

# Parametric Vibrations of Periodic Shells

B. Tomczyk, R. Mania

**Abstract**—Thin linear-elastic cylindrical circular shells having a micro-periodic structure along two directions tangent to the shell midsurface (*biperiodic shells*) are object of considerations. The aim of this paper is twofold. First, we formulate an averaged non-asymptotic model for the analysis of parametric vibrations or dynamical stability of periodic shells under consideration, which has constant coefficients and takes into account the effect of a cell size on the overall shell behavior (*a length-scale effect*). This model is derived employing *the tolerance modeling procedure*. Second we apply the obtained model to derivation of *frequency equation* being a starting point in the analysis of parametric vibrations. The effect of the micro-structure length on this frequency equation is discussed.

**Keywords**—Micro-periodic shells, mathematical modeling, length-scale effect, parametric vibrations

## I. INTRODUCTION

THE object of consideration are thin linear-elastic cylindrical circular shells having a periodically inhomogeneous structure along two directions tangent to the shell midsurface. By periodic inhomogeneity we shall mean periodically variable shell thickness and/or periodically variable inertial and elastic properties of the shell material. Shells of this kind are termed *biperiodic*. As an example we can mention cylindrical shells with periodically spaced families of thin ribs shown in Fig.1. The period of heterogeneity are assumed to be very large compared with the maximum shell thickness and very small as compared to the midsurface curvature radius as well as the smallest characteristic length dimension of the shell midsurface.

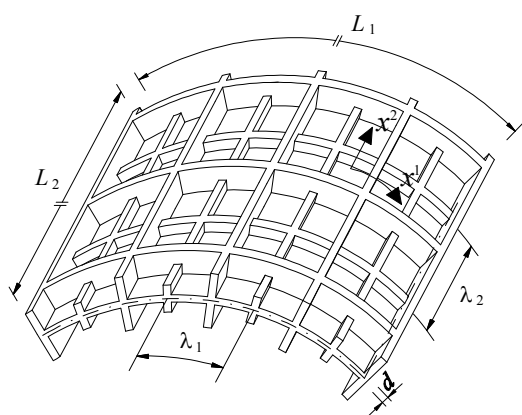


Fig. 1 An example of biperiodically stiffened cylindrical shell

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For shells of this kind it is interesting to analyze the effect of the periodicity cell size on the overall shell behavior (called *the length-scale effect*). However, the exact equations of the shell theory involve highly oscillating, non-continuous, periodic coefficients and hence they are too complicated to apply to investigations of engineering problems. That is why a lot of different approximate modeling methods for shells of this kind have been proposed. Periodic cylindrical shells are usually described using *homogenized models* derived by means of *asymptotic methods*. These models represent certain equivalent structures with constant or slowly varying rigidities and averaged mass densities, cf. [1]-[3]. Unfortunately, in models of this kind *the effect of the period lengths* on the overall shell behavior is neglected in the first approximation which is usually employed.

The periodically densely ribbed shells are also modelled as homogeneous orthotropic structures, cf. [4]. These orthotropic models are also incapable of describing many phenomena (e.g. the dispersion of waves and the existence of higher-order motions and higher free vibration frequencies dependent on a cell size) observed mainly in the dynamics and dynamic stability of periodic structures. In order to analyze the length-scale effect in some special dynamic or/and stability problems, the new averaged non-asymptotic models of thin cylindrical shells with a periodic micro-heterogeneity either along two directions tangent to the shell midsurface (*biperiodic structure*) or along one direction (*uniperiodic structure*) have been proposed by Tomczyk in a series of papers, e.g. [5]-[11], and also in books [12]-[16]. These, so called, *the tolerance models* have been obtained by applying *the non-asymptotic tolerance averaging technique*, proposed and discussed in monographs [17]-[20], to the known governing equations of Kirchhoff-Love theory of thin elastic shells (partial differential equations with functional highly oscillating non-continuous periodic coefficients). Contrary to starting equations, the governing equations of the tolerance models have coefficients which are constant or slowly-varying and depend on the period length of inhomogeneity. Hence, these models make it possible to investigate the effect of a cell size on the global shell dynamics and stability. This effect is described by means of certain extra unknowns called *fluctuation amplitudes* and by known *fluctuation shape functions* which represent oscillations inside the periodicity cell. In the papers and books, mentioned above, the applications of the proposed models to analysis of special problems dealing with dynamics as well as stationary and dynamical stability of *uniperiodically and biperiodically densely stiffened cylindrical shells* have been presented. *It was shown that the length-scale effect plays an important role in these problems and cannot be neglected.*

It has to be emphasized that the non-asymptotic tolerance models of shells with uni- and biperiodic structure have to be led out independently, because they are based on different modeling assumptions. The governing equations for uniperiodic shells are more complicated. It means that contrary to the asymptotic approach, *the uniperiodic shell is not a special case of biperiodic shell*.

For review of application of the tolerance approach to the modeling of different periodic and also non-periodic structures the reader is referred to [17]-[20].

The first aim of this contribution is to formulate a new mathematical non-asymptotic model for analysis of parametric vibrations and dynamical stability of periodically stiffened shells under consideration. This model will be derived by applying *a new approach to the tolerance modeling of micro-heterogeneous media* proposed by Woźniak in [19]. The second aim is to apply this model to investigate the effect of a microstructure size on the frequency equation being a starting point in the analysis of parametric vibrations (or dynamic stability) of periodic shells under consideration.

It should be mentioned that the periodic shells considered here are widely applied in civil engineering, most often as roof girders and bridge girders. They are also widely used as housings of reactors and tanks. Periodic shells having small length dimensions are elements of air-planes, ships and machines.

In the subsequent section the basic denotations, preliminary concepts and starting equations will be presented.

## II. PRELIMINARIES

In this paper we investigate linear-elastic thin circular cylindrical shells. The shells are reinforced by families of ribs, which are periodically and densely distributed in circumferential and axial directions. Shells of this kind are termed *biperiodic*. Example of such shell is shown in Fig. 1.

In order to describe the shell geometry define  $\Omega = (0, L_1) \times (0, L_2)$  as a set of points  $\mathbf{x} \equiv (x^1, x^2)$  in  $R^2$ ;  $x^1, x^2$  being the Cartesian orthogonal coordinates parametrizing region  $\Omega \subset R^2$ . Let  $O\bar{x}^1\bar{x}^2\bar{x}^3$  stand for a Cartesian orthogonal coordinate system in the physical space  $E^3$ . Points of  $E^3$  will be denoted by  $\bar{\mathbf{x}} \equiv (\bar{x}^1, \bar{x}^2, \bar{x}^3)$ . A cylindrical shell midsurface  $M$  is given by its parametric representation  $M \equiv \{\bar{\mathbf{x}} \in E^3 : \bar{\mathbf{x}} = \bar{\mathbf{r}}(x^1, x^2), (x^1, x^2) \in \Omega\}$ , where  $\bar{\mathbf{r}}(\cdot)$  is the smooth function such that  $\partial \bar{\mathbf{r}} / \partial x^1 \cdot \partial \bar{\mathbf{r}} / \partial x^2 = 0$ ,  $\partial \bar{\mathbf{r}} / \partial x^1 \cdot \partial \bar{\mathbf{r}} / \partial x^1 = 1$ ,  $\partial \bar{\mathbf{r}} / \partial x^2 \cdot \partial \bar{\mathbf{r}} / \partial x^2 = 1$ . It means that on  $M$  we have introduced the orthonormal parametrization and hence  $L_1, L_2$  are length dimensions of  $M$ . It is assumed that  $x^1$  and  $x^2$  are coordinates parametrizing the shell midsurface along the lines of its principal curvature and along its generatrix, respectively, cf. Fig. 1.

Subsequently, sub- and superscripts  $\alpha, \beta, \dots$  run over sequence  $1, 2$  and are related to midsurface parameters  $x^1, x^2$ ;

summation convention holds. The partial differentiation related to  $x^\alpha$  is represented by  $\partial_\alpha$ . Moreover, it is denoted  $\partial_{\alpha\dots\delta} \equiv \partial_\alpha \dots \partial_\delta$ . Differentiation with respect to time coordinate  $t \in [t_0, t_1]$  is represented by the overdot. Denote by  $a_{\alpha\beta}$  and  $a^{\alpha\beta}$  the covariant and contravariant midsurface first metric tensors; respectively. For the introduced parametrization  $a_{\alpha\beta} = a^{\alpha\beta} = \delta^{\alpha\beta}$  are the unit tensors.

