end{array}\right],$$

and from $n_{k+1} \leq n_k + 1$ the matrix $E_k^{k+2-n_{k+1}}$ is simplified to the form (32). Thus the condition $n_{k+1} \leq n_k + 1$ is introduced only for having the compact formulas (14)-(26).

As will be seen later, the case $n_{k+1} = 2$ is of special interest and it is formulated in the form of the following Corollary

Corollary 1. Let in Theorem 1, $n_{k+1} = 2$. Then the following relations hold for the estimator $\tilde{x}_{k+1}(2)$ satisfying Definition 1,

$$\tilde{x}_{k+1}(2) = \tilde{x}_k(n_k) + K_{k+1}(2)[z_{k+1}^* - H_{k+1}^*\tilde{x}_k(n_k)], \quad (38)$$
$$z_{k+1}^* = z_{k+1} - \Sigma_{21}\Sigma_{11}^{-1}z_k, \quad (39)$$
$$H_{k+1}^* = H_{k+1} - \Sigma_{21}\Sigma_{11}^{-1}H_k, \quad (40)$$
$$K_{k+1}(2) = [P_k(n_k)H_{k+1}^T - N_{k+1}]\tilde{\Sigma}_{22}, \quad (41)$$
$$\tilde{\Sigma}_{22} = [\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12}]^{-1}, \quad (42)$$
$$\Sigma_{11} = R_k - H_kP_k(n_k)H_k^T, \Sigma_{12} = R_{k,k+1} - H_kN_{k+1}, \quad (43)$$

where N_{k+1}, Σ_{22} are defined in Theorem 1.

Mention that the structure of the filter (38)-(43) is similar to that of the Gevers-Kailath filter [6].

Corollary 2. Let $\tilde{x}_k(n_k) = \hat{x}_k$. For $n_{k+1} = k+1$, Theorem 1 yields $\tilde{x}_{k+1} = \hat{x}_{k+1}$ and the following equality holds

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$$\Delta\Sigma_{22} := -\Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12} = L_1 - L_2,$$

$$L_1 := \bar{K}_k^T V_k^{1,-1} H_k^1 P_k H^{1,T} V_k^{1,-1}\bar{K}_k, L_2 := \bar{K}_k^T V_k^{1,-1}\bar{K}_k. \quad (44)$$

Here $\bar{K}_k = \bar{K}_k^1$ and $\bar{K}_k^i$ is defined by (24).

The equality (44) will be used in the further.

III. Time-varying system state

Consider the filtering problem in its general formulation (1)(3)-(5). A natural way to generalize (12) in this case is to introduce the class of recursive filters

$$\tilde{x}_{k+1}(n_{k+1}) = A_k\tilde{x}_{k+1/k}(n_{k+1}) + B_k z_{k+1}^{k+2-n_{k+1}}, \quad \tilde{x}_{k+1/k}(n_{k+1}) := \Phi_k\tilde{x}_k(n_k) \quad (45)$$

where $\{\tilde{x}_k(n_k)\}$ is a sequence of filtered estimates for the system state x_k, $k = 1, 2, ...$. The results to be presented below can be established by the same technique as done in section 2.1.

Introduce the notations: Let $\Phi(i, j)$ be the transition matrix for the system (1). Then [10]

$$x_j = \Phi(j,i)x_i - \sum_{l=j}^{i-1}\Phi(j, l+1)\Gamma_l w_l, i > j, \quad (46)$$

$$\tilde{H}_{k+1}^1 = (H^T(1, k+1), H^T(2, k+1), ..., H^T(k+1, k+1))^T, H(j,i) = H_j\Phi(j,i), \quad (47)$$

$$w(i, k+1) = v_i - \textstyle\sum_{l=i}^k H_i\Phi(i, l+1)\Gamma_l w_l, \sum_{l=k+1}^k = 0,$$

$$\tilde{w}_{k+1}^1 = [w^T(1, k+1), ..., w^T(k+1, k+1)]^T = (\eta_k^{1,T}, v_{k+1}^T)^T, \quad (48)$$

$$E(\tilde{w}_{k+1}^1) = 0, \quad E[\tilde{w}_{k+1}^1\tilde{w}_{k+1}^{1,T}] = W_{k+1}^1 \quad (49)$$

$$E(\eta_k^1) = 0, E[\eta_k^1\eta_k^{1,T}] = \Lambda_k^1. \quad (50)$$

Using the notations above we have

$$z_j = H(j, k+1)x_{k+1} + w(j, k+1), j = 1, 2, ..., k+1, \quad (51)$$

$$z_{k+1}^{k+2-n_{k+1}} = \tilde{H}_{k+1}^{k+2-n_{k+1}}x_{k+1} + \tilde{w}_{k+1}^{k+2-n_{k+1}}, \quad (52)$$

$$z_{k+1}^1 = \tilde{H}_{k+1}^1 x_{k+1} + \tilde{w}_{k+1}^1, \quad (53)$$

$$\tilde{H}_{k+1}^1 = [\bar{H}_k^{1,T}, H_{k+1}^T]^T, \bar{H}_k^1 = \tilde{H}_k^1\Phi(k, k+1), \quad (54)$$

$$\eta_k^1 = \tilde{w}_k^1 - \bar{H}_k^1\Gamma_k w_k. \quad (55)$$

In the further, according to [11] we will refer to the model (53) as of *high initial uncertainty* if the information on x_{k+1} is contained only in the observation vector z_{k+1}^1 (equivalently to assuming $M_0 = \infty$ - there is no a priori information on x_{k+1}). For the case (3)-(5)(7) are given, from (1) $x_{k+1} = \Phi(k+1, 0)x_0 + \sum_{l=0}^k\Phi(k+1, l+1)\Gamma_l w_l$ and

$$\bar{x}_{k+1/0} = \Phi(k+1, 0)\bar{x}_0, \; P_{k+1/0} = E[(\epsilon_{k+1/0})(\epsilon_{k+1/0})^T], \quad \epsilon_{k+1/0} := \bar{x}_{k+1/0} - x_{k+1}.$$

