World Academy of Science, Engineering and Technology
International Journal of Electronics and Communication Engineering
Vol:1, No:2, 2007

Recursive Wiener-Khintchine Theorem

Khalid M. Aamir[1] and Mohammad A. Maud[2]

***Abstract*—** Power Spectral Density (PSD) computed by taking the Fourier transform of auto-correlation functions (Wiener-Khintchine Theorem) gives better result, in case of noisy data, as compared to the Periodogram approach. However, the computational complexity of Wiener-Khintchine approach is more than that of the Periodogram approach. For the computation of short time Fourier transform (STFT), this problem becomes even more prominent where computation of PSD is required after every shift in the window under analysis. In this paper, recursive version of the Wiener-Khintchine theorem has been derived by using the sliding DFT approach meant for computation of STFT. The computational complexity of the proposed recursive Wiener-Khintchine algorithm, for a window size of N, is $O(N)$.

***Keywords*—** Power Spectral Density (PSD), Wiener-Khintchine Theorem, Periodogram, Short Time Fourier Transform (STFT), The Sliding DFT.

I. Introduction

THE discrete Fourier transform (DFT), for transforming the time signal into its frequency domain counterpart, is a popular signal analysis tool in science and engineering. Frequency and time are orthogonal. But some signals do have frequency components that change with time for example speech can be heard as having pitch that rises and falls over time. The solution to such problems is the Short Time Fourier transform (STFT) proposed by Gabor in 1946 [1][2]. STFT evaluates the way frequency content changes with time [3]. Moving a fixed data length window over the time series achieves this end. Each movement of the window corresponds to the passage of a unit time instant. Every time before moving the window, the discrete Fourier transform is taken through the fast Fourier transform (FFT) algorithm [3]. Thus, the transform of each window corresponding to a time instant is available for analysis. However, it is well known that the signal spectrum or the Power Spectral Density (PSD) computed through DFT does not give good results when the signal is corrupted by noise.

A recursive sliding DFT algorithm [4][5] has been proposed for the purpose of computing STFT efficiently. With this method, the computational complexity for calculating DFT of each window is $O(N)$ as compared to $O(N^2)$ for standard DFT computation and $O(N \log_2 N)$ for FFT.

The method of calculating the PSD from the Fourier transform of the autocorrelation functions, as proposed by Wiener-Khintchine's theorem (WKT), gives better estimate of the PSD as compared to DFT for the case of noisy signals. The Wiener-Khintchine theorem relating the autocorrelation functions $r_{xx}(\tau)$ to PSD is given by [6][7],

$$P(f) = \int_{-\infty}^{\infty} r_{xx}(\tau) e^{-j2\pi f\tau} d\tau \qquad (1)$$

where

$$r_{xx}(\tau) = E[x(t+\tau)x^*(t)] \qquad (2)$$

with * indicating complex conjugate.

Numerical computation of autocorrelation is done by the following equation,

$$r_{xx}(m) = \sum_{n=0}^{N-1} x[n+m]x[n] \qquad \text{for } 0 \le m \le N-1 \qquad (3)$$

This implies that computation of PSD from autocorrelation functions would be computationally expensive over computation of PSD from the signal through DFT (Periodogram). With the advantage of WKT over periodogram approach for noisy data, it seems appropriate to devise efficient algorithm for computing PSD through WKT.

In this paper, the recursive sliding window DFT algorithm [4], [5] has been used for efficient computation of the PSD recursively from the autocorrelation functions.

After giving a short review of the recursive sliding DFT in the next section, the recursive form of WKT has been derived in Section III. A discussion on the computational complexity of the proposed algorithm follows in Section IV. The paper ends with some simulation results.

II. Review of the Sliding DFT

Consider a time series signal,

$$x[n] = \{x_0, x_1, x_2, \cdots, x_{K-1}\} \text{ for } 0 \le n < K \qquad (4)$$

Define a window $y_1[n]$ of length N, where $N < K$, within the original time series $x[n]$ such that,

$$y_1[n] = \{x_0, x_1, x_2, \cdots, x_{N-1}\} \qquad (5)$$

Also define another window $y_2[n]$ of the same length as $y_1[n]$ except that it is one time sample shifted version.

$$y_2[n] = \{x_1, x_2, x_3, \cdots, x_N\} \qquad (6)$$

In the standard STFT algorithm, DFT $Y_1[k]$ for $k = 0, 1, \cdots, N-1$, is computed for the first window $y_1[n]$ and then the window is moved ahead by one sample. The new window is $y_2[n]$ and corresponding DFT is $Y_2[k]$. This process continues until all samples of the time series are exhausted.

