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Constructive Proof of the Existence of an Equilibrium in a Competitive Economy with Sequentially Locally Non-Constant Excess Demand Functions

Yasuhito Tanaka

Abstract—In this paper we will constructively prove the existence of an equilibrium in a competitive economy with sequentially locally non-constant excess demand functions. And we will show that the existence of such an equilibrium in a competitive economy implies Sperner's lemma. We follow the Bishop style constructive mathematics.

Keywords—Sequentially locally non-constant excess demand functions, Equilibrium in a competitive economy, Constructive mathematics

I. Introduction

It is well known that Brouwer's fixed point theorem can not be constructively proved[1]. Thus, the existence of an equilibrium in a competitive economy can not be constructively proved. Sperner's lemma which is used to prove Brouwer's theorem, however, can be constructively proved. Some authors have constructively presented an approximate version of Brouwer's theorem using Sperner's lemma. See [8] and [9]. Thus, Brouwer's fixed point theorem is constructively, in the sense of constructive mathematics à la Bishop, proved in its approximate version.

Also Dalen[8] states a conjecture that a uniformly continuous function f from a simplex into itself, with property that each open set contains a point x such that $x \neq f(x)$, which means $|x - f(x)| > 0$, and also at every point x on the boundaries of the simplex $x \neq f(x)$, has an exact fixed point. Recently [2] showed that the following theorem is equivalent to Brouwer's fan theorem.

> Each uniformly continuous function f from a compact metric space X into itself with at most one fixed point and approximate fixed points has a fixed point.

By reference to the notion of *sequentially at most one maximum* in [1] we consider a condition that a function is *sequentially locally non-constant*. Sequential local non-constancy, the condition in [8] and the condition that a function has *at most one fixed point* in [1] are mutually different.

In this paper we present a proof of the existence of an exact equilibrium in a competitive economy with sequentially locally non-constant excess demand functions. Also we will show that the existence of an equilibrium in a competitive economy with sequentially locally non-constant excess demand functions implies Sperner's lemma.

In [6] we have constructively proved Brouwer's fixed point theorem for sequentially locally non-constant and uniformly sequentially continuous functions without the fan theorem. It is a partial answer to Dalen's conjecture. Uniformly sequential continuity is weaker than uniform continuity. Thus, the result in [6] implies that we can constructively prove Brouwer's fixed point theorem for sequentially locally non-constant and uniformly continuous functions. In this paper we apply the procedure of the proof in that paper to the problem of the existence of an equilibrium in a competitive economy with sequentially locally non-constant excess demand functions.

II. Existence of an equilibrium in a competitive economy

In constructive mathematics a nonempty set is called an *inhabited* set. A set S is inhabited if there exists an element of S.

> Note that in order to show that S is inhabited, we cannot just prove that it is impossible for S to be empty: we must actually construct an element of S (see page 12 of [4]).

Also in constructive mathematics compactness of a set means *total boundedness with completeness*. First we present finite enumerability and ε-approximation to a set. A set S is *finitely enumerable* if there exist a natural number N and a mapping of the set $\{1, 2, \dots, N\}$ onto S. An ε-approximation to S is a subset of S such that for each $\mathbf{p} \in S$ there exists $\mathbf{q}$ in that ε-approximation with $|\mathbf{p} - \mathbf{q}| < \varepsilon$($|\mathbf{p} - \mathbf{q}|$ is the distance between $\mathbf{p}$ and $\mathbf{q}$). S is totally bounded if for each $\varepsilon > 0$ there exists a finitely enumerable ε-approximation to S. Completeness of a set, of course, means that every Cauchy sequence in the set converges.

Let $\mathbf{p} = (p_0, p_1, \dots, p_n)$ be a point in an n-dimensional simplex Δ, and consider a function φ from Δ to itself. Denote the i-th components of $\mathbf{p}$ and $\varphi(\mathbf{p})$ by p_i and φ_i or $\varphi_i(\mathbf{p})$.

According to [4] we have the following result.

Lemma 1: If Δ is an n-dimensional simplex, for each $\varepsilon > 0$ there exist totally bounded sets $H_1, \dots, H_n$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^{n} H_i$.