This information can be represented by the additional "observation" $z^* := \bar{x}_{k+1/0}$ (Comment 2) and instead of (53) we have the following model

$$z_{k+1}^{(1)} = \tilde{H}_{k+1}^{(1)}x_{k+1} + \tilde{w}_{k+1}^{(1)}. \quad (56)$$

$$z_{k+1}^{(1)} = \left\{\begin{array}{c}\bar{x}_{k+1/0}\\ ......\\ z_{k+1}^1\end{array}\right\}, \tilde{H}_{k+1}^{(1)} = \left\{\begin{array}{c} I\\ ......\\ \tilde{H}_{k+1}^1\end{array}\right\}, \quad \tilde{w}_{k+1}^{(1)} = \left\{\begin{array}{c}\epsilon_{k+1/0}\\ ......\\ \tilde{w}_{k+1}^1\end{array}\right\}.$$

As the model (56) includes the information (3)-(5) in the form of z^*, it can be considered as that of high initial uncertainty. Later on, for simplicity we shall derive the filtering algorithms for the vector x_{k+1} in the model (53) remembering that the similar results can be deduced for x_{k+1} in the model (56).

Application of Theorem 1 to the model (53) leads to the following

Theorem 2. Consider the class of recursive filters (45) and let $\{\tilde{x}_{k+1}(n_{k+1})\}$ be a sequence of unbiased estimators for x_{k+1} such that each estimator $\tilde{x}_{k+1} = \tilde{x}_{k+1}(n_{k+1})$ is a function of the previous $\tilde{x}_k(n_k)$ and n_{k+1} last observations. Assume that these estimators are optimal in the sense of Definition 1. Then we have

$$\tilde{x}_{k+1}(n_{k+1}) = \Phi_k\tilde{x}_k(n_k) + K_{k+1}[z_{k+1}^* - H_{k+1}^*\Phi_k\tilde{x}_k(n_k)],$$
$$z_{k+1}^* = z_{k+1}^{k+2-n_{k+1}}, H_{k+1}^* = \tilde{H}_{k+1}^{k+2-n_{k+1}}$$

and K_{k+1} can be determined as done in the proof of Lemma 1.

Corollary 3. Let $\tilde{x}_{k+1/k} = \hat{x}_{k+1/k} = \Phi_k\hat{x}_k$. Then

$$\tilde{x}_{k+1}(n_{k+1}) = \hat{x}_{k+1/k} + K_{k+1}[z_{k+1}^* - \tilde{H}_{k+1}^*\hat{x}_{k+1/k}],$$
$$\hat{x}_{k+1/k} = [\tilde{H}_k^{1,T}\Lambda_k^{1,-1}\tilde{H}_k^1]^{-1}\tilde{H}_k^{1,T}\Lambda_k^{1,-1}z_k^1, \; \Lambda_k^i = E[\eta_k^i\eta_k^{i,T}],$$
$$z_{k+1}^* = z_{k+1} - \Sigma_{21}\Sigma_{11}^{-1}z_k^{k+2-n_{k+1}},$$
$$H_{k+1}^* = H_{k+1} - \Sigma_{21}\Sigma_{11}^{-1}\tilde{H}_{k+1}^{k+2-n_{k+1}},$$
$$K_{k+1} = (M_{k+1}H_{k+1}^T - N_{k+1})\tilde{\Sigma}_{22},$$
$$\tilde{\Sigma}_{22} = [\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12}]^{-1},$$
$$\Sigma_{22} = R_{k+1} + H_{k+1}M_{k+1}H_{k+1}^T - H_{k+1}N_{k+1} - (H_{k+1}N_{k+1})^T,$$
$$\Sigma_{11} = \Lambda_k^{k+2-n_{k+1}} - \tilde{H}_k^{k+2-n_{k+1}}E_k^{k+2-n_{k+1}}$$
$$\Sigma_{12} = \bar{K}_k^{k+2-n_{k+1}} - \tilde{H}_k^{k+2-n_{k+1}}N_{k+1},$$
$$N_{k+1} = E[e_{k+1/k}v_{k+1}^T],$$
$$E_k^{k+2-n_{k+1}} = E[e_{k+1/k}\eta_k^{k+2-n_{k+1},T}],$$
$$e_{k+1/k} := \hat{x}_{k+1/k} - x_{k+1},$$
$$\bar{K}_k^{k+2-n_{k+1}} = E[\eta_k^{k+2-n_{k+1}}v_{k+1}^T],$$
$$P_{k+1}(n_{k+1}) = M_{k+1} - K_{k+1}(M_{k+1}H_{k+1}^T - N_{k+1})^T,$$
$$M_{k+1} = [\tilde{H}_k^{1,T}\Lambda_k^{1,-1}\tilde{H}_k^1]^{-1},$$
$$\eta_k^i = (\eta_i^T, ..., \eta_k^T)^T = (w^T(i, k+1), ..., w^T(k+1, k+1))^T.$$

