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A New Derivative-Free Quasi-Secant Algorithm For Solving Non-Linear Equations

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Abstract—Most of the nonlinear equation solvers do not converge always or they use the derivatives of the function to approximate the root of such equations. Here, we give a derivative-free algorithm that guarantees the convergence. The proposed two-step method, which is to some extent like the secant method, is accompanied with some numerical examples. The illustrative instances manifest that the rate of convergence in proposed algorithm is more than the quadratically iterative schemes.

Keywords—Non-linear equation, iterative methods, derivative-free, convergence.

I. Introduction

WITH the booming growth of science and technology, the need of some faster methods which guarantee the convergence is a challenge. Finding the simple root of the non-linear equations is very important in numerical analysis and has many applications in engineering and other applied sciences. After the classical Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

for finding the simple zeros of the following nonlinear equation

$$f(x) = 0, \tag{1}$$

many methods have been proposed. Almost all of them have their own limitations and convincing points. To explain more, the Newton's method converges quadratically. As well as condition $f'(x) \neq 0$ in the neighborhood of root x^* is severe indeed for its convergence so its applications is restricted [1].

Recently some methods for improving the order of convergence had been proposed [3,4]. Although the calculation of the higher derivatives is so time consuming, but all of the improvements consist one or two additional iteration of the function or its derivatives to boost up the order of convergence, see for example [8]. To gain a better understanding of this issue, see the new family of seventh-order methods that has been studied in [5].

Ostrowski's method [9], which is defined as below and is a fourth-order method

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = y_n - \frac{f(y_n)(x_n - y_n)}{f(x_n) - 2f(y_n)}, \tag{2}$$

has been improved and a new variant of it with eighth-order convergence for solving non-linear equations is proposed in [7]. These are the interesting works in this field, but all of these methods contain the calculation of more derivatives which is indeed time-consuming and are failed for a non-differentiable function in the neighborhood of the root. On the other hand, the methods do not guarantee the convergence for any starting point x_0.

The Secant method do not use the derivatives of the function, but its convergence order is $\left(1+\sqrt{5}\right)/2$. Another derivative-free method, Inverse Quadratic Interpolation (IQI) [10] which is totally better than derivative-free Muller's method of the second-order can be defined as follows.

IQI converges quadratically and does not use any derivative of the function, but the computational cost of this method is so much. Since, this is a three-point method. IQI is a similar generalization of the Secant method to parabolas. However, the parabola is of form $x = p(y)$ instead of $y = p(x)$, as in Muller's method. This parabola will intersect the x-axis in a single point, so there is no ambiguity in finding the new value of the root per iteration (unlike the Muller's method). The second-degree polynomial $x = p(y)$ that passes through the three points, i.e. three previous guesses, (a, A), (b, B), (c, C) is

$$p\left(y\right) = a\frac{\left(y-B\right)\left(y-C\right)}{\left(A-B\right)\left(A-C\right)} + b\frac{\left(y-A\right)\left(y-C\right)}{\left(B-A\right)\left(B-C\right)} + c\frac{\left(y-A\right)\left(y-B\right)}{\left(C-A\right)\left(C-B\right)}.$$

If $q = f(a)/f(b)$, $r = f(c)/f(b)$, and $s = f(c)/f(a)$. Then we have

$$p\left(0\right) = c - \frac{r\left(r-q\right)\left(c-b\right) + \left(1-r\right)s(c-a)}{\left(q-1\right)\left(r-1\right)\left(s-1\right)},$$

Note that, this is an example of Lagrange interpolation.

If we assume $a = x_i$, $b = x_{i+1}$, $c = x_{i+2}$, and $A = f(x_i)$, $B = f(x_{i+1})$, $C = f(x_{i+2})$, then the next guess $x_{i+3} = p(0)$ is

$$x_{i+3} = x_{i+2} - \frac{r\left(r-q\right)\left(x_{i+2}-x_{i+1}\right) + \left(1-r\right)s\left(x_{i+2}-x_i\right)}{\left(q-1\right)\left(r-1\right)\left(s-1\right)}$$

where

$$q = \frac{f(x_i)}{f(x_{i+1})},$$

$$r = \frac{f(x_{i+2})}{f(x_{i+1})},$$

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and

$$s = f(x_{i+2})/f(x_i).$$

Definition 1. The efficiency index of an iterative scheme for solving the non-linear equation (1) when p is the order of method is defined as:

$$p^{1/\kappa},$$

where κ is the number of function evaluations per iteration.