Y. Tanaka is with the Faculty of Economics, Doshisha University, Kyoto, 602-8580, Japan, e-mail: (yasuhito@mail.doshisha.ac.jp).

[1] [5] provided a *constructive* proof of Brouwer's fixed point theorem. But it is not constructive from the view point of constructive mathematics à la Bishop. It is sufficient to say that one dimensional case of Brouwer's fixed point theorem, that is, the intermediate value theorem is non-constructive. See [3] or [8].

The notion that f has at most one fixed point in [2] is defined as follows;

Definition 1 (At most one fixed point): For all $\mathbf{p}, \mathbf{q} \in \Delta$, if $\mathbf{p} \neq \mathbf{q}$, then $\varphi(\mathbf{p}) \neq \mathbf{p}$ or $\varphi(\mathbf{q}) \neq \mathbf{q}$.

By reference to the notion of *sequentially at most one maximum* in [1], we define the property of *sequential local non-constancy* as follow;

Definition 2 (Sequential local non-constancy of functions): There exists $\bar{\varepsilon} > 0$ with the following property. For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $H_1, H_2, \dots, H_m$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^m H_i$, and if for all sequences $(\mathbf{p}_n)_{n\geq 1}$, $(\mathbf{q}_n)_{n\geq 1}$ in each H_i, $|\varphi(\mathbf{p}_n) - \mathbf{p}_n| \longrightarrow 0$ and $|\varphi(\mathbf{q}_n) - \mathbf{q}_n| \longrightarrow 0$, then $|\mathbf{p}_n - \mathbf{q}_n| \longrightarrow 0$.

If φ is a uniformly continuous function from Δ to itself, according to [8] and [9] it has an approximate fixed point. This means

$$\text{For each } \varepsilon > 0 \text{ there exists } x \in \Delta \text{ such that } |\mathbf{p} - \varphi(\mathbf{p})| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\inf_{\mathbf{p}\in\Delta} |\mathbf{p} - \varphi(\mathbf{p})| = 0.$$

Then,

$$\inf_{\mathbf{p}\in H_i} |\mathbf{p} - \varphi(\mathbf{p})| = 0,$$

for some H_i such that $\cup_{i=1}^n H_i = \Delta$. Since n is finite, we can find such an H_i.

Now we show the following lemma.

Lemma 2: Let φ be a uniformly continuous function from Δ to itself. Assume $\inf_{\mathbf{p}\in H_i} \varphi(\mathbf{p}) = 0$. If the following property holds:

> For each $\varepsilon > 0$ there exists $\eta > 0$ such that if $\mathbf{p}, \mathbf{q} \in H_i$, $|\varphi(\mathbf{p}) - \mathbf{p}| < \eta$ and $|\varphi(\mathbf{q}) - \mathbf{q}| < \eta$, then $|\mathbf{p} - \mathbf{q}| \leq \varepsilon$.

Then, there exists a point $\mathbf{r} \in H_i$ such that $\varphi(\mathbf{r}) = \mathbf{r}$.

Proof: Choose a sequence $(\mathbf{p}_n)_{n\geq 1}$ in H_i such that $|\varphi(\mathbf{p}_n) - \mathbf{p}_n| \longrightarrow 0$. Compute N such that $|\varphi(\mathbf{p}_n) - \mathbf{p}_n| < \eta$ for all $n \geq N$. Then, for $m, n \geq N$ we have $|\mathbf{p}_m - \mathbf{p}_n| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $(\mathbf{p}_n)_{n\geq 1}$ is a Cauchy sequence in H_i, and converges to a limit $\mathbf{r} \in H_i$. The continuity of φ yields $|\varphi(\mathbf{r}) - \mathbf{r}| = 0$, that is, $\varphi(\mathbf{r}) = \mathbf{r}$. ■