Corollary 4 (Case $n_{k+1} = 2$). Under the conditions of Corollary 3, for $n_{k+1} = 2$

$$\tilde{x}_{k+1}(2) = \hat{x}_{k+1/k} + K_{k+1}[z_{k+1}^* - H_{k+1}^*\hat{x}_{k+1/k}],$$
$$z_{k+1}^* = z_{k+1} - \Sigma_{21}\Sigma_{11}^{-1}z_k,$$
$$H_{k+1}^* = H_{k+1} - \Sigma_{21}\Sigma_{11}^{-1}H_{k+1,k},$$
$$K_{k+1} = (M_{k+1}H_{k+1}^T - N_{k+1})\tilde{\Sigma}_{22},$$

$$\tilde{\Sigma}_{22} = [\Sigma_{22}^0 - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12}]^{-1},$$
$$\Sigma_{11} = \Lambda_k^k - H_{k+1,k} M_{k+1} H_{k+1,k}^T,$$
$$\Sigma_{12} = \bar{K}_k^k - H_{k+1,k} N_{k+1}.$$

Other variables in Corollary 4 are defined as in Corollary 3.

Corollary 5. Under the conditions of Corollary 3, for $n_{k+1} = k+2$,

$$\tilde{x}_{k+1}(k+2) = \hat{x}_{k+1},$$
$$\Delta\Sigma_{22} := -\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} = L_1 - L_2$$

$$L_1 := \bar{K}_k^T \Lambda_k^{1,-1} \tilde{H}_k^1 M_{k+1} \tilde{H}_k^{1,T} \Lambda_k^{1,-1} \bar{K}_k,$$
$$L_2 := \bar{K}_k^T \Lambda_k^{1,-1} \bar{K}_k, \bar{K}_k = \bar{K}_k^1. \quad (57)$$

Comment 3. It is important to stress that Eq. (46) is valid under hypothesis of existence of the inverse of the fundamental matrix of the system dynamics. That is usually the case when the discretized system is obtained from a differential equation. Even then for many practical filtering problems, the difficulties arise when the matrix Φ_k exists only in a numerical form or when it is of very high dimension as a result of discretization from the set of partial differential equations [4]. In such situations, instead of Eq. (46) it would be better to express $x_j, j = k+1-n_{k+1}, ..., k+1$ as a function of $x_{k+1-n_{k+1}}$. By this way we need only to integrate the direct model to generate the predictor for the system state x_{k+1}.

IV. Properties of the filter

Property 1. For all $n_{k+1}, 1 \leq n_{k+1} \leq n_{k+1}$,

$$\text{tr } P_{k+1}(n_{k+1}) \leq \text{tr } P_k(n_k),$$

i.e. the estimator $\tilde{x}_{k+1}$ is better than $\tilde{x}_k$ in MMS sense.

The proof of this fact follows immediately from (26) since the matrix $K_{k+1}(n_{k+1})[P_k(n_k)H_{k+1}^T - N_{k+1}]$ is non-negative definitive.

In the further the symbol $A \geq 0$ (or $A \leq 0$) signifies that the matrix A is non-negative (or non-positive) definitive.

Property 2. For all $m_{k+1}, n_{k+1}, 1 \leq m_{k+1} \leq n_{k+1} \leq k+2$ the following inequality holds

$$P_{k+1}(n_{k+1}) \leq P_{k+1}(m_{k+1}), \text{or} \quad (58)$$
$$\text{tr}P_{k+1}(n_{k+1}) \leq \text{tr}P_{k+1}(m_{k+1}) \quad (59)$$

In order to prove (59) we need some auxiliary results.

Lemma 2. Let S be a square matrix

$$S = \left\{ \begin{array}{cc} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{array} \right\},$$

where S_{11}, S_{22} are symmetric matrices of dimensions ($n \times n$) and ($m \times m$) respectively. Then $S \geq 0$ if and only if (iff) $S_{11} \geq 0$, $S_{11}S_{11}^+S_{12} = S_{12}, S_{22} - S_{12}^+ S_{11} S_{12} \geq 0$. Here the symbol "+" denotes the operation of pseudo-inversion of matrix.

Lemma 2 is the statement (a) in Theorem (9.1.6) of [1].

In what follows for simplicity we use the notation $\Sigma = \Sigma_{22}$, $\tilde{\Sigma} := \Sigma_{22}^{-1}$, Σ_{22} is defined by (17). The notation $C(m_{k+1})$ signifies that this matrix corresponds to the matrix C defined in the recursive filter with the estimates depending on m_{k+1} latest observations. Let

$$\Sigma(m_{k+1}) := \tilde{\Sigma}^{-1}(m_{k+1}).$$

Lemma 3. For all $n_{k+1} \geq m_{k+1}$,

$$\Sigma(n_{k+1}) - \Sigma(m_{k+1}) \leq 0. \quad (60)$$

Proof:

