

# 1–skeleton resolution of free simplicial algebras with given $CW$ –basis

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*Abstract*—In this paper we use the definition of  $CW$  basis of a free simplicial algebra. Using the free simplicial algebra, it is shown to construct free or totally free 2–crossed modules on suitable construction data with given a  $CW$ –basis of the free simplicial algebra. We give applications free crossed squares, free squared complexes and free 2–crossed complexes by using of 1(one) skeleton resolution of a step by step construction of the free simplicial algebra with a given  $CW$ –basis.

*Keywords*—Free crossed square, Free 2–crossed modules, Free simplicial algebra, Free square complexes, Free 2–crossed complexes  $CW$ –basis, 1–skeleton.

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## I. INTRODUCTION

Simplicial commutative algebras play an important role in homological algebra, homotopy theory and algebraic  $K$ –theory. The present article intends to study the 1(one) skeleton resolution of a step by step construction of a free simplicial algebra via André method’s, using with a given  $CW$ –basis.

This study will be appear using step by step of André’s method for a free simplicial algebra with a given  $CW$ –basis in the 1–skeleton. Brown [2] and Ellis [3] presented the definitions of coproduct and tensor product of groups. Coproduct structure of algebras was introduced by Shammu in [8]. Brown, Ellis and Shammu did not introduce to free simplicial algebra with a given  $CW$ –basis and also they never used these definitions in their researches.

The step by step construction was used by Arvasi and Porter in [1], but they did not build up this construction for the free simplicial algebra with a given  $CW$ –basis. For this reason, our method is completely different from their method and the soonest finding results show that likeness on free crossed square, squared complex and 2–crossed complex are basically structure of algebraic topology.

We reach some results on these structures of algebraic topology using the free simplicial algebra with a given  $CW$ –basis. Therefore our method is more trustworthy which is easily verified.

In this article we firstly introduce to the free simplicial algebra with a given  $CW$ –basis. In addition, we find some results using the free simplicial algebra with a given  $CW$ –basis. Secondly, we give 1–skeleton resolution of the free simplicial algebra with a given  $CW$ –basis  $\mathbb{A}^{(1)}$ . Furthermore,

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we present the details of calculation concerning with in the applications of coproduct and tensor product of crossed square, square complex and 2–crossed complex in the section 3 and 4.

## II. THE MOORE COMPLEX OF SIMPLICIAL ALGEBRA

Let us given a simplicial algebra  $\mathcal{A}$ , then we may definite the *Moore complex* of simplicial algebra as follows: The *Moore complex*  $(\mathbf{N}\mathcal{A}, \partial)$  of  $\mathcal{A}$  is the chain complex defined by  $(\mathbf{N}\mathcal{A})_n = \bigcap_{i=0}^{n-1} \text{Ker} d_i^n$  with  $\partial_n : \mathbf{N}\mathcal{A}_n \rightarrow \mathbf{N}\mathcal{A}_{n-1}$  induced from  $d_n^n$  by restriction.

The  $n^{\text{th}}$  homotopy module  $\pi_n(\mathcal{A})$  of  $\mathcal{A}$  is the  $n^{\text{th}}$  homology of the Moore complex of  $\mathcal{A}$ , i.e.,  $\pi_n(\mathcal{A}) \cong H_n(\mathbf{N}\mathcal{A}, \partial) = \bigcap_{i=0}^n \text{Ker} d_i^n / d_{n+1}^{n+1} (\bigcap_{i=0}^n \text{Ker} d_i^{n+1})$ .

## III. FREE SIMPLICIAL ALGEBRA WITH GIVEN $CW$ -BASIS

We recall that two definitions in [6] as follows in this section.

*Definition 3.1:* A simplicial algebra  $\mathbb{A}$  is called free if  
 (i)  $\mathcal{A}_n$  is a free algebra with a given basis, for every integer  $n \geq 0$ ,  
 (ii) the basis are stable under all degeneracy operators, i.e., for every pair of integers  $(i, n)$  with  $0 \leq i \leq n$  and every given generator  $x \in \mathcal{A}_n$  and the element  $s_i(x)$  is a given generator of  $\mathcal{A}_{n+1}$ .