Consider a competitive exchange economy. There are $n+1$ goods X_0, X_1, $\cdots$, X_n. n is a finite positive integer. The prices of the goods are denoted by $p_i(\geq 0)$, $i = 0, 1, \cdots, n$. Let $\bar{p} = p_0 + p_1 + \cdots + p_n$, and define

$$\bar{p}_i = \frac{p_i}{\bar{p}}, \ i = 0, 1, \cdots, n.$$

Denote anew $\bar{p}_0$, $\bar{p}_1$, $\cdots$, $\bar{p}_n$, respectively, by p_0, p_1, $\cdots$, p_n. Then,

$$p_0 + p_1 + \cdots + p_n = 1. \tag{1}$$

$\mathbf{p} = (p_0, p_1, \cdots, p_n)$ represents a point on an n-dimensional simplex. It is usually assumed that consumers' excess demand (demand minus supply) for each good is homogeneous of degree zero. It means that consumers' excess demand for each good is determined by relative prices of the goods, and above notation of the prices yields no loss of generality. We denote the vector of excess demands for the goods when the vector of prices is $\mathbf{p}$ by $\mathbf{f}(\mathbf{p}) = (f_1, f_2, \dots, f_n)$. We require the following condition;

$$\mathbf{pf}(\mathbf{p}) = p_0 f_0 + p_1 f_1 + \cdots + p_n f_n = 0 \ (\text{Walras Law}). \tag{2}$$

f_i is equal to the sum of excess demands of all consumers for the good X_i. By the budget constraint of each consumer, in a competitive exchange economy the sum of excess demands of all consumers for each good must be 0. Adding the budget constraints for all consumers yields (2). We assume that the excess demand function $\mathbf{f}(\mathbf{p})$ is uniformly continuous about the prices of the goods. Uniform continuity of $\mathbf{f}$ means that for any $\mathbf{p}$, $\mathbf{p}'$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{If } |\mathbf{p}' - \mathbf{p}| < \delta, \text{we have } |\mathbf{f}(\mathbf{p}') - \mathbf{f}(\mathbf{p})| < \varepsilon.$$

δ depends only on ε not on $\mathbf{p}$ and $\mathbf{p}'$. It implies that a slight price change yields only a slight excess demand change. An equilibrium in a competitive exchange economy is a state where excess demand for each good is not positive.

Next we assume the following condition.

Definition 3: (Sequential local non-constancy of excess demand functions) There exists $\bar{\varepsilon} > 0$ with the following property. For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $H_1, \dots, H_n$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^n H_i$, and if for all sequences $(\mathbf{p}_m)_{m\geq 1}$, $(\mathbf{q}_m)_{m\geq 1}$ in each H_i, $|\max(f_i(\mathbf{p}_m), 0)| \longrightarrow 0$ and $|\max(f_i(\mathbf{q}_m), 0)| \longrightarrow 0$ for all i, then $|\mathbf{p}_m - \mathbf{q}_m| \longrightarrow 0$.

Consider the following function from the set of price vectors $\mathbf{p} = (p_0, p_1, \cdots, p_n)$ to the set of $n+1$ tuples of real numbers $\mathbf{v} = (v_0, v_1, \cdots, v_n)$.

$$v_i = p_i + \max(f_i, 0) \text{ for all } i.$$

With this we define a function from an n-dimensional simplex Δ to itself $\varphi(\mathbf{p}) = (\varphi_0, \varphi_1, \cdots, \varphi_n)$ as follows;

$$\varphi_i = \frac{1}{v_0 + v_1 + \cdots + v_n} v_i, \text{ for all } i$$

Since $\varphi_i \geq 0, \ i = 0, 1, \cdots, n$ and

$$\varphi_0 + \varphi_1 + \cdots + \varphi_n = 1,$$

$(\varphi_0, \varphi_1, \cdots, \varphi_n)$ represents a point on Δ. By the uniform continuity of $\mathbf{f}$, φ is also uniformly continuous. From the sequential local non-constancy of excess demand functions we obtain the following results.

> For each ε such that $0 < \varepsilon < \bar{\varepsilon}$ if for all sequences $(\mathbf{p}_m)_{m\geq 1}$, $(\mathbf{q}_m)_{m\geq 1}$ in each H_i above defined $|\varphi(\mathbf{p}_m) - \mathbf{p}_m| \longrightarrow 0$ and $|\varphi(\mathbf{q}_m) - \mathbf{q}_m| \longrightarrow 0$, then $|\mathbf{p}_m - \mathbf{q}_m| \longrightarrow 0$.

Therefore, φ is a sequentially locally non-constant function.

Now we show the following theorem.