Let  $\tilde{d}(\mathbf{x})$  and  $r$  stand for the thickness stiffened shell and the constant midsurface curvature radius, respectively.

Denote by  $b_{\alpha\beta}$  the covariant midsurface second metric tensor. For the introduced parametrization  $b_{22} = b_{12} = b_{21} = 0$  and  $b_{11} = -r^{-1}$ .

Let  $\lambda_1$  and  $\lambda_2$  be the period lengths of the stiffened shell structure respectively in  $x^1$ - and  $x^2$ -directions, cf. Fig. 1. Define *the basic cell*  $\Delta$  and *the cell distribution*  $(\Omega, \Delta)$  assigned to  $\Omega = (0, L_1) \times (0, L_2) \subset R^2$  by means of:

$$\Delta \equiv [-\lambda_1/2, \lambda_1/2] \times [-\lambda_2/2, \lambda_2/2],$$

$$(\Omega, \Delta) \equiv \{\Delta(x^1, x^2) \equiv (x^1, x^2) + \Delta, (x^1, x^2) \in \bar{\Omega}\},$$

where point  $(x^1, x^2)$  is a center of a cell  $\Delta(x^1, x^2)$  and  $\bar{\Omega}$  is a closure of  $\Omega$ .

The diameter  $\lambda \equiv \sqrt{(\lambda_1)^2 + (\lambda_2)^2}$  of  $\Delta$  is assumed to satisfy conditions:  $\lambda / \tilde{d}_{\max} \gg 1$ ,  $\lambda / r \ll 1$  and  $\lambda / \min(L_1, L_2) \ll 1$ . Hence, the diameter will be called *the microstructure length parameter*. In every cell  $\Delta(\mathbf{x})$  we introduce local coordinates  $z^1, z^2$  along the  $x^1$ - and  $x^2$ -directions, respectively, with the 0-point at the center of the cell. It means that the cell  $\Delta$  has two symmetry axes: for  $z^1 = 0$  and  $z^2 = 0$ . Thus, inside the cell, the geometrical, elastic and inertial properties of the stiffened shell are described by symmetric (i.e. even) functions of  $\mathbf{z} \equiv (z^1, z^2) \in [-\lambda_1/2, \lambda_1/2] \times [-\lambda_2/2, \lambda_2/2]$ .

Denote by  $u_\alpha = u_\alpha(\mathbf{x}, t)$ ,  $w = w(\mathbf{x}, t)$ ,  $\mathbf{x} \in \Omega$ ,  $t \in (t_0, t_1)$ , the midsurface shell displacements in directions tangent and normal to  $M$ , respectively. Elastic properties of the shell are described by shell stiffness tensors  $D^{\alpha\beta\gamma\delta}(\mathbf{x})$ ,  $B^{\alpha\beta\gamma\delta}(\mathbf{x})$ . Let  $\mu(\mathbf{x})$  stand for a shell mass density per midsurface unit area. Let  $f^\alpha(\mathbf{x}, t)$ ,  $f(\mathbf{x}, t)$  be external forces per midsurface unit area, respectively tangent and normal to  $M$ . We denote by  $\bar{N}^{\alpha\beta}(t)$  the time-dependent compressive membrane forces.

Functions  $\mu(\mathbf{x})$ ,  $D^{\alpha\beta\gamma\delta}(\mathbf{x})$ ,  $B^{\alpha\beta\gamma\delta}(\mathbf{x})$  and  $\tilde{d}(\mathbf{x})$ ,  $\mathbf{x} \in \Omega$ , are assumed to be  $\Delta$ -periodic with respect to arguments  $x^1, x^2$ .

It is assumed that the in the general case the behavior of the stiffened shell under consideration is described by the action functional

$$A(u_\alpha, w) = \int_0^{L_1} \int_0^{L_2} \int_0^{t_1} L(\mathbf{x}, \partial_\beta u_\alpha, u_\alpha, \dot{u}_\alpha, \partial_{\alpha\beta} w, \partial_\alpha w, w, \dot{w}) dt dx^2 dx^1, \quad (1)$$

where lagrangian  $L$  is highly oscillating function with respect to  $\mathbf{x}$  and has the well-known form (cf. [21])

$$L = \frac{1}{2} (D^{\alpha\beta\gamma\delta} \partial_\beta u_\alpha \partial_\delta u_\gamma + 2r^{-1} D^{\alpha\beta 11} w \partial_\beta u_\alpha + r^{-2} D^{1111} w w + B^{\alpha\beta\gamma\delta} \partial_{\alpha\beta} w \partial_{\gamma\delta} w + \bar{N}^{\alpha\beta}(t) \partial_\alpha w \partial_\beta w - \mu a^{\alpha\beta} \dot{u}_\alpha \dot{u}_\beta - \mu \dot{w}^2) - f^\alpha u_\alpha - f w. \quad (2)$$

Obviously, in the above formula we have taken into account that  $b_{11} = -r^{-1}$ .

The principle of stationary action applied to  $A$  leads to the following system of Euler-Lagrange equations

$$\begin{aligned} \partial_\beta \frac{\partial L}{\partial(\partial_\beta u_\alpha)} - \frac{\partial L}{\partial u_\alpha} + \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{u}_\alpha} &= 0, \\ -\partial_{\alpha\beta} \frac{\partial L}{\partial(\partial_{\alpha\beta} w)} + \partial_\alpha \frac{\partial L}{\partial(\partial_\alpha w)} - \frac{\partial L}{\partial w} + \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{w}} &= 0. \end{aligned} \quad (3)$$

After combining (3) with (2) the above system can be written in the form

$$\begin{aligned} \partial_\beta (D^{\alpha\beta\gamma\delta} \partial_\delta u_\gamma) + r^{-1} \partial_\beta (D^{\alpha\beta 11} w) - \mu a^{\alpha\beta} \ddot{u}_\beta + f^\alpha &= 0, \\ r^{-1} D^{\alpha\beta 11} \partial_\beta u_\alpha + \partial_{\alpha\beta} (B^{\alpha\beta\gamma\delta} \partial_\gamma w) + r^{-2} D^{1111} w - \bar{N}^{\alpha\beta} \partial_{\alpha\beta} w + \mu \ddot{w} - f &= 0. \end{aligned} \quad (4)$$

We observe that (4) coincides with the well-known governing equations of simplified Kirchhoff-Love second-order theory of thin elastic shells, cf. [21]. In the above equations displacements  $u_\alpha = u_\alpha(\mathbf{x}, t)$ ,  $w = w(\mathbf{x}, t)$  are the basic unknowns. For periodic shells coefficients of lagrangian  $L$  and hence also of (4) are highly oscillating non-continuous functions depending on  $\mathbf{x}$  with a period  $\lambda$ . That is why equations (3) (or their explicit form (4)) cannot be directly applied to investigations of engineering problems. Our aim is to “replace” these equations by equations with constant coefficients depending on the microstructure size. To this end we will apply the *tolerance modeling technique* given by Woźniak in [19]. Next, the obtained model equations will be applied to derive the *frequency equation* being a starting point in the analysis of parametric vibrations and dynamical shell stability. The effect of a cell size on the form of this frequency equation will be discussed.

To make the analysis more clear, in the subsequent section we shall outline the basic concepts and the main assumptions of the *tolerance averaging approach* following book [19].

### III. MODELING APPROACH

Following monograph [19], we outline below the basic concepts and assumptions which will be used in the course of modeling procedure.

#### A. Basic Concepts

The fundamental concepts of the tolerance modeling are those of *tolerance* determined by *tolerance parameter*, *cell distribution*, *tolerance periodic function* and its two special cases: *slowly-varying and highly-oscillating functions*. The tolerance approach is based on the notion of the *averaging of tolerance periodic function*.

The main statement of the modeling procedure is that every measurement as well as numerical calculation can be realized in practice only within a certain accuracy defined by *tolerance parameter*  $\delta$  being a positive constant.

The concept of *cell distribution*  $(\Omega, \Delta)$  assigned to  $\Omega = (0, L_1) \times (0, L_2)$  has been introduced in Section II.