*Definition 3.2:* Let  $\mathbb{A}$  be a free simplicial algebra (as above). A subset  $\mathfrak{A} \subset \mathbb{A}$  will be called a  *$CW$  – basis* for  $\mathbb{A}$  if

- (a)  $\mathfrak{A}_n = \mathfrak{A} \cap \mathcal{A}_n$  freely generates  $\mathcal{A}_n$  for all  $n \geq 0$ ,
- (b)  $\mathfrak{A}$  is closed under degeneracies, i.e.,  $x \in \mathfrak{A}_n$  implies  $s_i(x) \in \mathfrak{A}_{n+1}$  for all  $0 \leq i \leq n$ ,
- (c) if  $x \in \mathfrak{A}_n$  is non-degenerate, then  $d_i(x) = e_{n-1}$ , ( $e_{n-1}$ , the identity element of  $\mathcal{A}_{n-1}$ ) for all  $0 \leq i < n$ .

Let  $\mathbb{A}$  be a free simplicial algebra with a given  $CW$ –basis,  $\mathfrak{A}$ , then  $X_0 = \mathfrak{A}_0$  freely generates  $\mathcal{A}_0$ , that is,  $\mathcal{A}_0 = \mathcal{A}[X_0]$ . If we continue process of Definition 3.2, then  $\mathfrak{A}_1$  freely generates  $\mathcal{A}_1$  and let  $s_0(X_0) \subseteq \mathfrak{A}_1$  and also if  $Y_1 = \mathfrak{A}_1 \setminus s_0(X_0)$ , then  $d_i(y) = 0$  for  $0 \leq i < 1$  where  $y \in Y_1$ . Therefore let  $\mathcal{A}_1 = \mathcal{A}[s_0(X_0) \cup Y_1] \cong \mathcal{A}[s_0(X_0)] * \mathcal{A}[Y_1]$ , where “ $*$ ” is free product of  $\mathcal{A}[s_0(X_0)]$  and  $\mathcal{A}[Y_1]$ . For  $\mathcal{A}_2$ , let  $s_0(\mathfrak{A}_1) \cup s_1(\mathfrak{A}_1) \subseteq \mathfrak{A}_2$  and also if  $y \in Y_2 = \mathfrak{A}_2 \setminus \bigcup_{i=0}^1 s_i(\mathfrak{A}_1)$ , then  $d_0(y) = d_1(y) = 0$  and additionally  $y$  is in  $N\mathcal{A}_2$ . In general, note that if  $Y_n = \mathfrak{A}_n \setminus \bigcup_{i=0}^{n-1} s_i(\mathfrak{A}_i)$  then  $Y_n \subseteq N\mathcal{A}_n$  and  $Y_n$  normally generates  $N\mathcal{A}_n$ .

Thus we can give 1–skeleton resolution of  $\mathbb{A}^1$  of the free simplicial algebra with a given  $CW$ –basis. The 1–skeleton  $\mathbb{A}^{(1)}$  of the free simplicial resolution of an algebra  $A$  can be built by adding new indeterminate for instance, if there is an one to one correspondence with  $\Omega^0$  as of generator for  $\pi_1(\mathbb{A})$ ,  $\mathcal{A}_1^{(1)} = \mathcal{A}_1^{(0)}[X_0] = \mathcal{A}[s_0(X_0) \cup Y_1] \cong \mathcal{A}[s_0(X_0)] * \mathcal{A}[Y_1]$ , where with the face maps and degeneracy map

$$\mathcal{A}[s_0(X_0) \cup Y_1] \begin{array}{c} \xrightarrow{d_0, d_1} \\ \xleftarrow{s_0} \end{array} \mathcal{A}[X_0] \xrightarrow{d_0^0} A$$

here  $\mathcal{A}(X_0) \xrightarrow{d_0^0} A$  is an augmentation map and  $s_0, d_0^1$  and  $d_1^1$  are given by

$$d_1^1(y_1) = b_1 \in \text{Ker}d_0^0, \quad d_0^1(y_1) = 0, y_1 \in Y_1, \\ x_0 = s_0(x_0) \text{ for } x_0 \in X_0$$