Theorem 1: In a competitive exchange economy, if the excess demand functions for the goods are uniformly continuous about their prices and satisfy the Walras law and the condition of sequential local non-constancy, then there exists an equilibrium.

Proof: Assume $\inf_{\mathbf{p}\in H_i} |\mathbf{p}-\varphi(\mathbf{p})| = 0$. Choose a sequence $(\mathbf{r}_m)_{m\geq 1}$ in $H_i \subset \Delta$ such that $|\varphi(\mathbf{r}_m)-\mathbf{r}_m| \longrightarrow 0$. We will prove the following condition.

For each $\varepsilon > 0$ there exists $\eta > 0$ such that if $\mathbf{p},\mathbf{q} \in H_i$, $|\varphi(\mathbf{p})-\mathbf{p}| < \eta$ and $|\varphi(\mathbf{q})-\mathbf{q}| < \eta$, then $|\mathbf{p}-\mathbf{q}| \leq \varepsilon$.

Assume that the set

$$K = \{(\mathbf{p},\mathbf{q}) \in H_i \times H_i : \ |\mathbf{p}-\mathbf{q}| \geq \varepsilon\}$$

is nonempty and compact[2]. Since the mapping $(\mathbf{p},\mathbf{q}) \longrightarrow \max(|\varphi(\mathbf{p})-\mathbf{p}|, |\varphi(\mathbf{q})-\mathbf{q}|)$ is uniformly continuous, we can construct an increasing binary sequence $(\lambda_m)_{m\geq 1}$ such that

$$\lambda_m = 0 \Rightarrow \inf_{(\mathbf{p},\mathbf{q})\in K} \max(|\varphi(\mathbf{p})-\mathbf{p}|, |\varphi(\mathbf{q})-\mathbf{q}|) < 2^{-m},$$

$$\lambda_m = 1 \Rightarrow \inf_{(\mathbf{p},\mathbf{q})\in K} \max(|\varphi(\mathbf{p})-\mathbf{p}|, |\varphi(\mathbf{q})-\mathbf{q}|) > 2^{-m-1}.$$

It suffices to find m such that $\lambda_m = 1$. In that case, if $|\varphi(\mathbf{p})-\mathbf{p}| < 2^{-m-1}$, $|\varphi(\mathbf{q})-\mathbf{q}| < 2^{-m-1}$, we have $(\mathbf{p},\mathbf{q}) \notin K$ and $|\mathbf{p}-\mathbf{q}| \leq \varepsilon$. Assume $\lambda_1 = 0$. If $\lambda_m = 0$, choose $(\mathbf{p}_m,\mathbf{q}_m) \in K$ such that $\max(|\varphi(\mathbf{p}_m)-\mathbf{p}_m|, |\varphi(\mathbf{q}_m)-\mathbf{q}_m|) < 2^{-m}$, and if $\lambda_m = 1$, set $\mathbf{p}_m = \mathbf{q}_m = \mathbf{r}_m$. Then, $|\varphi(\mathbf{p}_m)-\mathbf{p}_m| \longrightarrow 0$ and $|\varphi(\mathbf{q}_m)-\mathbf{q}_m| \longrightarrow 0$, so $|\mathbf{p}_m-\mathbf{q}_m| \longrightarrow 0$. Computing M such that $|\mathbf{p}_M-\mathbf{q}_M| < \varepsilon$, we must have $\lambda_M = 1$. Note that φ is a sequentially locally non-constant uniformly continuous function from Δ to itself. Thus, in view of Lemma 2.2 we have completed the proof of the existence of a point which satisfies

$$\mathbf{p} = \varphi(\mathbf{p}). \tag{3}$$

Let $\mathbf{p}^* = (p_0^*, p_1^*, \dots, p_n^*)$ be one of the points which satisfy (3). Let us consider the relationship between the price and excess demand for each good in that case. From the definitions of φ and $\mathbf{v}$, and $\sum_{j=0}^n v_j = 1 + \sum_{j=0}^n \max(f_j, 0)$(because $\sum_{j=0}^n p_j^* = 1$), (3) means