A bounded integrable function  $f(\cdot)$  defined on  $\bar{\Omega} = [0, L_1] \times [0, L_2]$  (which can also depend on  $t$  as a parameter) is called *tolerance periodic* with respect to cell  $\Delta$  and tolerance parameter  $\delta$ , if roughly speaking, its values in an arbitrary cell  $\Delta(\mathbf{x})$  can be approximated, with sufficient accuracy, by the corresponding values of a certain  $\Delta$ -periodic function  $f_x(\mathbf{z})$ ,  $\mathbf{z} \in \Delta(\mathbf{x})$ ,  $\mathbf{x} \in \bar{\Omega}$ . Function  $f_x$  is a  $\Delta$ -periodic approximation of  $f$  in  $\Delta(\mathbf{x})$ . This condition has to be fulfilled by all derivatives of  $f$  up to the  $R$ -th order, i.e. by all its derivatives which occur in the problem under consideration; in the problem analyzed here  $R$  is equal either 1 or 2. In this case we shall write  $f \in TP_\delta^R(\Omega, \Delta)$ . It has to be emphasized that for periodic structures being object of considerations in this paper function  $f_x(\mathbf{z})$ ,  $\mathbf{z} \in \Delta(\mathbf{x})$ ,  $\mathbf{x} \in \bar{\Omega}$  has the same analytical form in every cell  $\Delta(\mathbf{x})$ ,  $\mathbf{x} \in \bar{\Omega}$ . Hence,  $f_x(\cdot)$  is independent of  $\mathbf{x}$ . In the general case, i.e. for tolerance periodic structures (i.e. structures which in small neighborhoods of  $\Delta(\mathbf{x})$  can be approximately regarded as periodic)  $f_x = f_x(\mathbf{x}, \mathbf{z})$ ,  $\mathbf{z} \in \Delta(\mathbf{x})$ ,  $\mathbf{x} \in \bar{\Omega}$ .

Subsequently we will denote by  $\partial \equiv (\partial_1, \partial_2)$  the gradient operator in  $\Omega$  and by  $\partial^k f(\cdot)$ ,  $k = 0, 1, \dots, R$ , the  $k$ -th gradient of function  $f(\cdot)$  defined in  $\Omega$ , where  $\partial^0 f(\cdot) \equiv f(\cdot)$ . Let  $f_x^{(k)}(\mathbf{z})$ ,  $\mathbf{z} \in \Delta(\mathbf{x})$  be a periodic approximation of  $\partial^k f \in TP_\delta^R(\Omega, \Delta)$  in cell  $\Delta(\mathbf{x})$ ,  $\mathbf{x} \in \bar{\Omega}$ ,  $k = 0, 1, \dots, R$ ,  $f_x^0(\cdot) \equiv f_x(\cdot)$ .

A continuous bounded differentiable function  $v(\mathbf{x})$  defined on  $\bar{\Omega} = [0, L_1] \times [0, L_2]$  (which can also depend on  $t$  as a parameter) is called *slowly-varying* with respect to cell  $\Delta$  and tolerance parameter  $\delta$ , if

$$\begin{aligned} v(\mathbf{x}) &\in TP_\delta^R(\Omega, \Delta), \\ v_x^{(k)}(\mathbf{z}) &= \partial^k v(\mathbf{x}), \quad k = 0, 1, \dots, R, \quad \mathbf{z} \in \Delta(\mathbf{x}), \mathbf{x} \in \bar{\Omega}. \end{aligned} \quad (5)$$

It means that periodic approximation  $v_x^{(k)}$  of  $\partial^k v(\cdot)$  in

$\Delta(\mathbf{x})$  is a constant function for every  $\mathbf{x} \in \bar{\Omega}$ . Under the above conditions we shall write  $v \in SV_{\delta}^R(\Omega, \Delta)$ .

Function  $h(\mathbf{x})$  defined in  $\bar{\Omega} = [0, L_1] \times [0, L_2]$  is called the highly-oscillating function with respect to cell  $\Delta$  and tolerance parameter  $\delta$ ,  $h \in HO_{\delta}^R(\Omega, \Delta)$ , if

$$\begin{aligned} h(\mathbf{x}) &\in TP_{\delta}^R(\Omega, \Delta), \quad h_{\mathbf{x}}^k(\mathbf{z}) = \partial^k h_{\mathbf{x}}(\mathbf{z}), \\ (\forall v(\mathbf{x}) \in SV_{\delta}^R(\Omega, \Delta)) (f(\mathbf{x}) = h(\mathbf{x})v(\mathbf{x}) &\in TP_{\delta}^R(\Omega, \Delta)), \\ f_{\mathbf{x}}^{(k)}(\mathbf{z}) &= \partial^k h_{\mathbf{x}}(\mathbf{z})v(\mathbf{x}), \quad k = 0, 1, \dots, R, \quad \mathbf{z} \in \Delta(\mathbf{x}), \mathbf{x} \in \bar{\Omega}. \end{aligned} \quad (6)$$

Condition (6)<sub>4</sub> states that in calculations of  $k$ -th gradient  $f_{\mathbf{x}}^{(k)}(\mathbf{z})$  of periodic approximation  $f_{\mathbf{x}}(\mathbf{z})$  of function  $f = hv$ , terms  $h_{\mathbf{x}}(\mathbf{z})\partial^k v(\mathbf{x})$  and terms including gradients  $\partial^s h_{\mathbf{x}}(\mathbf{z})$ , where  $s < k$ , are negligibly small as compared with terms  $\partial^k h_{\mathbf{x}}(\mathbf{z})v(\mathbf{x})$ ,  $k = 1, 2, \dots, R$ , and hence can be neglected.

In the problem considered here we also deal with the highly-oscillating functions which are  $\Delta$ -periodic, i.e. they are special cases of the highly-oscillating tolerance  $\Delta$ -periodic functions, defined above. Set of the highly-oscillating  $\Delta$ -periodic functions is denoted by  $h \in HO^R(\Omega, \Delta)$ . Let  $h(\mathbf{x})$  be a highly-oscillating  $\lambda$ -periodic function defined in  $\bar{\Omega}$  which is continuous together with its gradients  $\partial^k h$ ,  $k = 1, \dots, R-1$ , and has either continuous or a piecewise continuous bounded gradient  $\partial^R h$ . Periodic function  $h(\cdot)$  will be called the fluctuation shape function, if it depends on  $\lambda$  as a parameter and satisfies conditions (6)<sub>3</sub> and (6)<sub>4</sub>, (in (6)<sub>4</sub>  $\partial^k h_{\mathbf{x}}(\mathbf{z})$  has to be replaced by  $\partial^k h(\mathbf{z})$ ), together with conditions:

$$\begin{aligned} \partial^k h &\in O(\lambda^{R-k}), \quad k = 0, 1, \dots, R, \quad \partial^0 h \equiv h, \\ \int_{\Delta(\mathbf{x})} \partial^k h(\mathbf{z})d\mathbf{z} &= 0, \quad \mathbf{z} \in \Delta(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad k = 1, 2, \dots, R, \\ \int_{\Delta(\mathbf{x})} \mu(\mathbf{z})h(\mathbf{z})d\mathbf{z} &= 0, \quad \mathbf{z} \in \Delta(\mathbf{x}), \end{aligned} \quad (7)$$

where  $\mu$  is a certain positive valued  $\lambda$ -periodic function defined in  $\Omega$ . In stationary problems, condition (7)<sub>3</sub> is replaced by  $\int_{\Delta(\mathbf{x})} h(\mathbf{z})d\mathbf{z} = 0$ .

Let  $f(\cdot) \in TP_{\delta}^R(\Omega, \Delta)$ . By the averaging of tolerance periodic function  $f \equiv \partial^0 f$  and its derivatives  $\partial^k f$ ,  $k = 1, 2, \dots, R$ , we shall mean function  $\langle \partial^k f \rangle(\mathbf{x})$ ,  $\mathbf{x} \in \bar{\Omega}$ , defined by

$$\langle \partial^k f \rangle(\mathbf{x}) \equiv \frac{1}{|\Delta|} \int_{\Delta(\mathbf{x})} f_{\mathbf{x}}^{(k)}(\mathbf{x}, \mathbf{z})d\mathbf{z}, \quad k = 0, 1, \dots, R, \quad \mathbf{z} \in \Delta(\mathbf{x}), \mathbf{x} \in \bar{\Omega}. \quad (8)$$

For periodic media periodic approximation  $f_{\mathbf{x}}^{(k)}$  of  $\partial^k f$  in  $\Delta(\mathbf{x})$  is independent of argument  $\mathbf{x}$  and  $\langle \partial^k f \rangle$  is constant. For tolerance periodic media  $\langle \partial^k f \rangle$  is a smooth slowly-varying function of  $\mathbf{x}$ .