Thus 1–skeleton resolution  $\mathbb{A}^{(1)}$  looks like:

$$\mathbb{A}^{(1)} : \dots \mathcal{A}_2 \begin{array}{c} \xrightarrow{d_0, d_1, d_2} \\ \xleftarrow{s_1, s_0} \end{array} \mathcal{A}_1 \begin{array}{c} \xrightarrow{d_1, d_0} \\ \xleftarrow{s_0} \end{array} \mathcal{A}_0 \xrightarrow{f} A/I.$$

where  $\mathcal{A}_2^{((1),2)} = \mathcal{A}[s_1 s_0(X_0) \cup s_1(Y_1) \cup s_0(Y_1)]$ ,  $\mathcal{A}_1^{((1),1)} = \mathcal{A}[s_0(X_0) \cup Y_1]$  and  $\mathcal{A}_0^{((1),0)} = \mathcal{A}[X_0]$ .

### A. Applications of Coproduct and Tensor Product of Crossed Square

In the background of  $CW$ –complexes, Ellis [3] presented an extraordinary definition of the top group in (totally) the free crossed square using topological methods. A free simplicial algebra with a given  $CW$ –basis is the algebraic analogue of a  $CW$ –complex. Therefore one expect that the similar result to hold in that setting. Now we give to application of tensor product for crossed square of 1–skeleton resolution of the free simplicial algebra with a given  $CW$ –basis. The functor from the category of simplicial algebras to the crossed cubes is defined as

$$\mathfrak{M}(-, 2) : \text{SimpAlg} \rightarrow \text{Ccrn}^n$$

in [1]. If for  $n = 2$  we apply this functor to the 1–skeleton resolution  $\mathbb{A}^{(1)}$  of the free simplicial algebra with a given  $CW$ –basis  $\mathcal{A}$ , we get  $\mathfrak{M}(\mathbb{A}^{(1)}, 2)$  which is the free crossed square is

$$\mathfrak{M}(\mathbb{A}^{(1)}, 2) \cong \begin{array}{ccc} N\mathcal{A}_2^{((1),2)} / \partial_3(N\mathcal{A}_3^{((1),3)} \cap I_3) & \xrightarrow{\partial_2} & \text{Ker } d_0^{((1),2)} \\ \partial_2' \downarrow & & \downarrow \mu \\ \text{Ker } d_1^{((1),2)} & \xrightarrow{\mu'} & \mathcal{A}_1^{((1),1)} \end{array}$$

together with the  $h$ –map

$$h: \text{Ker}d_1^{((1),2)} \times \text{Ker}d_0^{((1),2)} \longrightarrow N\mathcal{A}_2^{((1),2)} / \partial_3(N\mathcal{A}_3^{((1),3)} \cap I_3)$$

given by

$$x \otimes y = h(x, y) = s_1 y_0 (s_1 y_1 - s_0 y_1) \pmod{\partial_3(N\mathcal{A}_3^{((1),3)} \cap I_3)}$$

where “ $\otimes$ ” is defined as coproduct or tensor product, and let  $I_n$  be the ideal generated by the degenerate elements of  $\mathcal{A}_n$  (for  $n = 3$ ). Then

$$\mathcal{A}_2^{((1),2)} = \mathcal{A}[s_1 s_0(X_0) \cup s_1(Y_1)],$$

$$N\mathcal{A}_2^{((1),2)} = \bigcap_{i=0}^1 \text{Ker}d_i^1 = [s_1(Y_1)]^+ \cap Z',$$

$$\text{Ker}d_0^1 = [s_1(Y_1)]^+,$$

$$\text{Ker}d_1^1 = Z' = \{s_1(y_1) - s_0(y_1) : y_1 \in Y_1\};$$

$$\mathcal{A}_3^{((1),3)} = \mathcal{A}[s_2 s_1 s_0(X_0) \cup s_2 s_1(Y_1) \cup s_2 s_0(Y_1) \cup s_1 s_0(Y_1)],$$