It is not hard to see that one can represent

$$\Sigma_{21}(m_{k+1})\Sigma_{11}^{-1}(m_{k+1})\Sigma_{12}(m_{k+1}) =$$
$$\Sigma_{21}(n_{k+1})\Sigma_{11}^{n_{k+1}}(m_{k+1})\Sigma_{12}(n_{k+1}), \quad (61)$$
$$\Sigma_{11}^{n_{k+1}}(m_{k+1}) = \left\{ \begin{array}{cc} 0 & 0 \\ 0 & \Sigma_{11}^{-1}(m_{k+1}) \end{array} \right\}. \quad (62)$$

The obtained expressions (61)(62) allow us to present the difference $\Sigma(m_{k+1}) - \Sigma(n_{k+1})$ in the form

$$\Delta\Sigma_{k+1}(n;m) = \Sigma(m_{k+1}) - \Sigma(n_{k+1}) =$$
$$\Sigma_{12}^T(n_{k+1}) A \Sigma_{12}(n_{k+1}), \quad (63)$$
$$A := \Sigma_{11}^{-1}(n_{k+1}) - \Sigma_{11}^{n_{k+1}}(m_{k+1}). \quad (64)$$

Denote by $\tilde{\Sigma}_{11}(n_{k+1})$ the inverse matrix for $\Sigma_{11}(n_{k+1})$,

$$\Sigma_{11}^{-1}(n_{k+1}) = \tilde{\Sigma}_{11}(n_{k+1}) = \left\{ \begin{array}{cc} \tilde{\Sigma}_1 & \tilde{\Sigma}_2 \\ \tilde{\Sigma}_3 & \tilde{\Sigma}_4 \end{array} \right\}. \quad (65)$$

The formulas (65)(62) imply

$$A = \left\{ \begin{array}{cc} \tilde{\Sigma}_1 & \tilde{\Sigma}_2 \\ \tilde{\Sigma}_2^T & \tilde{\Sigma}_4 - \Sigma_{11}^{-1}(m_{k+1}) \end{array} \right\}.$$

The Lemma will be proven if we can show that $A \geq 0$. According to Lemma 2, we need to establish

(a) $\tilde{\Sigma}_1 \geq 0$,

(b) $\tilde{\Sigma}_1 \tilde{\Sigma}_1^+ \tilde{\Sigma}_2 = \tilde{\Sigma}_2$,

(c) $[\tilde{\Sigma}_4 - \Sigma_{11}^{-1}(m_{k+1})] - \tilde{\Sigma}_2^T \tilde{\Sigma}_1^+ \tilde{\Sigma}_2 \geq 0$.

First mention that from the existence of $\Sigma_1^{-1}(n_{k+1})$ we have $\Sigma_1(n_{k+1}) > 0$. The structure (34) implies then $\Sigma_{11}(n_{k+1}) > 0$ hence $\tilde{\Sigma}_{11}(n_{k+1}) > 0$. This fact and (65) prove $\tilde{\Sigma}_1 > 0$ and we have (a), (b). It remains to show (c). In fact the left-hand side of (c) is equal to 0. First mention that by construction,

$$\Sigma_{11}(n_{k+1}) = [S_{ij}]_{i,j=1}^2$$

where S_{ij} are block matrices of appropriate dimension, with $S_{22} = \Sigma_{11}(m_{k+1})$. Now, as $\tilde{\Sigma}_{11}^{-1}(n_{k+1}) = \Sigma_{11}(n_{k+1})$, according to Lemma on inversion of block matrix [10] we have $S_{22} = \Sigma_{11}(m_{k+1}) = (\tilde{\Sigma}_4 - \tilde{\Sigma}_2^T \tilde{\Sigma}_1^{-1} \tilde{\Sigma}_2)^{-1}$ or $\Sigma_{11}^{-1}(m_{k+1}) = \tilde{\Sigma}_4 - \tilde{\Sigma}_2^T \tilde{\Sigma}_1^{-1} \tilde{\Sigma}_2$ or $[\tilde{\Sigma}_4 - \Sigma_{11}^{-1}(m_{k+1})] - \tilde{\Sigma}_2^T \tilde{\Sigma}_1^{-1} \tilde{\Sigma}_2 = 0$. Thus all three conditions (a)-(c) hold. ■

The property 2 follows immediately from the formulas similar to (26)(16)(17) and Lemma 3.

Analogously one can establish the properties 1-2 for the filter in section 3.

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V. On equivalence of two estimators based on different numbers of last observations

Consider two estimators $\tilde{x}_{k+1}(n_{k+1})$ and $\tilde{x}_{k+1}(m_{k+1})$ derived according to Theorem 1 on the basis of n_{k+1} and m_{k+1} last observations respectively, with $n_{k+1} > m_{k+1}$.

Definition 2. Two estimators $\tilde{x}_{k+1}(n_{k+1})$ and $\tilde{x}_{k+1}(m_{k+1})$ are called equivalent if the following equality holds

$$\text{tr}P_{k+1}(n_{k+1}) = \text{tr}P_{k+1}(m_{k+1}). \tag{66}$$

The Definition 2 is correct since the solution of the problem in Theorem 1 (Definition 1) is unique (see Comment 1).

In what follows a new definition, equivalent to Definition 2, will be introduced. This new definition allows us to easier examine the equivalence of two estimators. First we need some preliminary results.