Let  $f(\mathbf{x}, \partial^k g(\mathbf{x}))$ ,  $k = 0, 1, \dots, R$  be a composite function defined in  $\Omega$  such that  $f(\mathbf{x}, \partial^k g(\mathbf{x})) \in HO_{\delta}^0(\Omega, \Delta)$ ,  $g(\mathbf{x}) \in TP_{\delta}^R(\Omega, \Delta)$ . The tolerance averaging of this function is defined by

$$\langle f(\cdot, \partial^k g(\cdot)) \rangle(\mathbf{x}) \equiv \frac{1}{|\Delta|} \int_{\Delta(\mathbf{x})} f_{\mathbf{x}}(\mathbf{x}, \mathbf{z}, g_{\mathbf{x}}^{(k)}(\mathbf{x}, \mathbf{z}))d\mathbf{z}, \quad \mathbf{z} \in \Delta(\mathbf{x}), \mathbf{x} \in \bar{\Omega}. \quad (9)$$

For periodically micro-heterogeneous shells under consideration function  $f_{\mathbf{x}}$  is independent of  $\mathbf{x}$  and  $\langle f(\cdot, \partial^k g(\cdot)) \rangle$  is constant. It can be seen, that definition (8) is a special case of definition (9).

In the tolerance modeling of dynamic problems for periodic shells we also deal with mean (constant) value  $\langle f \rangle$  of  $\Delta$ -periodic integrable function  $f(\cdot)$  defined by

$$\langle f(\mathbf{z}) \rangle \equiv \frac{1}{|\Delta|} \int_{\Delta(\mathbf{x})} f(\mathbf{z})d\mathbf{z}, \quad \mathbf{z} \in \Delta(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega}. \quad (10)$$

More general definitions of these concepts are given in [19] and also in [18], [20].

#### B. Modeling Assumption

The fundamental assumption imposed on the lagrangian under consideration in the framework of the tolerance averaging approach is called the micro-macro decomposition. It states that the displacement fields occurring in this lagrangian have to be the tolerance periodic functions in  $\mathbf{x}$ . Hence, they can be decomposed into unknown averaged displacements being slowly-varying functions in  $\mathbf{x}$  and fluctuations represented by known highly-oscillating functions called fluctuation shape functions and by unknown fluctuation amplitudes being slowly-varying in  $\mathbf{x}$ .

### IV. TOLERANCE MODEL EQUATIONS

#### A. Modeling Procedure

The tolerance modeling procedure for Euler-Lagrange equations (3) is realized in two steps.

The first step is the tolerance averaging of action functional (1). To this end let us introduce two systems of linear independent highly-oscillating functions, called the fluctuation shape functions, being  $\lambda$ -periodic in  $\mathbf{x} \equiv (x^1, x^2)$ :  $h^A(\mathbf{x}) \in HO^1(\Omega, \Delta)$ ,  $A = 1, \dots, n$  and  $g^A(\mathbf{x}) \in HO^2(\Omega, \Delta)$ ,  $A = 1, \dots, N$ . These functions are assumed to be known in every problem under consideration. Agree with (7), they have to satisfy conditions:

$h^a \in O(\lambda)$ ,  $\lambda \partial_\alpha h^a \in O(\lambda)$ ,  
 $g^A \in O(\lambda^2)$ ,  $\lambda \partial_\alpha g^A \in O(\lambda^2)$ ,  $\lambda^2 \partial_{\alpha\beta} g^A \in O(\lambda^2)$ ,  
 $\langle \mu h^a \rangle = \langle \mu g^A \rangle = 0$  and  $\langle \mu h^a h^b \rangle = \langle \mu g^A g^B \rangle = 0$  for  
 $a \neq b$ ,  $A \neq B$ , where  $\mu(\cdot)$  is the shell mass density being a  $\lambda$ -  
 periodic function with respect to  $\mathbf{x}$ .

In dynamic problems, functions  $h^a(\mathbf{x})$ ,  $g^A(\mathbf{x})$  represent ei-  
 ther the principal modes of free periodic vibrations of the cell  
 $\Delta(\mathbf{x})$  or physically reasonable approximation of these modes.  
 Hence, they can be obtained as solutions to certain periodic  
 eigenvalue problems describing free periodic vibrations of the  
 cell, cf. [13]. In stationary problems, these functions can be  
 treated as the shape functions resulting from the finite element  
 periodic discretization of the cell.

Now, we have to introduce the micro-macro decomposition  
 of displacements  $u_\alpha(\mathbf{x}, t) \in TP_\delta^1(\Omega, \Delta)$ ,  $w(\mathbf{x}, t) \in TP_\delta^2(\Omega, \Delta)$ ,  
 $\mathbf{x} \in \Omega$ ,  $t \in (t_0, t_1)$ , which in the problem under consideration  
 is assumed in the form

$$\begin{aligned} u_\alpha(\mathbf{x}, t) &= u_{h\alpha}(\mathbf{x}, t) = u_\alpha^0(\mathbf{x}, t) + h^a(\mathbf{x}) U_\alpha^a(\mathbf{x}, t), \quad a = 1, \dots, n, \\ w(\mathbf{x}, t) &= w_g(\mathbf{x}, t) = w^0(\mathbf{x}, t) + g^A(\mathbf{x}) W^A(\mathbf{x}, t), \quad A = 1, \dots, N, \end{aligned} \quad (11)$$

where

$$\begin{aligned} u_\alpha^0(\mathbf{x}, t), U_\alpha^a(\mathbf{x}, t) &\in SV_\delta^1(\Omega, \Delta) \subset TP_\delta^1(\Omega, \Delta), \\ w^0(\mathbf{x}, t), W^A(\mathbf{x}, t) &\in SV_\delta^2(\Omega, \Delta) \subset TP_\delta^2(\Omega, \Delta), \end{aligned} \quad (12)$$

and where summation convention over  $a$  and  $A$  holds. Func-  
 tions  $u_\alpha^0, w^0$ , called *averaged variables*, and functions  
 $U_\alpha^a, W^A$ , called *fluctuation amplitudes*, are the *new unknowns*  
 being slowly-varying in  $\mathbf{x}$ .

Since lagrangian (2) is a highly-oscillating function with re-  
 spect to  $\mathbf{x}$ ,  $L \in HO_\delta^0(\Omega, \Delta)$ , then there exists its periodic ap-  
 proximation in every  $\Delta(\mathbf{x})$ . The periodic approximation of  $L$   
 is obtained by replacing displacements  $u_\alpha, w$  and their deriva-  
 tives occurring in (2) by periodic approximations of these  
 functions. These approximations are calculated applying mi-  
 cro-macro decomposition (11) and bearing in mind properties  
 of the slowly-varying and highly-oscillating functions given by  
 means of (5), (6). Then, using tolerance averaging formula (9)  
 we arrive at function  $\langle L_{hg} \rangle$  being *the tolerance averaging of*  
*lagrangian (2) in  $\Delta(\mathbf{x})$  under micro-macro decomposition*  
 (11). The obtained result has the form