One of the well-known methods which guarantee the convergence is Brent iterative scheme. Brent method [2], is a hybrid method. It is most desirable to combine the property of guaranteed convergence, from the Bisection method, with the property of fast convergence from the more sophisticated methods. It was originally proposed by Dekker and Van Wijngaarden in the 1960s.

The method is applied to a continuous function f and an interval bounded by a and b, where $f(a)f(b) < 0$. Brent's method keeps track of a current point x_i that is best in the sense of backward error, and a bracket $[a_i, b_i]$ of the root. Roughly speaking, the IQI method is attempted, and the result is used to replace one of x_i, a_i, b_i if (one) the backward error improves and (two) the bracketing interval is cut at least in half. If not, the Secant method is attempted with the same goal. If it fails as well, a Bisection method step is taken, guaranteeing that the uncertainty is cut at least in half. MATLAB's comment fzero implements a version of Brent's method, along with a preprocessing step, to discover a good initial bracketing interval if (one) is not provided by the user.

The stopping criterion is of a mixed forward/backward error type. The algorithm terminates when the change from x_i to the new point x_{i+1} is less that $2\varepsilon_{machine} max\,(1, x_i)$ or when the backward error $|f(x_i)|$ achieves mac hine zero.

The preprocessing step is not triggered if the user provides an initial bracketing interval.

Recently, the Cauchy method

$$x_{n+1} = x_n - \frac{2}{1+\sqrt{1-2L_f(x_n)}}\frac{f(x_n)}{f'(x_n)},$$

$$L_f(x_n) = \frac{f''(x_n)f(x_n)}{[f'(x_n)]^2}, \tag{3}$$

of local order three has been improved [6] as below, and the efficiency index of the obtained derivative-free method increased to 2.618. For removing the second derivative of (3), some variants of Cauchy's method are obtained by the approach of replacing the second derivative with the values of the function on different points. This derivative-free iterative scheme is defined as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{f(y_n) - f(x_n)}(y_n - x_n)$$

where

$$y_{n+1} = x_{n+1} - \frac{f(x_{n+1})}{f(y_n) - f(x_n)}(y_n - x_n).$$

But the problem of this method is that its convergence is not guaranteed as well.

In this paper, the important characteristic of the new method is that per iteration, it requires four evaluations of the function (without any evaluation of the derivatives) and it guarantees the convergence while even the aforementioned methods do not converge always or the order of convergence is lower. Some numerical examples are given to show the efficiency of the new method. From a practical standpoint, it is interesting to improve the well-known methods. So in the other words in this work, we simply increase the order of convergence of a method that guarantees the convergence.

The following two-step iterative scheme [11], for solving nonlinear equations converges quadratically to the simple root and the convergence is guaranteed.

Theorem 1. Suppose the equation (1) has a simple root x^* on an interval (a, b). Then the following iterative scheme converges quadrayically

$$x_{n+1} = x_n - \frac{2(x_n - x_{n-1})f(x_n)}{f(2x_n - x_{n-1}) - f(x_{n-1})} \tag{4}$$

where $x_{-1} = a$ and $x_0 = \frac{(a+b)}{2}$. and the convergence is guaranteed.