An efficient recursive-sliding window DFT for computing the STFT has been derived by Jacobsen and Lyons [5]. The

---

[1] K. M. Aamir is a graduate student at Lahore University of Management Sciences, Lahore, Pakistan. (e-mail: kmaamir@ lums.edu.pk)

[2] M. A. Maud is currently the Dean of Quality Assurance at University of Management and Technology, Lahore, Pakistan. (e-mail: mamaud@umt.edu.pk).

computational complexity of this approach is $O(N)$ for a window size of N. It is given by,

$$Y_2[k] = e^{j2\pi k/N}\left(Y_1[k] + x_N - x_0\right) \tag{7}$$

III. Recursive Form of Wiener-Khintchine Theorem

Define $s_{xx}(m)$ as the autocorrelation function of $y_1[n]$ (eq. (5)) and $r_{xx}(m)$ as the autocorrelation function of $y_2[n]$ (eq. (6)). Define $S_{xx}(k)$ and $R_{xx}(k)$ as the corresponding DFT's of the autocorrelation functions. Thus the expressions, ignoring the constant multiplier $1/N$, can be written as follows.

$$s_{xx}(m) = \sum_{n=0}^{N-1} y_1[n+m]y_1[n] \qquad \text{for } 0 \le m \le N-1$$

$$s_{xx}(m) = y_1[m]y_1[0] + \sum_{n=1}^{N-1} y_1[n+m]y_1[n]$$

$$s_{xx}(m) - y_1[m]y_1[0] = \sum_{n=1}^{N-1} y_1[n+m]y_1[n] \tag{8}$$

As the window moves ahead by one sample, the autocorrelation would become,

$$r_{xx}(m) = \sum_{n=0}^{N-1} y_2[n+m]y_2[n] \qquad \text{for } 0 \le m \le N-1$$

$$\begin{aligned} r_{xx}(m) &= y_2[N+m-1]y_2[N-1] \\ &\quad + \sum_{n=0}^{N-2} y_2[n+m]y_2[n] \end{aligned} \tag{9}$$

Since eq. (8) and eq. (9) have the same summands so,

$$\begin{aligned} r_{xx}(m) &= s_{xx}(m) - y_1[m]y_1[0] \\ &\quad + y_2[N+m-1]y_2[N-1] \end{aligned}$$

Taking DFT on both the sides,

$$\begin{aligned} R_{xx}(k) &= S_{xx}(k) - y_1[0]\,\mathrm{DFT}[y_1[m]] \\ &\quad + y_2[N-1]\,\mathrm{DFT}[y_2[N+m-1]] \end{aligned} \tag{10}$$

$$\mathrm{DFT}[y_1[m]] = \text{DFT of previous window} = Y_1[k] \tag{11}$$

And,

$$\mathrm{DFT}[y_2[N+m-1]] = \sum_{m=0}^{N-1} y_2[N+m-1]e^{-j2\pi mk/N} \tag{12}$$

Let $m = N - n - 1$, putting in eq. (12),

$$\begin{aligned} \mathrm{DFT}[y_2[N+m-1]] &= \sum_{n=0}^{N-1} y_2[n]\exp\left[-j2\pi\frac{N-n-1}{N}k\right] \\ &= \exp\left[-j2\pi\frac{N-1}{N}k\right]\sum_{n=0}^{N-1} y_2[n]\exp\left[j2\pi nk/N\right] \\ &= \exp\left[j2\pi k/N\right]\left[\sum_{n=0}^{N-1} y_2^*[n]\exp\left[-j2\pi nk/N\right]\right]^* \end{aligned} \tag{13}$$

For real signals, $y_2^*[n] = y_2[n]$, so

$$\begin{aligned} \mathrm{DFT}[y_2[N+m-1]] &= e^{j2\pi k/N}\left[\mathrm{DFT}[y_2[m]]\right]^* \\ &= e^{j2\pi k/N}Y_2^*(k) \end{aligned} \tag{14}$$

Putting eq. (11) and (14) in eq. (10).

$$\begin{aligned} R_{xx}(k) &= S_{xx}(k) - y_1[0]\,\mathrm{Y}_1(k) \\ &\quad + y_2[N-1]e^{j2\pi k/N}\,\mathrm{Y}_2^*(k) \end{aligned} \tag{15}$$

Thus, PSD of the current window can be computed in N computations from the PSD of the previous window using the sliding DFT in $O(N)$.