$$N\mathcal{A}_3^{((1),3)} = \bigcap_{i=0}^2 \text{Ker}d_i^{((1),3)} = [s_2 s_1(Y_1)]^+ \cap Z^+ \cap Z_1^+,$$

$$\text{Ker}d_0^{((1),3)} = [s_2 s_1(Y_1)]^+,$$

$$\text{Ker}d_1^{((1),3)} = Z^+, \quad \text{Ker}d_2^{((1),3)} = Z_1^+,$$

$$Z = \{s_2 s_1(x) - s_2 s_0(x) \mid y_1 \in Y_1\} \text{ and} \\ Z_1 = \{s_2 s_0(y_1) - s_1 s_0(y_1) : y_1 \in Y_1\}.$$

Now we can conclude that if we use a  $CW$ –basis of the free simplicial algebra, we calculate the 1–skeleton resolution of a  $CW$ –basis of the free simplicial algebra. Thus we get the result as follows.

**Corollary 3.3:** Let  $\mathbb{A}^{(1)}$  be the 1–skeleton resolution of the free simplicial algebra with a given  $CW$ –basis. If the free crossed square  $\mathfrak{M}(\mathbb{A}^{(1)}, 2)$  as described above, then

$$N\mathcal{A}_2^{((1),2)} / \partial_3(N\mathcal{A}_3^{((1),3)} \cap I_3) \cong \text{Ker}d_1^{((1),2)} \otimes_{\mathcal{A}_1^{((1),1)}} \text{Ker}d_0^{((1),2)}.$$

**Proof:** The proof is clear since

$$x \otimes y = h(x, y) = s_1 y_0 (s_1 y_1 - s_0 y_1). \text{ See [1] in details. } \square$$

**Corollary 3.4:** If the simplicial algebra  $\mathbb{A}$  equals to its 1–skeleton, so that  $\mathbb{A} = \mathbb{A}^{(1)}$ , then

$$\pi_2(\mathbb{A}) = \text{Ker}(\text{Ker}d_1^{((1),1)} \otimes_{\mathcal{A}_1^{((1),1)}} \text{Ker}d_0^{((1),1)}) \rightarrow \mathcal{A}_0^{((1),0)}.$$

**Proof:** We prove for  $k \geq 1$ . If  $\mathbb{A}^{(k)}$  is  $k$ –skeleton resolution of the free simplicial algebra with a given  $CW$ –basis,  $\mathbb{A}$ , for  $k \geq 1$ , then

$$\pi_k(\mathbb{A}^{(k)}) = \text{Ker}(N\mathcal{A}_k^{((k),k)} / \partial_{k+1}(N\mathcal{A}_{k+1}^{((k),k+1)}) \rightarrow \mathcal{A}_{k+1}^{(k)}).$$

By taking  $k = 1$  and the previous corollary, we get the result.  $\square$

IV. APPLICATIONS OF COPRODUCT AND TENSOR PRODUCT OF SQUARED COMPLEX AND 2-CROSSED COMPLEX

Now we can give some of applications of 1–skeleton resolution of the free simplicial algebra with a given CW–basis such as square complex and 2–crossed complex in some structure of algebraic topology. Let  $\mathbb{A}^{(1)}$  be the 1–skeleton resolution of the free simplicial algebra with a given CW–basis of an algebra. By Corollary 3.3, we know that

$$\begin{array}{ccc} \text{Kerd}_1^{((1,1))} \otimes_{\mathcal{A}_1^{(1)}} \text{Kerd}_0^{((1,1))} & \longrightarrow & \text{Kerd}_0^{((1,1))} \\ \downarrow & & \downarrow \\ \text{Kerd}_1^{((1,1))} & \longrightarrow & \mathcal{A}_1^{((1,1))} \end{array}$$