$$\frac{p_i^* + \max(f_i, 0)}{1 + \sum_{j=0}^n \max(f_j, 0)} = p_i^*$$

Let $\gamma = \sum_{j=0}^n \max(f_j, 0)$. Then, we have

$$\max(f_i, 0) = \gamma p_i^*.$$

From $\sum_{j=0}^n p_j^* = 1$ there is a k such that $p_k^* > 0$. If for all such k $\max(f_k, 0) = f_k = \gamma p_k^* > 0$ holds, that is, excess demands for all goods with positive prices are positive, we can not cancel out $p_k^* f_k > 0$ because the price of any good can not be negative, and the Walras law (2) is violated. Thus, we have $\gamma = 0$ and

$$\max(f_i, 0) = 0 \text{ for each } i. \tag{4}$$

This means that excess demand for each good is not positive. Such a state is an *equilibrium* in a competitive economy. In the equilibrium when $p_i > 0$ we must have $f_i = 0$ because if $f_i < 0$ we have $p_i f_i < 0$, and then the Walras law is violated. We have completed the proof of the existence of an equilibrium in a competitive economy with sequentially locally non-constant excess demand functions. ■

[2]See Theorem 2.2.13 of [4].

III. From the Existence of a Competitive Equilibrium to Sperner's Lemma

In this section we will derive Sperner's lemma from the existence of an equilibrium in a competitive economy[3]. Let partition an n-dimensional simplex Δ. Let K be the set of small n-dimensional simplices of Δ constructed by partition. Vertices of these small simplices of K are labeled with the numbers 0, 1, 2, …, n according to the following rules.

1) The vertices of Δ are respectively labeled with 0 to n. We label a point $(1, 0, \dots, 0)$ with 0, a point $(0, 1, 0, \dots, 0)$ with 1, a point $(0, 0, 1 \dots, 0)$ with 2, …, a point $(0, \dots, 0, 1)$ with n. That is, a vertex whose k-th coordinate $(k = 0, 1, \dots, n)$ is 1 and all other coordinates are 0 is labeled with k for all $k \in \{0, 1, \dots, n\}$.
2) If a vertex of K is contained in an $n-1$-dimensional face of Δ, then this vertex is labeled with some number which is the same as the number of a vertex of that face.
3) If a vertex of K is contained in an $n-2$-dimensional face of Δ, then this vertex is labeled with some number which is the same as the number of a vertex of that face. And similarly for cases of lower dimension.
4) A vertex contained inside of Δ is labeled with an arbitrary number among 0, 1, …, n.

Denote vertices of an n-dimensional simplex of K by $x^0, x^1, \dots, x^n$, the j-th component of x^i by x_j^i, and the label of x^i by $l(x^i)$. Let τ be a positive number which is smaller than $x_{l(x^i)}^i$ for all i, and define a function $f(x^i)$ as follows[4].

$$f(x^i) = (f_0(x^i), f_1(x^i), \dots, f_n(x^i)),$$

and

$$f_j(x^i) = \begin{cases} x_j^i - \tau & \text{for } j = l(x^i), \\ x_j^i + \frac{\tau}{n} & \text{for } j \neq l(x^i). \end{cases} \tag{5}$$

f_j denotes the j-th component of f. From the labeling rules $x_{l(x^i)}^i > 0$ for all x^i, and so $\tau > 0$ is well defined. Since $\sum_{j=0}^n f_j(x^i) = \sum_{j=0}^n x_j^i = 1$, we have

$$f(x^i) \in \Delta.$$

We extend f to all points in the simplex by convex combinations of its values on the vertices of the simplex. Let y be a point in the n-dimensional simplex of K whose vertices are $x^0, x^1, \dots, x^n$. Then, y and $f(y)$ are represented as follows;

$$y = \sum_{i=0}^n \lambda_i x^i, \text{ and } f(y) = \sum_{i=0}^n \lambda_i f(x^i), \ \lambda_i \geq 0, \ \sum_{i=0}^n \lambda_i = 1.$$

It is clear that f is uniformly continuous. We verify that f is sequentially locally non-constant.

1) Assume that a point z is contained in an $n-1$-dimensional small simplex δ^{n-1} constructed by partition of an $n-1$-dimensional face of Δ such that its i-th coordinate is $z_i = 0$. Denote the vertices of δ^{n-1} by

[3]Our result in this section is a variant of Uzawa equivalence theorem ([7]) which (classically) states that the existence of a competitive equilibrium and Brouwer's fixed point theorem are equivalent.