IV. Algorithm

The algorithm for recursive computation of the PSD based on WKT is given below

- Set size of time window = N
- Initialization
  - Set: a = first sample of the signal
  - Pad $N - 1$ zeros at the start of the signal.
  - Define a vector: $r_{xx}[n] = 0$, $n = 0$ to $N-1$
  - Set $r_{xx}[1] = a^2$
  - Define a vector: $S_{xx}[k] = a^2$, $k = 0$ to $N-1$
  - Calculate $\mathrm{Y}_1[k] = a \times \exp[-j\pi(N-1)k/N]$ for $k = 0$ to $N - 1$
- For $n = 1$ to K– 1
  - Define: x_o = the first sample of the previous window
  - Define: x_N = the last sample of the current window
  - Calculate $Y_2[k]$ = DFT of the new window using $Y_1[k]$, calculated by using the SDFT
  - For $k = 0$ to $N-1$
    - Calculate
      $$R_{xx}(k) = S_{xx}(k) - x_o \times \mathrm{Y}_1[k] + x_N \times e^{j2\pi k/N} \times \mathrm{Y}_2^*[k]$$
  - Update, for $k = 0$ to $N - 1$
    - Set $Y_1[k] = Y_2[k]$
    - Set $S_{xx}[k] = R_{xx}[k]$

V. Complexity of the Proposed Algorithm

Assume that at some instant the PSD and the DFT of the previous window are available. Using these past values, we can compute the PSD of the current window, for real valued signals, in $2N$ complex multiplications and $4N$ real multiplications which is equivalent to $3N$ complex multiplications, out of which N will be used to calculate the DFT of the current window only. The number of complex additions required is $5N/2$. It will need storage of $2N$ complex numbers, N for complex values of DFT of previous window and N for complex values of PSD of previous window. This is the requirement for each recursion. The loop runs K– 1 times. If we take computations during initialization into account, then $2N + 1$ real multiplications or equivalently $(2N+1)/4$ complex multiplications take place during initialization. So overall complexity (multiplications) of the algorithm will be,

$$3N(\mathrm{K}-1) + \frac{2N+1}{4} = 3N\mathrm{K} - \left(4N - \frac{2N+1}{4}\right) < 3N\mathrm{K} \tag{16}$$

Thus, the complexity for computation of the recursive PSD based on WKT for the complete data sequence is $O(\mathrm{K}N)$. This indicates a major improvement over the existing techniques.

VI. Simulation Results

The following autoregressive process of order 2 has been generated,

$$y[n] = 0.4y[n-1] - 0.93y[n-2] + w[n]$$

where $w[n]$ is a white Gaussian process of zero mean and unit variance. A measurement noise was added to the sequence thus generated resulting in an SNR of 8.8673 dB. Figure 1 shows the comparison of using WKT versus periodogram. The advantage of calculating PSD using Wiener-Khintchine theorem over the PSD using FFT is evident.

Acknowledgment

This research was fully funded by Higher Education Commission (HEC), Islamabad, Pakistan. This support is gratefully acknowledged.