is free square. Thus the free squared complex is

$$\begin{array}{ccc} & \xrightarrow{\lambda'} & \overline{N\mathcal{A}_1^{((1,1))}} \\ \dots \rightarrow c \rightarrow \overline{N\mathcal{A}_1^{((1,1))}} \otimes_{\mathcal{A}_1^{((1,1))}} N\mathcal{A}_1^{((1,1))} & & \xrightarrow{\mu'} \mathcal{A}_1^{((1,1))} (*) \\ & \xrightarrow{\lambda} & N\mathcal{A}_1^{((1,1))} \xrightarrow{\mu} \end{array}$$

where  $\mathcal{A}_2^{((1,2))} = \mathcal{A}[s_1s_0(X_0) \cup s_1(Y_1) \cup s_0(Y_1)]$ ,  $\mathcal{A}_1^{((1,1))} = \mathcal{A}[s_0(X_0) \cup Y_1]$  and  $\mathcal{A}_0^{((1,0))} = \mathcal{A}[X_0]$ , where  $N\mathcal{A}_1^{((1,1))} = \text{Kerd}_0^1 = [Y_1]^+$ ,  $\overline{N\mathcal{A}_1^{((1,1))}} = \text{Kerd}_1^{((1,1))} = [s_1(y_1) - s_0(y_1)] = Z'$ ,

$$C = \frac{N\mathcal{A}_3^{((1,3))}}{(N\mathcal{A}_3^{((1,3))} \cap I_3) + d_4(N\mathcal{A}_4^{((1,4))} \cap I_4)},$$

$$\mathcal{A}_3^{((1,3))} = \mathcal{A}[s_2s_1s_0(X_0) \cup s_2s_1(Y_1) \cup s_2s_0(Y_1) \cup s_1s_0(Y_1)],$$

$$N\mathcal{A}_3^{((1,3))} = \bigcap_{i=0}^2 \text{Kerd}_i^1 = [s_2s_1(Y_1)]^+ \cap Z^+ \cap Z_1^+$$

$$\text{Kerd}_0^{((1,3))} = [s_2s_1(Y_1)]^+,$$

$$\text{Kerd}_1^{((1,3))} = Z^+, \quad Z = \{s_2s_1(y_1) - s_2s_0(y_1), y_1 \in Y_1\}$$

$$\text{and } \text{Kerd}_2^{((1,3))} = Z_1^+, \quad Z_1 = \{s_2s_0(y_1) - s_1s_0(y_1) : y_1 \in Y_1\}.$$

$$\mathcal{A}_4^{((1,4))} = \mathcal{A}[s_3s_2s_1s_0(X_0) \cup s_3s_2s_1(Y_1) \cup s_3s_2s_0(Y_1) \cup s_3s_1s_0(Y_1) \cup s_2s_1s_0(Y_1)],$$

$$N\mathcal{A}_4^{((1,4))} = \bigcap_{i=0}^3 \text{Kerd}_i^{((1,4))} = [s_3s_2s_1(Y_1)]^+ \cap Z_2^+ \cap Z_3^+ \cap Z_4^+,$$

$$\text{Kerd}_0^{((1,4))} = [s_3s_2s_1(Y_1)]^+ \quad Z_2^+ = \text{Kerd}_1^{((1,4))}, \quad Z_2 = \{s_3s_2s_1(y_1) - s_3s_2s_0(y_1), y_1 \in Y_1\},$$

$$Z_3 = \{s_3s_2s_0(y_1) - s_3s_1s_0(y_1), y_1 \in Y_1\}, \quad Z_4^+ = \text{Kerd}_3^{((1,4))},$$

since  $Z_4 = \{s_3s_1s_0(y_1) - s_2s_1s_0(y_1) : y_1 \in Y_1\}$  by a free square complex, we mean one in which the crossed square is free, and in which each  $C_n$  is free algebra with a given CW–basis for  $n \geq 3$ .

Crossed modules techniques give a very efficient way to get the information on a homotopy type. If they clearly correspond to 1–types (and hence topological 2–types) then this is called as the algebraic model. We have recalled from Grandjean and Vale’s work [4], in 1986 that 2–crossed modules correspond to 2–types (and hence topological 3–types) is also defined as algebraic model.