Lemma 4. Let the matrix $C = A - B \geq 0$ be symmetric. Then

$$\text{tr } C = 0 \text{ iff } A = B.$$

Proof: It is well known if S is an orthonormal matrix then tr $(S^TCS) =$ tr (C). Let S be orthonormal matrix diagonalizing C. Then tr $(C) =$ tr $(S^TCS) =$ tr $(\Lambda) = \sum_{i=1}^n \lambda_i(C)$ where $\lambda_i(C)$ are the eigenvalues of C, $\lambda_i(C) \geq 0$, n is the dimension of C. Suppose tr $(C) = 0$. Then $\sum_{i=1}^n \lambda_i(C) = 0$ or $\lambda_i(C) = 0$, $\forall i = 1, ..., n$. The last means that $C = 0$ since the number of non-zero eigenvalues of C is equal to the rank of C. Hence $A - B = 0$ or $A = B$.

The inverse implication is trivial. ∎

According to Property 2, for $n_{k+1} > m_{k+1}$ we have

$$\text{tr } [P_{k+1}(n_{k+1})] \leq \text{tr } [P_{k+1}(m_{k+1})] \text{ or}$$
$$P_{k+1}(m_{k+1}) - P_{k+1}(n_{k+1}) \geq 0.$$

Taking into account Lemma 4 and the properties of $P_{k+1}(n_{k+1})$, Definition 2 can be replaced by the following

Definition 3. Two estimators $\tilde{x}_{k+1}(n_{k+1})$ and $\tilde{x}_{k+1}(m_{k+1})$ are called equivalent if the following equality holds

$$P_{k+1}(n_{k+1}) = P_{k+1}(m_{k+1}). \tag{67}$$

Lemma 5. Let $A - B \geq 0$ be symmetric matrix. Then $C^T(A - B)C = 0$ iff $(A - B)C = 0$.

Proof: According to the definition of a symmetric non-negative definitive matrix [1] there exists a matrix D such that $A - B = D^TD$. But then $C^TD^TDC = 0$ and this takes place iff $DC = 0$. We have then $D^TDC = 0$ or $(A - B)C = 0$. The inverse implication is trivial since from $D^TDC = 0$ it follows $C^TD^TDC = 0$. ∎

Proposition 1. Two estimators $\tilde{x}_{k+1}(n_{k+1})$ and $\tilde{x}_{k+1}(m_{k+1})$, obtained from Corollary 2, are equivalent iff

$$\Delta\Sigma_{k+1}(n; m)\Sigma^{-1}(m_{k+1})[P_k(n_k)H_{k+1}^T - N_{k+1}]^T = 0 \tag{68}$$

where $\Delta\Sigma_{k+1}(n; m)$ is defined by (63)(64).

Proof: Inserting $P_{k+1}(n_{k+1}), P_{k+1}(m_{k+1})$ into (67), from (26) it follows

$$A[\Sigma^{-1}(n_{k+1}) - \Sigma^{-1}(m_{k+1})]A^T = 0,$$
$$A := P_k(n_k)H_{k+1}^T - N_{k+1}$$

According to Lemma 3, $\Sigma(n_{k+1}) \leq \Sigma(m_{k+1})$ hence $\Sigma^{-1}(n_{k+1}) - \Sigma^{-1}(m_{k+1}) \geq 0$. Using Lemma 5 one can see that

$$[\Sigma^{-1}(n_{k+1}) - \Sigma^{-1}(m_{k+1})]A^T = 0. \tag{69}$$

As

$$\Sigma^{-1}(n_{k+1}) - \Sigma^{-1}(m_{k+1})$$
$$= -\Sigma^{-1}(n_{k+1})\Delta\Sigma_{k+1}(n; m)\Sigma^{-1}(m_{k+1}),$$

the condition (68) follows from substituting of the last equality into (69). ∎

Proposition 2. Two estimators $\tilde{x}_{k+1}(n_{k+1})$ and $\tilde{x}_{k+1}(m_{k+1})$, obtained from Corollary 3, are equivalent iff

$$\Delta\Sigma_{k+1}(n; m)\Sigma^{-1}(m_{k+1})[M_{k+1}H_{k+1}^T - N_{k+1}]^T = 0 \tag{70}$$

where $\Delta\Sigma_{k+1}(n; m)$ are defined as in Proposition 1, M_{k+1}, N_{k+1} are computed from Proposition 1 too.

VI. Applications

A. Optimal in MMS filtering with the Markovian observation noise

Consider the filtering problem (1)(2) with the following conditions

$$Q_{ij} = Q_i\delta_{ij}, K_{xw}(i) = 0, K_{xv}(i) = 0, K_{vw}(i) = 0, \forall i, j, \tag{71}$$
$$v_{i+1} = \Psi_i v_i + \xi_i, \tag{72}$$

where v_0 is uncorrelated with $\{w_i\}, \{xi_i\}$, x_0 is a random vector, $\{\xi_i\}$ is a white random sequence with

$$E(v_0) = 0, E(v_0v_0^T) = R_0 \tag{73}$$
$$E(\xi_i) = 0, E(\xi_i\xi_i^T) = \Xi_i\delta_{ij} \tag{74}$$

The filtering problem (1)(2)(71)-(74) is studied in [6]. On the basis of the results in section 3 we will derive here the solution to the filtering problem (1)(2)(71)-(74).