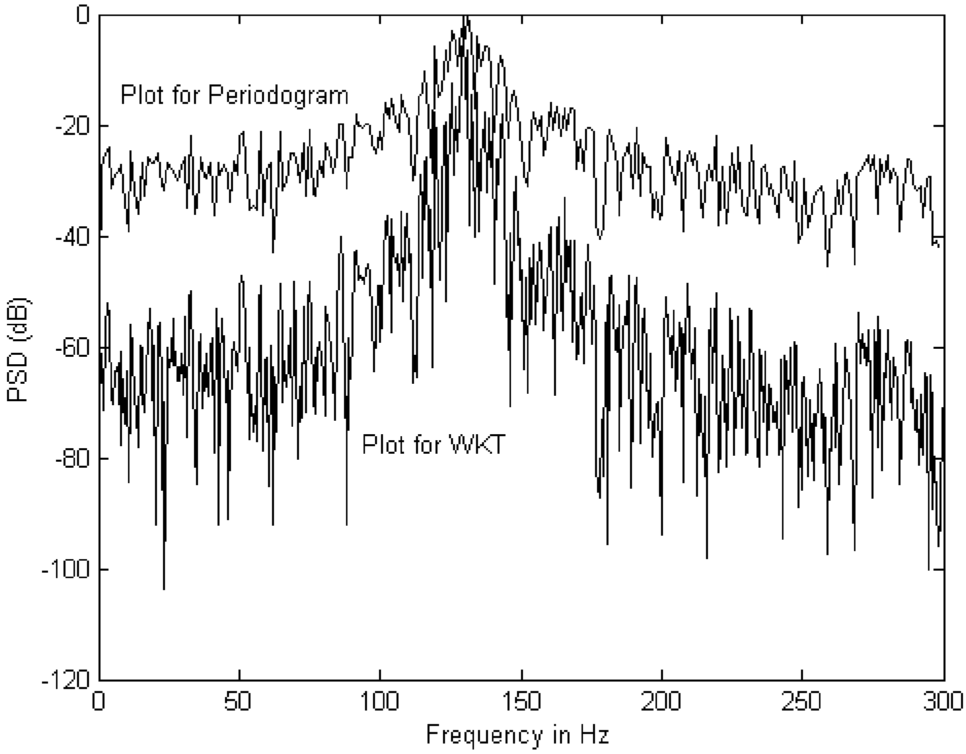

Figure 1 Comparison of PSD through Periodogram and WKT for SNR = 8.8673 db. It is clear that PSD of Wiener-Khintchine Theorem is better.

References

[1] W. T. Cochran, J. W. Cooley et al., "What is the Fast Fourier Transform?" *IEEE Transactions on Audio and Electroacoustics*, Vol. AU-15, No. 2, June 1967.

[2] M. T. Heideman, D. H. Johnson and C. S. Burrus, "Gauss and the History of the Fast Fourier Transform," *IEEE ASSP Magazine*, October 1984.

[3] J. C. Goswami, and A. K Chan, "Fundamentals of Wavelets: Theory, Algorithms and Applications," John Wiley & Sons, Inc., USA, 1999.

[4] H. V. Sorensen and C. S. Burrus, "Efficient Computation of Short Time Fast Fourier Transform," *Proc. ICASSP'88*, April 1988, pp. 1894-1897.

[5] E. Jacobsen, and R. Lyons, "The sliding DFT," *Signal Processing Magazine*, IEEE, Vol. 20 , No. 2 , Mar 2003, pp. 74 - 80.

[6] E. A. Robinson, "A Historical Perspective of Spectrum Estimation," Proceedings of the IEEE, vol. 70, No. 9, Sep.1982, pp. 885-907.

[7] A. Papoulis, "Levinson's Algorithm, Wold's Decomposition and Spectral Estimation," SIAM Review, vol. 27, No. 3, Sep. 1985, pp. 405-441.

Khalid M. Aamir He did his M.Sc. (Electronics) in 1997 and M.S.(Systems Engineering) in 1999, both from Quaid-e-Azam University, Islamabad, Pakistan. He is currently a Ph.D. candidate in Computer Engineering at Department of Computer Sciences, Lahore University of Management Sciences, Lahore, Pakistan.

He worked as Scientific Officer from 1999 to 2001 and since then he is Senior Scientific Officer at Informatics Complex of Computers and Control, Islamabad, Pakistan. His research interests include: Higher Order Statistics, Statistical Signal Processing, Adaptive Signal Processing, Digital Control Systems, System Identification.

Mohammad A. Maud He completed his PhD in 1986 from Stevens Institute of Technology, NJ, USA. Currently he is serving as the Dean of Quality Assurance at University of Management and Technology, Lahore, Pakistan. His research interests include: Statistical Signal Processing, Adaptive Signal Processing, Digital Control Systems, System Identification.