If  $C_1 \rightarrow C_0$  is a crossed module in a crossed complex  $(\dots \rightarrow C_n \rightarrow \dots \rightarrow C_2 \rightarrow C_1 \rightarrow C_0)$ , then we say that a crossed complex is called a 1–crossed complex and still if  $C_2 \rightarrow C_1 \rightarrow C_0$  is a 2–crossed module in a crossed complex, then we say that a crossed complex is also called a 2–crossed complex. Additionally  $C_1 \rightarrow C_0$  and  $C_2 \rightarrow C_1 \rightarrow C_0$  are said to be “tail” of crossed complex. Furthermore, totally free 2–crossed complex of group was defined by A. Mutlu and T. Porter in [7], and 2–crossed complex of algebra was defined by A. Mutlu in [5].

Now we can give the definition in [5] as follows.

**Definition 4.1:** A 2–crossed complex of algebras is a sequence of algebras

$$C : \quad \dots \rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \dots \rightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

in which

(i)  $C_0$  acts on  $C_n$ ,  $n \geq 1$ , the action of  $\partial C_1$  is trivial on  $C_n$  for  $n \geq 3$ ;

(ii) for each  $\partial_n$  this is a  $C_0$ –algebra homomorphism and  $\partial_i \partial_{i+1} = 0$  for all  $i \geq 1$ ;

and

(iii)  $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$  is a 2–crossed module.

A 2–crossed complex  $C$  will be said to be free if for  $n \geq 3$ , the  $C_0/\partial C_1$ –module,  $C_n$  are free and the 2–crossed complex at the base is also free 2–crossed module. Additionally it will be totally free if the base 2–crossed module is totally free (see [5]).

Now we can examine a free 2–crossed complex of the free simplicial algebra. Firstly  $\mathbb{A}^{(1)}$  be the 1–skeleton resolution of the free simplicial algebra with a given CW–basis. Then free 2–crossed complex is

$$\dots \xrightarrow{\partial_5} C_5 \xrightarrow{\partial_4} C_4 \xrightarrow{\partial_3} C_3 \xrightarrow{\partial_2} N\mathcal{A}_1^{((1,1))} \xrightarrow{\partial_1} N\mathcal{A}_0^{((1,0))}$$

where  $C_3 = \frac{N\mathcal{A}_2^{((1,2))}}{(N\mathcal{A}_2^{((1,2))} \cap I_2) + d_3(N\mathcal{A}_3^{((1,3))} \cap I_3)}$ , and for each  $C_n$  is the free simplicial algebra with a given CW–basis, then

$$C_n = \begin{cases} N\mathcal{A}_n & \text{for } n = 0, 1 \\ N\mathcal{A}_2/d_3(N\mathcal{A}_3 \cap I_3) & \text{for } n = 2 \\ N\mathcal{A}_n/(N\mathcal{A}_n \cap I_n) + d_{n+1}(N\mathcal{A}_{n+1} \cap I_{n+1}) & \text{for } n \geq 3 \end{cases}$$

is induced by the differential of  $N\mathcal{A}$  with  $\partial_n$  (see [7], [1]).

We thus have a functor

$$\rho : \mathfrak{StSimpAlg}_{\leq 1} \longrightarrow \mathfrak{FreeSqComp}$$

where  $\mathfrak{StSimpAlg}_{\leq 1}$  is the category of the 1–skeleton of the free simplicial algebra with a given CW–basis and  $\mathfrak{FreeSqComp}$  is the category of the free squared complex.

The homotopy module  $\pi_n(\rho)$ , for  $n \geq 1$ , the squared complex  $(*)$  is defined to be the homology module of the complex

$$\dots \xrightarrow{\partial_6} C_5 \xrightarrow{\partial_5} C_4 \xrightarrow{\partial_4} L \xrightarrow{\partial_3} M \times N \xrightarrow{\partial_2} \mathcal{A} \longrightarrow 0$$

with  $\partial_3(l) = (-\lambda'l, \lambda l)$  and  $\partial_2(m, n) = \mu(m) + \mu'(n)$ . The axioms of a crossed square guarantee that  $\partial_3$  and  $\partial_2$  are homomorphism with  $\partial_4(C_4)$  ideal in  $\text{Ker}(\partial_3)$ ,  $\partial_3(L)$  ideal in  $\text{Ker}(\partial_2)$ , and  $\partial_2(M \times N)$  ideal in  $\mathcal{A}$  respectively. Clearly  $\pi_n(\rho) = \text{Ker}\partial_n/\text{Im}\partial_{n+1}$ .