1) The filter for time-invariant system state: Denote by $\Psi(i, j)$ the transition matrix for the system (72). We have then

$$v_i = \Psi(i, j)v_j + \sum_{l=j}^{i-1} \Psi(i, l + 1)\xi_l, i \geq j + 1. \tag{75}$$

Lemma 6. The following relations hold

$$R_{ij} = E(v_iv_j^T) = \Psi(i, j)R_j, i \geq j + 1, \tag{76}$$
$$R_{j+1} = E(v_{j+1}v_{j+1}^T) = \Psi_jR_j\Psi_j^T + \Xi_j. \tag{77}$$

The Lemma 6 is proven by direct calculation of R_{ij}, R_{j+1} using the formulas (71)-(74), (75).

We proceed now to demonstrate that the condition (68) holds for $\tilde{x}(n_k) = \hat{x}_k$, $n_{k+1} = k + 2, m_{k+1} = 2$. Let us first compute $\Sigma(k + 2), \Sigma(2)$.

Make use of (44) for computing $\Sigma(k + 2)$. From Lemma 6 and Eq. (24) one has

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$$\bar{K}_k^T = [\Psi(k+1,0)R_0, ..., \Psi(k+1,k)R_k] = \Psi_k[\bar{K}_{k-1}^T, R_k]$$

For $V_k := V_k^1 = (V_{k,1}^T, V_{k,2}^T)^T$, V_k^1 is defined in (11), $V_{k,2}$ has p rows, it is not hard to see that

$$\bar{K}_k^T = \Psi_k V_{k,2}. \qquad (78)$$

Let $V_k^{-1} := (\tilde{V}_{k,1}^T, \tilde{V}_{k,2}^T)^T$. Then

$$V_{k,2}\tilde{V}_{k,1} = 0, V_{k,2}\tilde{V}_{k,2} = I \qquad (79)$$

hence

$$\bar{K}_k^T V_k^{-1} = (0, \Psi_k). \qquad (80)$$

Substituting (80) into (44) yields

$$\Sigma(k+1) = \Sigma_{22} + \Psi_k H_k P_k (\Psi_k H_k)^T - \Psi_k R_k \Psi_k^T. \qquad (81)$$

Consider $\Sigma(2)$ defined by (42)-(43). By Lemma 6 and (22) it follows

$$N_{k+1} = P_k H_k \Psi_k^T, \qquad (82)$$
$$\Sigma_{12}(2) = (R_k - H_k P_k H_k^T)\Psi_k^T. \qquad (83)$$

Now the equality $\Sigma(k+1) = \Sigma(2)$ holds by inserting (83)(43) into (42), taking into account (81). Thus $\Delta\Sigma_{k+1}(k+1;2) = 0$ the relationship (68) is valid in this case. It means, in virtue of Proposition 1, that $\tilde{x}_{k+1}(2) = \tilde{x}_{k+1}(k+1) = \hat{x}_{k+1}$, i.e. this Corollary yields in this case the optimal in MMS filter for $n_i = 2, \forall i$. We have hence

Theorem 3. The optimal in MMS filter for the filtering problem (1)(2)(7)(71)-(74) is given in the form

$$\hat{x}_{k+1} = \hat{x}_k + K_{k+1}(z_{k+1}^* - H_{k+1}^* \hat{x}_k),$$
$$K_{k+1} = P_k H_{k+1}^{*,T} \Xi_k = P_k H_{k+1}^{*,T}[H_{k+1}^* P_k H_{k+1}^{*,T} + \Xi_k]^{-1},$$
$$P_{k+1} = [P_k^{-1} + H_{k+1}^{*,T} \Xi_k H_{k+1}^*]^{-1},$$
$$H_{k+1}^* = H_{k+1} - \Psi_k H_k,$$
$$z_{k+1}^* = z_{k+1} - \Psi_k z_k.$$

2) The filter for time-varying system state: With the notations (46)(46)(48)-(50) and the assumption (71)-(74), from Corollary 3 we have (for simplicity, $W_k := W_k^1$, $\Lambda_k := \Lambda_k^1$)

$$\Lambda_k = W_k + \Delta W_k, \Delta W_k =$$
$$\tilde{H}_k^1 \Phi(k,k+1)\Gamma_k Q_k (\tilde{H}_k^1 \Phi(k,k+1)\Gamma_k)^T, \qquad (84)$$
$$W_k = V_k + \Delta W_k, \qquad (85)$$
$$\Delta W_k = \left\{ \begin{array}{cc} \Delta W_{11} & \Delta W_{12} \\ \Delta W_{12}^T & \Delta W_{22} \end{array} \right\}, \qquad (86)$$

where $\Delta W_{12}, \Delta W_{22}$ are zero matrices, ΔW_{22} is of $(p \times p)$; $\tilde{H}_k^1$ is defined by (54) and $\bar{K}_k$ remains as before and is defined by (78). From Matrix inversion lemma [7],

$$\Lambda_k^{-1} = W_k^{-1}[I - (I + \Delta W_k W_k^{-1})^{-1} \Delta W_k W_k^{-1}] =$$
$$W_k^{-1} B_1 = W_k^{-1}[I - B_2 \Delta W_k W_k^{-1}], \qquad (87)$$
$$B_1 := I - (I + \Delta W_k W_k^{-1})^{-1} \Delta W_k W_k^{-1}, \qquad (88)$$
$$B_2 := (I + \Delta W_k W_k^{-1})^{-1} \qquad (89)$$

and the matrix W_k^{-1} is equal to

$$W_k^{-1} = V_k^{-1}[I - \Delta V_k (I + V_k^{-1} \Delta V_k)^{-1} V_k^{-1}]. \qquad (90)$$