$$\begin{aligned} \langle L_{hg} \rangle &= \langle (\partial_\beta u_\alpha^0, u_\alpha^0, u_\alpha^0, U_\alpha^a, U_\alpha^a, \partial_{\alpha\beta} w^0, \partial_\alpha w^0, w^0, W^A, \dot{w}^0, \dot{W}^A) \\ &= \frac{1}{2} [ \langle D^{\alpha\beta\gamma\delta} \partial_\beta u_\alpha^0 \partial_\delta u_\gamma^0 + 2 \langle D^{\alpha\beta\gamma\delta} \partial_\delta h^a \partial_\beta u_\alpha^0 U_\gamma^a + \\ &+ \langle D^{\alpha\beta\gamma\delta} \partial_\beta h^a \partial_\delta h^b \rangle U_\gamma^a U_\alpha^b + \\ &+ 2r^{-1} (\langle D^{\alpha\beta 11} \rangle \partial_\beta u_\alpha^0 w^0 + \langle D^{\alpha\beta 11} \partial_\beta h^a \rangle w^0 U_\alpha^a + \\ &+ \langle D^{\alpha\beta 11} g^A \rangle \partial_\beta u_\alpha^0 W^A + \langle D^{\alpha\beta 11} \partial_\beta h^a g^A \rangle U_\alpha^a W^A) + \\ &+ r^{-2} (\langle D^{1111} \rangle w^0 w^0 + 2 \langle D^{1111} g^A \rangle w^0 W^A + \\ &+ \langle D^{1111} g^A g^B \rangle W^A W^B) + \langle B^{\alpha\beta\gamma\delta} \rangle \partial_{\alpha\beta} w^0 \partial_{\gamma\delta} w^0 + \\ &+ 2 \langle B^{\alpha\beta\gamma\delta} \partial_{\gamma\delta} g^A \rangle \partial_{\alpha\beta} w^0 W^A + \\ &+ \langle B^{\alpha\beta\gamma\delta} \partial_{\alpha\beta} g^A \partial_{\gamma\delta} g^B \rangle W^A W^B + \\ &+ \bar{N}^{\alpha\beta} \partial_\alpha w^0 \partial_\beta w^0 - \bar{N}^{\alpha\beta} \langle \partial_\alpha g^A \partial_\beta g^B \rangle W^A W^B + \\ &- \langle \mu \rangle a^{\alpha\beta} u_\alpha^0 \dot{u}_\beta^0 - \langle \mu \rangle (\dot{w}^0)^2 + \\ &- \langle \mu h^a h^b \rangle a^{\alpha\beta} \dot{U}_\alpha^a \dot{U}_\beta^b - \langle \mu g^A g^B \rangle \dot{W}^A \dot{W}^B ] + \\ &- \langle f^\alpha \rangle u_\alpha^0 - \langle f^\alpha h^a \rangle U_\alpha^a - \langle f \rangle w^0 - \langle f g^A \rangle W^A. \end{aligned} \quad (13)$$

Due to periodic structure of the shell, averages  $\langle \cdot \rangle$  on the  
 right-hand side of (13) are constant and calculated by means of  
 (10). The underlined terms in (13) depend on the microstruc-  
 ture length parameter  $\lambda$ . Functional

$$A_{hg}(u_\alpha^0, U_\alpha^a, w^0, W^A) = \int_0^{L_1} \int_0^{L_2} \int_0^{t_1} \langle L_{hg} \rangle dt dx^2 dx^1, \quad (14)$$

where  $\langle L_{hg} \rangle$  is given by (13), is called *the tolerance averag-*  
*ing of starting functional (1) under decomposition (11)*.

The second step in the tolerance modeling of Euler-  
 Lagrange equations (3) is to apply the principle of stationary  
 action to  $A_{hg}$  given above. The principle of stationary action  
 applied to  $A_{hg}$  leads to the system of Euler-Lagrange equa-  
 tions for  $u_\alpha^0, w^0, U_\alpha^a, W^A$  as the basic unknowns. The explicit  
 form of these equations will be given in the next subsection.

### B. Tolerance Model Equations

In the previous subsection, applying the tolerance averaging  
 of the starting lagrangian (2) and then using the principle of  
 stationary action to tolerance averaged action functional (14)  
 defined by means of averaged lagrangian (13), we have arrived  
 at the Euler-Lagrange equations, which explicit form can be  
 written as constitutive equations

$$\begin{aligned}
 N^{\alpha\beta} &= \langle D^{\alpha\beta\gamma\delta} \rangle \partial_\delta u_\gamma^0 + \langle D^{\alpha\beta\gamma\delta} \partial_\delta h^b \rangle U_\gamma^b + \\
 &\quad + r^{-1} \langle D^{\alpha\beta 11} \rangle w^0 + \langle D^{\alpha\beta 11} g^B \rangle W^B, \\
 M^{\alpha\beta} &= \langle B^{\alpha\beta\gamma\delta} \rangle \partial_\delta w^0 + \langle B^{\alpha\beta\gamma\delta} \partial_\delta g^B \rangle W^B, \\
 h^{ab} &= \langle \partial_\alpha h^a D^{\alpha\beta\gamma\delta} \rangle \partial_\delta u_\gamma^0 + \langle \partial_\alpha h^a D^{\alpha\beta\gamma\delta} \partial_\delta h^b \rangle U_\gamma^b + \\
 &\quad + r^{-1} \langle \partial_\alpha h^a D^{\alpha\beta 11} \rangle w^0 + \langle \partial_\alpha h^a D^{\alpha\beta 11} g^B \rangle W^B, \\
 G^A &= r^{-1} \langle g^A D^{11\gamma\delta} \rangle \partial_\delta u_\gamma^0 + r^{-2} \langle g^A D^{1111} \rangle w^0 + \\
 &\quad + \langle \partial_\gamma g^A B^{\alpha\beta\gamma\delta} \rangle \partial_\alpha w^0 + r^{-1} \langle g^A D^{11\beta\gamma} \partial_\beta h^b \rangle U_\gamma^b + \\
 &\quad + [\langle \partial_\alpha g^A B^{\alpha\beta\gamma\delta} \partial_\gamma g^B \rangle + r^{-2} \langle g^A D^{1111} g^B \rangle] W^B,
 \end{aligned} \tag{15}$$

and the dynamic equilibrium equations

$$\begin{aligned}
 \partial_\alpha N^{\alpha\beta} - \langle \mu \rangle a^{\alpha\beta} \ddot{u}_\alpha^0 + \langle f^\beta \rangle &= 0, \\
 \partial_\alpha M^{\alpha\beta} + r^{-1} N^{11} + \bar{N}^{\alpha\beta} \partial_\alpha w^0 + \langle \mu \rangle \dot{w}^0 - \langle f \rangle &= 0, \\
 \langle \mu h^a h^b \rangle a^{\alpha\beta} \ddot{U}_\alpha^a + h^{ab} \langle f^\beta h^a \rangle &= 0, \quad a, b = 1, 2, \dots, n, \\
 \langle \mu g^A g^B \rangle \dot{W}^B + G^A - \bar{N}^{\alpha\beta} \langle \partial_\alpha g^A \partial_\beta g^B \rangle W^B + \\
 - \langle f g^A \rangle &= 0, \quad A, B = 1, 2, \dots, N.
 \end{aligned} \tag{16}$$

Equations (15) and (16) together with *micro-macro decomposition* (11) and *physical reliability conditions* (12) constitute the tolerance model for analysis of parametric vibrations and selected dynamic stability problems for biperiodically stiffened shells under consideration.

### C. Discussion of Results

The characteristic features of the derived model are:

a) In contrast to starting equations (4) with discontinuous, highly oscillating and periodic coefficients, the tolerance model equations (15) and (16) proposed here have constant coefficients. Moreover, some of them depend on microstructure length parameter  $\lambda$  (underlined terms). Hence, the tolerance model makes it possible to describe the effect of length scale on the global shell behavior.

b) The number and form of boundary conditions for unknown macro-displacements  $u_\alpha^0, w^0$  are the same as in the classical shell theory governed by equations (4). There are no extra boundary conditions for unknown fluctuation amplitudes  $U_\alpha^a, W^A$  which are governed by the ordinary differential equations involving only time derivatives.

c) Decomposition (11) and hence also resulting tolerance equations (15) and (16) are uniquely determined by the postulated *a priori*  $\Delta$ -periodic fluctuations shape functions  $h^A(\mathbf{x}) \in HO^1(\Omega, \Delta)$ ,  $a = 1, \dots, n$  and  $g^A(\mathbf{x}) \in HO^2(\Omega, \Delta)$ ,  $A = 1, \dots, N$ , which can be obtained as solutions to certain periodic eigenvalue problems describing free periodic vibrations of the cell, cf. [13].

d) Solutions to selected initial/boundary value problems formulated in the framework of the tolerance model have a physical sense only if conditions (12) hold for the pertinent

tolerance parameter  $\delta$ . These conditions can be also used for the *a posteriori* evaluation of tolerance parameter  $\delta$  and hence, for the verification of the physical reliability of the obtained solutions.

e) It is easy to prove that for a homogeneous shell and homogeneous initial conditions for fluctuation amplitudes, the resulting equations generated by tolerance averaged Lagrange function (13) reduce to the starting equations (4) generated by Lagrange function (2).