Note that the homotopy module  $\pi_n(\rho(\mathbb{A}^{(1)}))$  of the squared complex

$$\rho(\mathbb{A}^{(1)}) = \left( C_n, \left( \begin{array}{ccc} \overline{N\mathcal{A}_1^{((1),1)}} \otimes_{\mathcal{A}_1^{(1)}} N\mathcal{A}_1^{((1),1)} & \longrightarrow & N\mathcal{A}_1^{((1),1)} \\ \downarrow & & \downarrow \\ \overline{N\mathcal{A}_1^{((1),1)}} & \longrightarrow & \mathcal{A}_1^{(1)} \end{array} \right) \right)$$

is the homology module of the complex

$$\dots \xrightarrow{\partial_6} C_5 \xrightarrow{\partial_5} C_4 \xrightarrow{\partial_4} C_3 \xrightarrow{\lambda} \overline{N\mathcal{A}_1^{((1),1)}} \xrightarrow{\partial} \mathcal{A}_1^{((1),1)} \longrightarrow 0.$$

This is also the 2-crossed complex where  $C_3 = \overline{N\mathcal{A}_1^{((1),1)}} \otimes_{\mathcal{A}_1^{(1)}} N\mathcal{A}_1^{((1),1)}$ .

By the (totally) free square complex, we mean that the crossed square is (totally) free, and also each  $C_n$  is the free simplicial algebra with a given CW-basis for  $n \geq 3$ .

Now we can give some theorems, such as the following theorem and corollary obviously use the definition of a CW-basis.

Moreover we give some results concerning with the free simplicial algebra with a given CW-basis by using in the 1-skeleton resolution for an algebra.

**Theorem 4.2:** The category of squared complexes of algebras is equivalent to the category of the 2-crossed complexes of algebras.

**Proof:** The category of crossed square of algebra is equivalent to the category of 2-crossed module of algebra.  $\square$

Thus we may obtain the result of the free version of Theorem 4.2 where the proof of its is trivial between equivalent two categories as follows.

**Corollary 4.3:** The category of the free squared complexes of algebras is equivalent to the category of the free 2-crossed complexes of algebras.  $\square$

#### REFERENCES

- [1] Z. ARVASI and T. PORTER . *Freeness conditions for 2-Crossed modules of Commutative Algebras*. Applied Categorical Structure **181** (1998), 426-448.
- [2] R. BROWN. *Coproduct of crossed P-modules, application to second homotopy groups and to the homology of groups*. Topology **23** (1984), 337-345.
- [3] G.J. ELLIS. *Crossed squares and combinatorial homotopy*. Math. Z. **214** (1993), 93-110.
- [4] A.R. GRANDJEÁN and M.J. VALE . *2-Modulos Cruzados en la Cohomologia de André-Quillen*. Memorias de la Real Academia de Ciencias. **22** (1986).
- [5] A. MUTLU . *Free 2-crossed complexes of simplicial algebras*. Mathematical and Computational Applications **5(1)** (2000), 13-22.
- [6] A. MUTLU, B. MUTLU AND E. USLU *To construction of free simplicial algebras with given CW-basis*. International Mathematical Forum, **4** (30) (2009), 1489 - 1495.
- [7] A. MUTLU AND T. PORTER , *Freeness conditions for 2-crossed modules and complexes*. Theory and Applications Categories **4** (8) (1998), 174-194.
- [8] N.M. SHAMMU. *Algebraic and Categorical Structure of Category of Crossed Modules of Algebras* Ph.D. Thesis, University of Wales, Bangor (1992).