As (79)(86) imply $\bar{K}_k^T V_k^{-1} \Delta V_k = 0$, taking into account (90)(80) we have

$$\bar{K}_k^T W_k^{-1} = (0, \Psi_k), \qquad (91)$$
$$\bar{K}_k^T \Lambda_k^{-1} = (0, \Psi_k) B_1. \qquad (92)$$

Let $\tilde{H} := \tilde{H}_k^1$. From (88)(84) it implies

$$B_1 \tilde{H} = \tilde{H} - B_2 \Delta W_k W_k^{-1} \tilde{H} = \tilde{H} - B_2 B_4 = \tilde{H} - B_3, \qquad (93)$$
$$B_4 := \Delta W_k W_k^{-1} \tilde{H} = \tilde{H} \Gamma Q \Gamma^T (\Phi P \Phi^T)^{-1}, \qquad (94)$$

where

$$B_3 := B_2 B_4 = \tilde{H} \Gamma Q \Gamma^T (\Phi P \Phi^T)^{-1} - B_5 B_4 =$$
$$\tilde{H}[I - \Phi P \Phi^T M^{-1} \Gamma Q \Gamma^T (\Phi P \Phi^T)^{-1}] \Gamma Q \Gamma^T (\Phi P \Phi^T)^{-1} =$$
$$\tilde{H}[I + \Gamma Q \Gamma^T (\Phi P \Phi^T)^{-1}]^{-1} \Gamma Q \Gamma^T (\Phi P \Phi^T)^{-1} =$$
$$\tilde{H}[I - (I + \Gamma Q \Gamma^T (\Phi P \Phi^T)^{-1}]^{-1} = \tilde{H}[I - \Phi P \Phi^T M^{-1}].$$

Thus

$$B_3 = \tilde{H}[I - \Phi P \Phi^T M^{-1}]. \qquad (95)$$

In derivation of (95), for simplicity, the sub-index k is omitted for the matrices $\Phi, P, \Gamma ...$ and the following formulas have been used

$$M = M_{k+1} = \Phi P \Phi^T + \Gamma Q \Gamma^T, \qquad (96)$$
$$B_2 := I - B_5 = I - \tilde{H} \Phi P \Phi^T M^{-1} \Gamma Q \Gamma^T \tilde{H}^T W^{-1},$$
$$\Phi P \Phi^T M^{-1} = [I - \Gamma Q \Gamma^T (\Phi P \Phi^T)^{-1}]^{-1}.$$

with M_{k+1} defined in Corollary 3. The proof of (96) will be given later. Taking into account (95), the formula (93) is equivalent to

$$B_1 \tilde{H} = \tilde{H} \Phi P \Phi^T M^{-1}. \qquad (97)$$

Return to (57). Taking into account (95)(97) one can transform L_1 in (57) into

$$L_1 = (0, \Psi) B_1 \tilde{H} M \tilde{H}^T B_1^T (0, \Psi)^T =$$
$$(0, \Psi) \tilde{H} \Phi P \Phi^T M^{-1} M M^{-1} \Phi P \Phi^T \tilde{H}^T (0, \Psi)^T \text{ or}$$

$$L1 = \Psi H_k P \Phi^T P M^{-1} \Phi P H_k^T \Psi^T. \qquad (98)$$

Let us calculate $B_2 \Delta W_k$ using (94)(95),

$$B_2 \Delta W_k = B_2 \tilde{H} \Gamma Q \Gamma^T \tilde{H}^T =$$
$$B_2 \tilde{H} \Gamma Q \Gamma^T (\Phi P \Phi^T)^{-1} \Phi P \tilde{H}^T =$$
$$B_3 \Phi P \tilde{H}^T = \tilde{H}[I - \Phi P \Phi^T M^{-1}] \Phi P \tilde{H}^T \text{ or}$$

$$B_2 \Delta W_k = \tilde{H}[P - P \Phi^T M^{-1} \Phi P] \tilde{H}^T. \qquad (99)$$

The formula (99) can be used for simplifying L_2 as follows

$$L_2 = \bar{K}_k^T \Lambda_k^{-1} \bar{K}_k = \bar{K}_k^T W_k^{-1} [W_k - B_2 \Delta W_k] W_k^{-1} \bar{K}_k =$$
$$= (0, \Psi)[W_k - H^1 (P - P \Phi^T M^{-1} \Phi P H^{1,T}](0, \Psi)^T =$$

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$$= \Psi R_k \Psi^T - \Psi H_k (P - P\Phi^T M \Phi P) H_k^T \Phi_k^T$$

therefore $\Delta\Sigma_{22}(k+1) := L_1 - L_2 = \Psi_k(H_k P H_k - R_k)\Psi_k^T$.