## V. ASYMPTOTIC MODEL EQUATIONS

The asymptotic model equations can be derived directly from the tolerance model equations (15), (16) by neglecting the underlined terms which depend on the microstructure length parameter  $\lambda$ . Hence, after calculating the fluctuation amplitudes by means of

$$\begin{aligned}
 U_\gamma^b &= -(G^{-1})_{\gamma\eta}^{bc} [\langle \partial_\alpha h^c D^{\alpha\eta\mu\delta} \rangle \partial_\delta u_\mu^0 + \rho^{-1} \langle \partial_\alpha h^c D^{\alpha\eta 11} \rangle w^0], \\
 w^A &= -(S^{-1})^{AB} \langle \partial_\alpha g^B B^{\alpha\beta\gamma\delta} \rangle \partial_\gamma w^0,
 \end{aligned} \tag{17}$$

where

$$G_{\beta\gamma}^{ab} = \langle \partial_\alpha h^a D^{\alpha\beta\gamma\delta} \partial_\delta h^b \rangle, \quad S^{AB} = \langle \partial_\alpha g^A B^{\alpha\beta\gamma\delta} \partial_\gamma g^B \rangle$$

and denoting

$$\begin{aligned}
 D_h^{\alpha\beta\gamma\delta} &\equiv \langle D^{\alpha\beta\gamma\delta} \rangle - \langle D^{\alpha\beta\eta\chi} \partial_\chi h^a \rangle (G^{-1})_{\eta\zeta}^{ab} \langle \partial_\chi h^b D^{\zeta\gamma\delta} \rangle, \\
 B_h^{\alpha\beta\gamma\delta} &\equiv \langle B^{\alpha\beta\gamma\delta} \rangle - \langle B^{\alpha\beta\mu\zeta} \partial_\mu g^A \rangle (S^{-1})^{AB} \langle \partial_\mu g^B B^{\mu\zeta\gamma\delta} \rangle,
 \end{aligned}$$

we arrive at the asymptotic model equations for unknowns  $u_\alpha^0(\mathbf{x}, t), w^0(\mathbf{x}, t)$

$$\begin{aligned}
 D_h^{\alpha\beta\gamma\delta} \partial_\beta u_\gamma^0 + r^{-1} D_h^{\alpha\beta 11} \partial_\beta w^0 - \langle \mu \rangle a^{\alpha\beta} \ddot{u}_\alpha^0 + \langle f^\alpha \rangle &= 0, \\
 B_h^{\alpha\beta\gamma\delta} \partial_\alpha w^0 + r^{-1} D_h^{11\gamma\delta} \partial_\delta u_\gamma^0 + r^{-2} D_h^{1111} w^0 + \\
 + \bar{N}^{\alpha\beta} \partial_\alpha w^0 + \langle \mu \rangle \dot{w}^0 - \langle f \rangle &= 0.
 \end{aligned} \tag{18}$$

Equations (18) have to be considered together with decomposition of  $u_\alpha(\mathbf{x}, t), w(\mathbf{x}, t)$  in  $\Omega \times (t_0, t_1)$  given by (11) with  $U_\alpha^a, W^A$  calculated by means of (17). Coefficients in (18) are constant in contrast to coefficients in (4) which are discontinuous, highly oscillating and periodic. Notice that equations (18) are not able to describe the length-scale effect on the overall shell dynamics and stability being independent of the microstructure size.

It has to be emphasized that equations (18) coincide with the consistent asymptotic model equations derived in [16] from Euler-Lagrange equations (3) by applying a new approach to the asymptotic modeling of problems for micro-heterogeneous media. This approach is proposed in [19], [20].

The subsequent analysis dealing with a certain parametric vibration problem will be based on tolerance model equations (15), (16) and asymptotic model equations (18).

## VI. APPLICATIONS

Now, the tolerance model equations (15), (16) will be applied to derivation of *frequency equation* being a starting point in the analysis of parametric vibrations and dynamic stability of periodically stiffened shells under consideration. In order to evaluate the effect of a cell size on this equation, the results obtained from the tolerance model will be compared with those derived from an asymptotic model (18).

### A. Formulation of the Problem

Let the shell, considered here, be closed and circular. It means that  $L_1 = 2\pi r$ , where  $r$  is the midsurface curvature radius. The shell is reinforced by  $n$  families of stiffeners, which are periodically and densely distributed in circumferential and axial directions; an example of such a shell is shown in Fig. 1. The stiffeners have constant cross-sections. Moreover, the gravity centers of the stiffener cross-sections are situated on the shell midsurface. It is assumed that both the shell and ribs are made of homogeneous isotropic materials.

We define  $\lambda \equiv \lambda_1 = \lambda_2$  as the period length of the stiffened shell structure. The period  $\lambda$  represents the distance between axes of two neighboring ribs belonging to the same family, cf. Fig. 1. It assumed that the following conditions hold:  $\lambda/d_{\max} \gg 1$ ,  $\lambda/r \ll 1$  and  $\lambda/L_1 \ll 1$ , where  $d_{\max}$  is the maximum height of stiffeners. We also assume that  $L_2 > L_1$  and hence  $\lambda/L_2 \ll 1$ . We recall that inside the cell  $\Delta \equiv [-\lambda/2, \lambda/2] \times [-\lambda/2, \lambda/2]$ , the geometrical, elastic and inertial properties of the stiffened shell are described by symmetric (i.e. even) functions of arguments  $\mathbf{z} \equiv (z^1, z^2) \in [-\lambda_1/2, \lambda_1/2] \times [-\lambda_2/2, \lambda_2/2]$ .

The stiffened shell under consideration will be treated as a shell with  $\lambda$ -periodically varying piece-wise constant properties: Young's modulus  $E(\mathbf{x})$ , mass density  $\mu(\mathbf{x})$  and thickness  $\tilde{d}(\mathbf{x})$ ,  $\mathbf{x} \in [0, L_1] \times [0, L_2]$ . Only Poisson's ratio  $\nu$  is constant. The stiffness tensors  $D^{\alpha\beta\gamma\delta}(\mathbf{x}), B^{\alpha\beta\gamma\delta}(\mathbf{x})$  are defined by:  $D^{\alpha\beta\gamma\delta}(\mathbf{x}) = D(\mathbf{x})H^{\alpha\beta\gamma\delta}$ ,  $B^{\alpha\beta\gamma\delta}(\mathbf{x}) = B(\mathbf{x})H^{\alpha\beta\gamma\delta}$ , where  $D(\mathbf{x}) = E(\mathbf{x})\tilde{d}(\mathbf{x})/(1-\nu^2)$ ,  $B(\mathbf{x}) = E(\mathbf{x})(\tilde{d}(\mathbf{x}))^3/(12(1-\nu^2))$ ,  $H^{\alpha\beta\gamma\delta} = 0.5[a^{\alpha\gamma}a^{\beta\delta} + a^{\alpha\delta}a^{\beta\gamma} + \nu(\epsilon^{\alpha\gamma}\epsilon^{\beta\delta} + \epsilon^{\alpha\delta}\epsilon^{\beta\gamma})]$ ;  $a^{\alpha\beta}$ ,  $\epsilon^{\alpha\gamma}$  are contravariant first midsurface tensor and Ricci bivector, respectively, related to orthonormal parametrization  $(x^1, x^2)$  introduced on  $M$ , cf. Section II. After some manipulations we obtain the following nonzero components of tensor  $H^{\alpha\beta\gamma\delta}$ :

$$H^{1111} = H^{2222} = 1, H^{1122} = H^{2211} = \nu, \\ H^{1212} = H^{1221} = H^{2121} = H^{2112} = (1-\nu)/2.$$

In every special problem under consideration, the forms of functions  $E(\mathbf{z})$ ,  $\mu(\mathbf{z})$ ,  $\tilde{d}(\mathbf{z})$ ,  $\mathbf{z} \in \Delta(\mathbf{x})$ , inside the cell have to be defined. Then, the averaged values of these functions have to be calculated by means of (10).

We assume that the shell is simply supported on edges  $x^2 = 0, x^2 = L_2$ , cf. [21].

In order to analyze parametric vibrations (or dynamic stability) the external forces will be neglected in the tolerance and asymptotic model equations. We will also neglect the forces of inertia in direction tangential to the shell midsurface.

We assume that the shell is uniformly compressed in axial direction by time-dependent forces  $\bar{N}(t) \equiv \bar{N}^{22}(t)$ ; hence  $\bar{N}^{12} = \bar{N}^{21} = \bar{N}^{11} = 0$ .