[4]We refer to [10] about the definition of this function.

$z^j,\ j=0,1,\dots,n-1$ and their i-th coordinate by z_i^j. Then, we have

$$f_i(z)=\sum_{j=0}^{n-1}\lambda_j f_i(z^j),\ \lambda_j\geq 0,\ \sum_{j=0}^{n}\lambda_j=1.$$

Since all vertices of δ^{n-1} are not labeled with i, (5) means $f_i(z^j)>z_i^j$ for all $j=\{0,1,\dots,n-1\}$. Then, there exists no sequence $(z(m))_{m\geq 1}$ such that $|f(z(m))-z(m)|\longrightarrow 0$ in an $n-1$-dimensional face of Δ.

2) Let z be a point in an n-dimensional simplex H_i. Assume that no vertex of H_i is labeled with i. Then

$$f_i(z)=\sum_{j=0}^{n}\lambda_j f_i(x^j)=z_i+\left(1+\frac{1}{n}\right)\tau, \qquad (6)$$

and so $z\neq f(z)$. Then, there exists no sequence $(z(m))_{m\geq 1}$ such that $|f(z(m))-z(m)|\longrightarrow 0$ in H_i.

3) Assume that z is contained in a fully labeled n-dimensional simplex H_i, and rename vertices of H_i so that a vertex x^i is labeled with i for each i. Then,

$$f_i(z)=\sum_{j=0}^{n}\lambda_j f_i(x^j)=\sum_{j=0}^{n}\lambda_j x_i^j+\sum_{j\neq i}\lambda_j\frac{\tau}{n}-\lambda_i\tau$$
$$=z_i+\left(\frac{1}{n}\sum_{j\neq i}\lambda_j-\lambda_i\right)\tau \text{ for each } i.$$

Consider sequences $(z(m))_{m\geq 1}=(z(1),z(2),\dots)$, $(z'(m))_{m\geq 1}=(z'(1),z'(2),\dots)$ such that $|f(z(m))-z(m)|\longrightarrow 0$ and $|f(z'(m))-z'(m)|\longrightarrow 0$. Let $z(m)=\sum_{i=0}^{n}\lambda(m)_i x^i$ and $z'(m)=\sum_{i=0}^{n}\lambda'(m)_i x^i$. Then, we have

$$\frac{1}{n}\sum_{j\neq i}\lambda(m)_j-\lambda(m)_i\longrightarrow 0, \text{ and}$$
$$\frac{1}{n}\sum_{j\neq i}\lambda'(m)_j-\lambda'(m)_i\longrightarrow 0 \text{ for all } i.$$

Therefore, we obtain

$$\lambda(m)_i\longrightarrow\frac{1}{n+1}, \text{and } \lambda'(m)_i\longrightarrow\frac{1}{n+1}.$$

These mean

$$|z(m)-z'(m)|\longrightarrow 0.$$

Thus, f is sequentially locally non-constant

Now, using f, we construct an excess demand function as follows;

$$g_i(y)=f_i(y)-y_i\mu(y),\ i=0,1,\dots,n. \qquad (7)$$

$y\in\Delta$, and $\mu(y)$ is defined by

$$\mu(y)=\frac{\sum_{i=0}^{n}y_i f_i(y)}{\sum_{i=0}^{n}y_i^2}.$$

Each $g_i(y)$ is uniformly continuous, and satisfies the Walras law as shown below. Multiplying y_i (the i-th component of y) to (7) for each i, and adding them from 0 to n yields

$$\sum_{i=0}^{n}y_i g_i=\sum_{i=0}^{n}y_i f_i(y)-\mu(y)\sum_{i=0}^{n}y_i^2 \qquad (8)$$
$$=\sum_{i=0}^{n}y_i f_i(y)-\frac{\sum_{i=0}^{n}y_i f_i(y)}{\sum_{i=0}^{n}y_i^2}\sum_{i=0}^{n}y_i^2$$
$$=\sum_{i=0}^{n}y_i f_i(y)-\sum_{i=0}^{n}y_i f_i(y)=0.$$

Because of sequential local non-constancy of f, $g_i(y)$'s are sequentially locally non-constant as excess demand functions described as follows;.