II. MAIN RESULT

In the following we present a technique that consists in an iterative method in two steps

$$z_n = \varphi(x_n), \tag{5}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \tag{6}$$

where $\varphi x_n)$ is the two-step method of second-order as (4)

$$\varphi(x_n) = x_n - \frac{2(x_n - x_{n-1})f(x_n)}{f(2x_n - x_{n-1}) - f(x_{n-1})} \tag{7}$$

$$x_{-1} = a;\ x_0 = \frac{(a+b)}{2} \tag{8}$$

and $f'(z_n)$ defined as below to avoid one calculation of the first derivative of the function

$$f'(z_n) = \frac{f(z_n) - f(x_n)}{z_n - x_n} \tag{9}$$

Algorithm 1. The following two-step iterative scheme for the function $f : D = (a, b) \to \mathrm{R}$ converges to the root of (1).

$$z_n = x_n - \frac{2(x_n - x_{n-1})f(x_n)}{f(2x_n - x_{n-1}) - f(x_{n-1})},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{\frac{f(z_n) - f(x_n)}{z_n - x_n}}.$$

The stopping criterion is just like Brent's method. The algorithm terminates when the change from x_i to the new point x_{i+1} is less than

$$2\varepsilon_{mac\ hine} max(1, x_i)$$

or when the backward error $|f(x_i)|$ achieves machine zero.

III. NUMERICAL IMPLEMENTATIONS

We consider some examples in this section, by taking into account of the proposed method. All of the calculations are done with MATLAB, up to eight floating point.

Example 1. We solve the following equation on the real open interval $D = (0, 6)$. The root of the equation is 2. The starting point is $x_0 = 3$ in the Newton's method and Ostowski's method which need an initial guess to start the iteration.

$$f_1(x) = (x - 2)^4(x + 1)$$

TABLE I
THE APPROXIMATION OF THE ROOT BY SOME VARIANT METHODS FOR f_1, AFTER 20 ITERATIONS.

Method	Root Approximation	Analysis of Convergence
Newton's Method	2.00348128	Not-guaranteed
Ref [11]	2.00748897	Guaranteed
Ostrowski's Method	2.00000575	Not-guaranteed
Proposed Algorithm	2.00031922	Guaranteed

In the above example the proposed Algorithm convergence is guaranteed. As well as the required computational iteration to get the approximation up to two decimal points is 13, while in [11] it required 20 iterations. (Note that the choice of $x_0 = 4$ or worse in Newton's method Ostrowski's method derives to failure in root-finding).

Example 2. Now we consider the simple zero-finding of the following equation in $D = (1, 4)$. In the following table we provide the numbers of needed steps to gain the root for each method. (The starting point $x_0 = 3$ can be taken for Newton's method and Ostrowski's method.)

$$f_2(x) = 10xe^{-x^2} - 1, \qquad x^* = 1.67963061042845.$$

TABLE II
Numbers of needed steps by some variant methods for f_2 to gain the exact root.

Method	Number of Steps	Analysis of Convergence
Newton's Method	-	Divergent
Ref [11]	5	Guaranteed
Ostrowski's Method	-	Divergent
Proposed Algorithm	4	Guaranteed

In this example, the choice of $x_0 = 3$ in Newton's method and Ostrowski's method derives to failure in root-finding, whereas the convergence of the zero-finding in the proposed method is guaranteed. As well as, its convergence is more than [11].

Example 3. We consider the problem of simple zero-finding in the following continuous function in the interval $D = (2, 4.5)$. The starting point will be taken $x_0 = 4$ for iterative schemes which need an initial guess.

$$f_3(x) = e^{x^2+7x-30} - 1, \qquad x^* = 3.$$

TABLE III
Numbers of needed steps by some variant methods for f_3 to gain the exact root.

Method	Number of Steps	Analysis of Convergence
Newton's Method	-	Divergent
Ref [11]	7	Guaranteed
Ostrowski's Method	-	Divergent
Proposed Algorithm	5	Guaranteed

As the instance illustrates, whilst the other well-known methods such as Newton's method or Ostrowski's method diverge for the initial point, and the rate of convergence in (4) is low, the proposed algorithm converge faster and the convergence is guaranteed.

IV. CONCLUDING REMARKS

As we have seen, the new algorithm is more efficient and its order of convergence is not local, i.e., by allocating x_{-1}, x_0 as in (8), the convergence of the iterative scheme is guaranteed. The new algorithm which is generally a generalization of [11] includes four evaluation of the function. The remarkable fact of this new algorithm is that after the first iteration, one of the past evaluations of the function can be used in the new iteration, and then it reduces the computational cost of the algorithm per iteration.

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