Compute $\Delta\Sigma_{22}(2) := -\Sigma_{12}^T(2)\Sigma_{11}^{-1}(2)\Sigma_{12}(2)$ from Corollary 4. We have

$$E[w(k,k+1)w^T(k+1,k+1)] = \\ E\{(v_k - H_k \Gamma w_k)v_{k+1}^T\} = R_k \Psi^T, \\ N_{k+1} = M\tilde{H}^T \Lambda_k^{-1} \bar{K} = \Phi P H_k^T \Psi^T \text{ since} \\ \bar{K}^T \Lambda_k^{-1} \tilde{H} = (0,\Psi) B_1 \tilde{H} = (0,\Psi)\tilde{H}\Phi P \Phi^T M^{-1} = \\ \Psi H_k P \Phi^T M^{-1}$$

hence

$$\Sigma_{12}(2) = E[w(k,k+)w^T(k+1,k+1)] - H_k N_{k+1} = \\ (R_k - H_k P H_k^T)\Psi^T, \\ \Sigma_{11}(2) = \\ R_k + H_k \Phi^{-1} \Gamma Q \Gamma^T \Phi^{-1,T} H_k^T - H_k \Phi^{-1} M \Phi^{-1,T} H_k^T = \\ = R_k + H_k \Phi^{-1}(\Gamma Q \Gamma^T - M)\Phi^{-1,T} H_k^T = R_k - H_k P H_k^T.$$

It is seen that $\Delta\Sigma_{22}(2) = \Delta\Sigma_{22}(k+1)$ therefore $\Sigma(k+1) = \Sigma(2)$ or $\Delta\Sigma_{k+1}(k+1;2) = 0$ and Proposition 2 yields the optimal in MMS filter in this case. It remains to show that (96) is valid. Really, let $B := \Phi P \Phi^T$. Then according to (84)-(86) and Matrix inversion lemma,

$$M^{-1} = \tilde{H}^T \Lambda_k^{-1} \tilde{H} = \tilde{H}^T W_k^{-1}(I + \Delta W_k W_k^{-1})^{-1} \tilde{H} = \\ \tilde{H}^T W_k^{-1}[I - \tilde{H}(I + \Gamma Q \Gamma^T \tilde{H}^T W_k^{-1} \tilde{H})^{-1} \\ \Gamma Q \Gamma^T \tilde{H}^T W_k^{-1}]\tilde{H} = \\ B^{-1} - B^{-1}(I + \Gamma Q \Gamma^T B^{-1})^{-1} \Gamma Q \Gamma^T B^{-1} = (B + \Gamma Q \Gamma^T)^{-1}$$

which proves (96). In deducing the last relations we have used the relationship $(\Phi P \Phi^T)^{-1} = \Phi^{T,-1} \tilde{H}^T W_k^{-1} \tilde{H} \Phi^{-1}$. Analogously one can prove $\hat{x}_{k+1/k} = \Phi_k \hat{x}_k$. We have thus proven

Theorem 4. (Optimal in MMS filter for Markovian observational noise) Optimal in MMS filter for the filtering problem (1)(2)(71)-(74) is of the form

$$\hat{x}_{k+1} = \hat{x}_{k+1/k} + K_{k+1}(z_{k+1}^* - H_{k+1}^* \hat{x}_{k+1/k}), \\ \hat{x}_{k+1/k} = \Phi_k \hat{x}_k, \\ K_{k+1} = [M_{k+1} H_{k+1}^{*,T} + \Gamma_k Q_k \Gamma_k^T H^T(k,k+1)\Psi_k^T]\Sigma^{-1}(2), \\ \Sigma(2) = H_{k+1}^* M_{k+1} H_{k+1}^{*,T} + \Xi_k - H_{k+1}^* \Gamma_k Q_k \Gamma_k^T H_{k+1}^{*,T} + \\ H_{k+1} \Gamma_k Q_k \Gamma_k^T H_{k+1}^T, \\ z_{k+1}^* = z_{k+1} - \Psi_k z_k, \\ H_{k+1}^* = H_{k+1} - \Psi_k H(k,k+1), \\ M_{k+1} = \Phi_k P_k \Phi_k^T + \Gamma_k Q_k \Gamma_k^T, \\ P_{k+1} = \\ M_{k+1} - K_{k+1}[M_{k+1} H_{k+1}^{*,T} + \Gamma_k Q_k \Gamma_k^T H^T(k,k+1)\Psi_k^T]^T.$$

For definition of $H(k,k+1)$, see (47). We have then $H(k,k+1) = H_k \Phi_k^{-1}$ and for example,

$$H_{k+1}^* \hat{x}_{k+1/k} = (H_{k+1} - \Psi_k H(k,k+1))\hat{x}_{k+1/k} = \\ H_{k+1}\hat{x}_{k+1/k} - \Psi_k H_k \Phi_k^{-1} \Phi_k \hat{x}_k = H_{k+1}\hat{x}_{k+1/k} - \Psi_k H_k \hat{x}_k.$$

It is not hard to show that for the observation Markovian noise sequence of memory m the filter in Theorem 2 becomes MMS if we take the structure for the estimator at $k+1$ as a linear function of $\hat{x}_k$ and $m+1$ latest observations.

VII. Conclusions

An approximation approach to the solution of a linear filtering problems with correlated noises was presented. A new type of class of linear recursive filters is proposed together with definition of an optimal MMS estimator among the members of this class of filters. It was clear that the approximate filters have interesting and different properties to their truly optimal MMS filter. Thank to simplified recursive structure, a substantial reduction in computational burden and storage requirements is achieved compared to truly optimal MMS filter. This is important when there are non-Markovian noise processes. For the Markovian m memory noise sequence, the proposed sub-optimal filter will yield the truly optimal MMS estimates if the filter is chosen as a function of the last estimate $\hat{x}_k$ and $m+1$ last observations.

There are, no doubt, a wide variety of engineering problems to which the approximate filters are applicable and that could be worthy of further scrutiny. This is a subject we plan to set up, with emphasis on performing the approximate filter along with the truly optimal one, in order to show the main benefits of the proposed approximate approach.

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