1) In an $n-1$-dimensional face of Δ there exists no sequence $(z(m))_{m\geq 1}$ such that $|f(z(m))-z(m)|\longrightarrow 0$, and so there exists no sequence $g(z(m))_{m\geq 1}$ such that $\mu(z(m))\longrightarrow 1$ and $\max(g(z(m)),0)\longrightarrow 0$.
2) In an n-dimensional simplex which is not fully labeled there exists no sequence $(z(m))_{m\geq 1}$ such that $|f(z(m))-z(m)|\longrightarrow 0$, and so there exists no sequence $g(z(m))_{m\geq 1}$ such that $\mu(z(m))\longrightarrow 1$ and $\max(g(z(m)),0)\longrightarrow 0$.
3) In Δ for all sequences $(x(m))_{m\geq 1}$, $(y(m))_{m\geq 1}$ such that $|f(x(m))-x(m)|\longrightarrow 0$ and $|f(y(m))-y(m)|\longrightarrow 0$, we have $|x(m)-y(m)|\longrightarrow 0$. When $|f(x(m))-x(m)|\longrightarrow 0$, we have $\mu(x(m))\longrightarrow 1$ and $\max(g(x(m)),0)\longrightarrow 0$. Similarly, when $|f(y(m))-y(m)|\longrightarrow 0$, we have $\mu(y(m))\longrightarrow 1$ and $\max(g(y(m)),0)\longrightarrow 0$.

Therefore, g is sequentially locally non-constant, and there exists an equilibrium. Let $y^*=\{y_0^*,y_1^*,\dots,y_n^*\}$ be the equilibrium price vector. Then,

$$g_i(y^*)\leq 0 \text{ for all } i,$$

and

$$g_i(y^*)=0 \text{ for } i \text{ such that } y_i^*>0,$$

and so $f_i(y^*)=\mu(y^*)y_i^*$ for all such i. $\sum_{i=0}^{n}f_i(y^*)=\sum_{i=0}^{n}y_i^*=1$ implies $\mu(y^*)\leq 1$. On the other hand, $g_i(y^*)\leq 0$(for all i) means $\mu(y^*)\geq 1$. Thus, $\mu(y^*)=1$, and we obtain

$$f_i(y^*)=y_i^* \text{ for all } i. \qquad (9)$$

Let Δ^* be a simplex of K which contains y^*, and $y^0,y^1,\dots,y^n$ be the vertices of Δ^*. Then, y^* and $f(y^*)$ are represented as

$$y^*=\sum_{i=0}^{n}\lambda_i y^i \text{ and } f(y^*)=\sum_{i=0}^{n}\lambda_i f(y^i),\ \lambda_i\geq 0,\ \sum_{i=0}^{n}\lambda_i=1.$$

(5) implies that if only one y^k among $y^0,y^1,\dots,y^n$ is labeled with i, we have

$$f_i(y^*)-y_i^*=\sum_{j=0}^{n}\lambda_j y_i^j+\sum_{j=0,j\neq k}\lambda_j\frac{\tau}{n}-\lambda_k\tau-y_i^*=\left(\frac{1}{n}\sum_{j=0,j\neq k}\lambda_j-\lambda_k\right)$$

y_i^j is the i-th component of y^j.

Since $\tau > 0$, $f_i(y^*) = y_i^*$(for all i) is equivalent to

$$\frac{1}{n}\sum_{j=0,j\neq k}\lambda_j - \lambda_k = 0.$$

(9) is satisfied with $\lambda_k = \frac{1}{n+1}$ for all k. On the other hand, if no y^j is labeled with i, we have

$$f_i(y^*) = \sum_{j=0}^{n}\lambda_j y_i^j = y_i^* + \left(1 + \frac{1}{n}\right)\tau,$$

and then (9) can not be satisfied. Thus, for each i one and only one y^j must be labeled with i. Therefore, Δ^* must be a fully labeled simplex. We have completed the proof of Sperner's lemma.

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Yasuhito Tanaka is Professor at the Faculty of Economics, Doshisha University, 602-8580, Japan.