Let the investigated problem be rotationally symmetric with a period  $\lambda/r$ ; hence  $u_1^0 = U_1^a = 0$  and the remaining basic unknowns are only the functions of  $x^2$ -midsurface parameter. It has to be emphasized that the total displacement  $u_2, w$  in the micro-macro decomposition (11) are functions of both arguments, because the fluctuation shape functions depend on  $x^1$  and  $x^2$ .

For the sake of simplicity, we shall confine ourselves to the simplest form of the tolerance model in which  $a = n = A = N = 1$ . Hence, we introduce only two  $\lambda$ -periodic fluctuation shape functions  $h(\mathbf{z}) \equiv h^1(\mathbf{z}) \in HO^1(\Omega, \Delta)$  and  $g(\mathbf{z}) \equiv g^1(\mathbf{z}) \in HO^2(\Omega, \Delta)$ ,  $\mathbf{z} \equiv (z^1, z^2) \in \Delta(\mathbf{x})$ , which have to satisfy condition  $\langle \mu h \rangle = \langle \mu g \rangle = 0$ . Bearing in mind the symmetry of the cell geometry and symmetric distribution of the material properties inside the cell we assume that  $h(\mathbf{z})$  and  $g(\mathbf{z})$  are respectively *odd* and *even* functions of  $\mathbf{z}$ ; i.e.  $h(\mathbf{z})$  and  $g(\mathbf{z})$  are respectively *antisymmetric* and *symmetric* functions on the cell. *It assumed that these functions are known in the problem under consideration.* They can be obtained as solutions to certain periodic eigenvalue problems describing free periodic vibrations of the cell, cf. [13].

In the sequel denotations  $U_2(x^2, t) \equiv U_2^1(x^2, t)$ ,  $W(x^2, t) \equiv W^1(x^2, t)$  will be used.

Bearing in mind the conditions and denotations given above we will derive below the formula for *frequency equation* being a starting point in the analysis of parametric vibrations or dynamic stability of periodically stiffened shells under consideration. The effect of the microstructure size on this equation will be analyzed by using both the tolerance model governed by equations (15), (16) and the asymptotic model represented by equations (18).

### B. Analysis in the Framework of the Tolerance Model

Now, the system of equations (16) of the tolerance model is separated into independent equation for  $U_2(x^2, t)$ :

$\langle D^{\alpha 22\alpha}(\partial_\alpha h)^2 \rangle U_2 = 0$ , which yields  $U_2 = 0$ , and the system of three equations for  $u_2^0(x^2, t)$ ,  $w^0(x^2, t)$  and  $W(x^2, t)$ .

Under extra denotations

$$\begin{aligned} \tilde{D} &\equiv \langle D^{2222} \rangle, \quad \tilde{B} \equiv \langle B^{2222} \rangle, \quad \bar{D} \equiv \langle D^{1122} \rangle, \\ K^{22} &\equiv \langle B^{22\alpha\alpha} \partial_{\alpha\alpha} g \rangle, \quad S \equiv \langle B^{\alpha\beta\gamma\delta} \partial_{\alpha\beta} g \partial_{\gamma\delta} g \rangle, \quad E_{22} \equiv \lambda^{-2} \langle (\partial_2 g)^2 \rangle, \\ \tilde{\mu} &\equiv \langle \mu \rangle, \quad \bar{\mu} \equiv \lambda^{-4} \langle \mu(g)^2 \rangle, \end{aligned}$$

this system takes the following form

$$\begin{aligned} \tilde{D} \partial_2 u_2^0 + \bar{D} r^{-1} \partial_2 w^0 &= 0, \\ \bar{D} r^{-1} \partial_2 u_2^0 + \tilde{B} \partial_2 u_2^0 + \tilde{D} r^{-2} w^0 + \bar{N} \partial_2 w^0 + \\ + \tilde{\mu} \ddot{w}^0 + K^{22} \partial_2 W &= 0, \\ K^{22} \partial_2 w^0 + (S - \lambda^2 \bar{N}(t) E_{22}) W + \lambda^4 \bar{\mu} \ddot{W} &= 0, \end{aligned} \quad (19)$$

where some terms depend explicitly on microstructure length parameter  $\lambda$ . All coefficients of (19) are constant.

Separating variables  $x^2$  and  $t$ , the solutions to (19) satisfying boundary conditions for the simply supported shell on edges  $x^2 = 0$ ,  $x^2 = L_2$  can be assumed in the form

$$\begin{aligned} u_2^0(x^2, t) &= \sum_{m=1}^{\infty} T_m^u(t) \cos(\alpha_m x^2), \\ w^0(x^2, t) &= \sum_{m=1}^{\infty} T_m^w(t) \sin(\alpha_m x^2), \\ W(x^2, t) &= \sum_{m=1}^{\infty} T_m^W(t) \sin(\alpha_m x^2), \quad \alpha_m = m\pi / L_2. \end{aligned} \quad (20)$$

Substituting the right-hand sides of (20) into (19), taking into account that  $\lambda / L_2 \ll 1$ , i.e.  $\alpha_m \lambda \ll 1$ , and also  $\tilde{d}_{\max} / \lambda \ll 1$ ,  $\lambda / r \ll 1$  and hence neglecting some terms as small compared to 1, we obtain the equation for function  $T_m(t)$

$$\begin{aligned} \lambda^4 \tilde{\mu} \ddot{\ddot{T}}_m(t) + \\ + \lambda^4 \bar{\mu} \frac{d^2}{dt^2} \left[ \left( \tilde{B}(\alpha_m)^4 + \frac{\tilde{D}}{\rho^2} - \frac{\bar{D}^2}{\rho^2 \bar{D}} - \bar{N}(t)(\alpha_m)^2 \right) T_m(t) \right] + \\ + (S - \lambda^2 \bar{N}(t) E_{22}) \tilde{\mu} \frac{d^2}{dt^2} T_m(t) + \\ + \left[ (S - \lambda^2 \bar{N}(t) E_{22}) \left( \tilde{B}(\alpha_m)^4 + \frac{\tilde{D}}{\rho^2} - \frac{\bar{D}^2}{\rho^2 \bar{D}} - \bar{N}(t)(\alpha_m)^2 \right) + \right. \\ \left. - (K^{22})^2 (\alpha_m)^4 \right] T_m(t) = 0. \end{aligned} \quad (21)$$

We assume that compressive axial forces  $\bar{N}(t) \equiv \bar{N}^{22}(t)$  are given by  $\bar{N}(t) = \bar{N}_b \cos(pt)$ , where  $p$  is the oscillation frequency of these forces and  $\bar{N}_b$  is constant.

Let us denote  $\chi_m \equiv \tilde{B} + r^{-2}(\alpha_m)^{-4}(\tilde{D} - \bar{D}^2(\tilde{D})^{-1})$  and then introduce the following formulae

$$\begin{aligned} (\omega_m)^2 &\equiv (\alpha_m)^4 (\tilde{\mu})^{-1} [\chi_m - (K^{22})^2 S^{-1}], \quad \omega_*^2 \equiv S(\lambda^4 \bar{\mu})^{-1}, \\ \bar{N}_{cr,m} &\equiv (\alpha_m)^2 [\chi_m - (K^{22})^2 S^{-1}], \\ \bar{N}_{*cr,m} &\equiv S(\lambda^2 E_{22})^{-1} + (\alpha_m)^2 (K^{22})^2 S^{-1}, \quad \tilde{N}_{*cr} \equiv S(\lambda^2 E_{22})^{-1}, \\ 2\tilde{\mu}_m &\equiv \bar{N}_b (\bar{N}_{cr,m})^{-1}, \end{aligned} \quad (22)$$

where  $\omega_m$  and  $\omega_*$  are the  $m$ -th lower and new additional higher free vibration frequencies, respectively,  $\bar{N}_{cr,m}$  and  $\bar{N}_{*cr,m}$  are the  $m$ -th lower and the  $m$ -th new additional higher static critical forces, respectively,  $\tilde{N}_{*cr}$  is the approximation of  $\bar{N}_{*cr,m}$  and  $\tilde{\mu}_m$  is the modulation factor.

Using formulae (22), frequency equation (21) can be transformed into

$$\begin{aligned} \frac{d^4 T_m}{dt^4} + \omega_*^2 \left[ 1 - \frac{\bar{N}(t)}{\tilde{N}_{*cr}} \right] \frac{d^2 T_m}{dt^2} + \\ + \omega_*^2 (\omega_m)^2 \left[ 1 - \frac{\bar{N}(t)}{\bar{N}_{cr,m}} \right] \left[ 1 - \frac{\bar{N}(t)}{\bar{N}_{*cr,m}} \right] \frac{\bar{N}_{*cr,m}}{\tilde{N}_{*cr}} T_m = 0. \end{aligned}$$

Moreover, because in the most cases the following conditions hold:  $\bar{N} / \tilde{N}_{*cr} \ll 1$  and  $\bar{N} / \bar{N}_{*cr,m} \ll 1$ , then we finally obtain

$$\frac{d^4 T_m}{dt^4} + \omega_*^2 \frac{d^2 T_m}{dt^2} + \omega_*^2 \frac{\bar{N}_{*cr,m}}{\tilde{N}_{*cr}} \omega_m^2 [1 - 2\tilde{\mu}_m \cos(pt)] T_m = 0. \quad (23)$$

The above equation is a starting point of the analysis of parametric vibrations and dynamic stability of the periodic shells under consideration in the framework of the non-asymptotic tolerance model. It is worth noting that equation (23) is more general than the corresponding one proposed and discussed by Tomczyk in [13]. In contrast to frequency equation derived in [13], we obtain here not only the additional higher free vibration frequency  $\omega_*$  (formula (22)<sub>2</sub>), but also the additional higher critical force  $\bar{N}_{*cr,m}$  (formula (22)<sub>4</sub>).

It is easy to see, that some parameters in (23) depend on the period length  $\lambda$ . Hence, this equation makes it possible to investigate the length-scale effect on the parametric vibrations and dynamical stability of periodic shells. It must be emphasized that the obtained fourth order ordinary differential equation (23) is a certain generalization of the known Mathieu's equation, cf. [21]. It takes the form of the Mathieu's equation provided that in (23) the length-scale effect is neglected.

Contrary to classical Mathieu's equation being the second order ordinary differential equation for the unknown function



of time coordinate, for equation (23) additional initial conditions posed on higher order derivatives of function  $T$ , i.e. on  $\ddot{T}, \ddot{\dot{T}}$ , have to be formulated. Function  $T$  and its derivatives  $\dot{T}, \ddot{T}, \ddot{\dot{T}}$  can be treated as the macro-deflection, velocity, acceleration and higher-order acceleration, respectively.

In order to evaluate the obtained results let us analyze this same problem in the framework of a model without the length-scale effect, represented by (18).

#### C. Analysis in the Framework of the Asymptotic Model

Under assumptions introduced in Subsection VI.A, equations (18) yield

$$\begin{aligned} \tilde{D}\partial_{22}u_2^0 + \bar{D}r^{-1}\partial_2w^0 &= 0, \\ \bar{D}r^{-1}\partial_2u_2^0 + \tilde{B}\partial_{2222}w^0 + \tilde{D}r^{-2}w^0 + \bar{N}\partial_{22}w^0 + \tilde{\mu}\ddot{w}^0 + \\ + K^{22}\partial_{22}W &= 0, \\ K^{22}\partial_{22}w^0 + S W &= 0. \end{aligned} \quad (24)$$

The model obtained above is not able to describe the length-scale effect on the overall shell stability being independent of the period length  $\lambda$ .

The solutions to equations (24) can be assumed in the form (20). Substituting (20) into (24) and taking into account formulae (22) we arrive at the following frequency equation

$$\frac{d^2T_m}{dt^2} + (\omega_m)^2[1 - 2\tilde{\mu}_m \cos(pt)]T_m = 0. \quad (25)$$

All parameters of the above equation are independent of the cell size  $\lambda$ . In the framework of the asymptotic model it is not possible to determine the additional higher free vibration frequency and the additional higher critical forces, caused by the periodic structure of the shell. It is easy to see that (25) has a form of the known Mathieu's equation, which describes dynamic stability and parametric vibrations of different structures, cf. [21].

#### D. Comparison of Results and Conclusions

Summarizing the results obtained in this section the following conclusions can be formulated:

a) Contrary to asymptotic models commonly used for investigations of the shell parametric vibrations or of the shell stability, the non-asymptotic tolerance model derived here describes the effect of the period lengths (the length-scale effect) on the overall shell behavior.

b) Taking into account the length-scale effect, we arrive at the fourth-order ordinary differential frequency equation for the unknown function of time coordinate, cf. (23), which can be treated as a certain generalization of the known Mathieu's equation being the second-order ordinary differential equation, cf. [21]. Neglecting in (23) the terms with microstructure length parameter  $\lambda$  we obtain the "classical" Mathieu's equation. On the contrary, within the asymptotic model the known Mathieu's equation (25) is obtained.

c) In the framework of tolerance model governed by equations (19), the fundamental lower and new additional higher free vibration frequencies as well as the fundamental lower and new additional higher critical forces can be derived and investigated, cf. formulae (22). The lower free vibration frequencies and the lower critical forces coincide with those obtained from asymptotic model governed by equations (24). On the other hand, the higher free vibration frequency and higher critical forces caused by a periodic structure of the stiffened shell, cannot be determined using the asymptotic model.

## VII. FINAL REMARKS

Thin linear-elastic Kirchhoff-Love-type circular cylindrical shells with a micro-periodically inhomogeneous structure along the axial and circumferential directions are objects under consideration. Shells of this kind are termed *biperiodic*. As an example we can mention cylindrical shells with periodically and densely spaced families of longitudinal and circular stiffeners as shown in Fig.1. Dynamic and stability behavior of such shells are described by Euler-Lagrange equations (3) generated by the well known Lagrange function (2). The explicit form of (3), given by (4), coincides with the governing equations of the simplified Kirchhoff-Love second-order theory for elastic shells. For periodic shells coefficients of these equations are highly oscillating non-continuous periodic functions. That is why the direct application of equations (4) to investigations of specific engineering problems is non-effective even using computational methods.

The new mathematical non-asymptotic model for analysis of parametric vibrations and selected dynamic stability problems for periodic shells under consideration is formulated by applying a new approach to the tolerance modeling of micro-heterogeneous media proposed in [19]. Contrary to the "exact" shell equations (4) with highly oscillating non-continuous periodic coefficients, the tolerance model equations have coefficients which are constant or depend only on the time coordinate. Moreover, in contrast to the known asymptotic models commonly used to analysis of dynamics and stability of densely stiffened shells, the non-asymptotic tolerance model takes into account the effect of a cell size on the overall shell behavior (the length-scale effect). The tolerance model is represented by means of constitutive relations (15) and dynamic balance equations (16) for averaged shell displacements and fluctuation amplitudes as the basic unknowns as well as by means of micro-macro decomposition (11) of the total shell displacements and the physical reliability conditions (12) making it possible to determine a posteriori an accuracy of the obtained solutions to special problems. Decomposition (11) and hence also resulting tolerance equations (15) and (16) are uniquely determined by the known periodic linear independent fluctuation shape functions being solutions to certain periodic eigenvalue problems describing free vibrations of the cell, cf. [13].

Taking into account the effect of the microstructure length, i.e. of diameter  $\lambda$  of the basic cell, on dynamic stability of

thin periodically densely stiffened shells under consideration subjected to time-dependent compressive axial forces we arrive at the fourth-order ordinary differential frequency equation for the unknown function of time coordinate, which can be treated as a certain generalization of the known Mathieu's equation, cf. (23). It reduces to the Mathieu equation provided that the period length  $\lambda$  is neglected. On the contrary, within the asymptotic model the known Mathieu equation (25) is obtained.

In the framework of the tolerance model, proposed here, the fundamental lower and the new additional higher free vibration frequencies as well as the fundamental lower and the new additional higher critical forces can be calculated and analyzed, cf. formulae (22). The lower free vibration frequencies and the lower critical forces obtained from the tolerance model coincide with corresponding those derived from the asymptotic model. The higher free vibration frequencies and higher critical forces depend on the microstructure length  $\lambda$  and cannot be derived from the asymptotic models; they can be analyzed only in the framework of the tolerance models.

The investigations of the length-scale effect on parametric vibrations and on boundaries of dynamical instability regions for periodically stiffened shells under consideration based on comparison of results obtained from the new non-asymptotic model, proposed here, with results derived from the known asymptotic models as well as with numerical results acquired by using the known computer programs based on the finite element method (e.g. ANSYS) will be continued and presented in the next paper.

#### ACKNOWLEDGMENT

This contribution is supported by the Ministry of Science and Higher Education of Poland under grant No. NN506 251638.

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