

# Simulation of Large Deformations of Rubbers by the RKPM Method

M. Foroutan, H. Dalayeli, and M. Sadeghian

**Abstract**—In this paper processes including large deformations of a rubber with hyperelastic material behavior are simulated by the RKPM method. Due to the loss of kronecker delta properties in the mesh less shape functions, the imposition of essential boundary conditions consumes significant CPU time in mesh free computations. In this work transformation method is used for imposition of essential boundary conditions. A RKPM material shape function is used in this analysis. The support of the material shape functions covers the same set of particles during material deformation and hence the transformation matrix is formed only once at the initial stages. A computer program in MATLAB is developed for simulations.

**Keywords**—RKPM, large deformations, transformation, essential boundary conditions.

## I. INTRODUCTION

IN the modeling of large deformation processes by the FEM, considerable loss in accuracy arises when the elements in the mesh is extremely deformed. Also the growth of cracks can not be simulated by the finite element method easily. The traditional technique for solving these problems is to remesh the domain at every step during the simulation. However this technique requires the projection of field variables between meshes in successive stages of the problem which consumes significant CPU time. To overcome these difficulties a new class of methods has recently developed which do not require a mesh to discretize the problem domain. These are methods in which the approximate solution is constructed entirely in terms of asset of nodes, and no elements are needed to construct the discrete equations. This class of methods is called mesh less methods. Among the meshless methods EFG and RKPM are a most suitable for structural analyses. In EFG method proposed by Blytschco et al[1], moving least square proposed by Lancaster et al [2] is used for approximation of domain variable. Liu et al [3-5] introduced a new method called Reproducing Kernel Particle Method. They used a correction function in the kernel of integral transformation to impose the reproducing conditions. In RKPM computation the support of the kernel function must cover enough particles for the method to be stable [3,4]. Using the kernel function with a fixed support size could lead to instability when large deformations are occurred.

Chen et al[6] introduced a material kernel function that deforms with the material. In this work for large deformation

M. Foroutan is an assistant professor, Razi University, Kermanshah, Iran.  
 H. Dalayeli is an assistant professor, Maleke Ashtar University, Isfahan, Iran.

M. Sadeghian is a M.S. student, Razi University, Kermanshah, Iran.

analyses of hyperelastic materials these material kernel functions are used. The support of the material kernel function covers the same set of particles during the deformation and therefore instability is avoided in large deformation analyses. Due to the loss of kronecker delta properties in the mesh less shape functions, the imposition of essential boundary conditions consumes significant CPU time in mesh free computations. In this work transformation method, proposed by Chen et al [7] is used for imposition of essential boundary conditions. Using this method, CPU time consumption is decreased considerably.

In this work large deformations of a hyperelastic material are simulated in two processes, simple tension and constrained tension.

## II. GOVERNING EQUATIONS

### A. Hyperelastic Materials

Hyperelastic materials have path-independent behavior. For this materials strain energy density is a potential function that components of stress tensor are derived from it. The strain energy density is usually defined as a function of deformation gradient tensor and the first piola-Kirchhoff stress  $\mathbf{P}$  is derived from it as following:

$$\mathbf{P}^T = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} \quad \text{or} \quad P_{ji} = \frac{\partial W}{\partial F_{ij}} \quad (1)$$

For calculation of second Piola-Kirchhoff stress the above equation is rewritten as:

$$\mathbf{S} = \frac{\partial W(\mathbf{E})}{\partial \mathbf{E}} \quad \text{or} \quad S_{ij} = \frac{\partial W}{\partial E_{ij}} \quad (2)$$

Where  $\mathbf{S}$  is second Piola-Kirchhoff stress and  $\mathbf{E}$  is Green-Lagrange strain tensor and related to deformation gradient tensor  $\mathbf{F}$  as following:

$$E_{ij} = \frac{1}{2}(F_{ki}F_{kj} - \delta_{ij}) \quad (3)$$

Chen et al [8] have shown that strain energy density  $W$  in almost incompressible hyperelastic materials has two components. The first component is due to distortion and second one is due to stretch;

$$W(\bar{I}_1, \bar{I}_2, J) = \bar{W}(\bar{I}_1, \bar{I}_2) + \tilde{W}(J) \quad (4)$$

Where  $\bar{W}$  and  $\tilde{W}$  are defined as:

$$\bar{W}(\bar{I}_1, \bar{I}_2) = \sum_{m+n=1}^{\infty} A_{mn} (\bar{I}_1 - 3)^m (\bar{I}_2 - 3)^n \quad (5)$$

$$\tilde{W}(J) = \frac{k}{2} (J - 1)^2 \quad (6)$$

In the above equations  $\bar{I}_1, \bar{I}_2$  are reduced invariants proposed by Penn [9] to separate the distortion and stretch:

$$\bar{I}_1 = I_1 I_3^{-1/3} \quad (7)$$

$$\bar{I}_2 = I_2 I_3^{-2/3}$$

Where  $I_1, I_2, I_3$  are invariants of Green deformation tensor, defined as:

$$G_{ij} = 2E_{ij} + \delta_{ij} \quad (8)$$

$k$  is bulk modulus and  $J$  is defined as:

$$J = I_3^{1/2} \quad (9)$$

Therefore according to (2) second piola-Kirchhoff stress is calculated from strain energy density by:

$$S_{ij} = 2 \left[ \bar{q}_1 I_3^{-1/3} \left( \delta_{ij} - \frac{1}{3} I_1 G_{ij}^{-1} \right) + \bar{q}_2 I_3^{-2/3} \left( I_1 \delta_{ij} - G_{ij} - \frac{2}{3} I_2 G_{ij}^{-1} \right) \right] + P J G_{ij}^{-1} \quad (10)$$

where ;

$$\bar{q}_i = \frac{\partial \bar{W}}{\partial \bar{I}_i}, i=1,2, \quad P = \frac{\partial \tilde{W}}{\partial J} = k(J - 1) \quad (11)$$

### B. Principle of Minimum Potential Energy

According to this principle, the equilibrium condition is satisfied if and only if variation of potential energy is vanished:

$$\delta \Pi = \int_{\Omega_0} \delta F_{ij} \frac{\partial W(\mathbf{F})}{\partial F_{ij}} d\Omega_0 - \int_{\Omega_0} \delta u_i b_i d\Omega_0 - \int_{\Gamma_0} \delta u_i \bar{t}_i d\Gamma_0 = 0 \quad (12)$$

Where  $b_i$  and  $\bar{t}_i$  are body force and surface traction respectively. Note that all of integrals are calculated on the initial configuration  $\Omega_0$  of body. Substitution of  $\mathbf{P}$  from (1) in to (12) leads to:

$$\int_{\Omega_0} \delta F_{ij} P_{ji} d\Omega_0 = \int_{\Omega_0} \delta u_i b_i d\Omega_0 + \int_{\Gamma_0} \delta u_i \bar{t}_i d\Gamma_0 \quad (13)$$

### C. RKPM Approximation

In the RKPM method displacement vector is approximated by:

$$u_i^h(\mathbf{X}, t) = \sum_{I=1}^N \Phi_I(\mathbf{X}) u_{iI}(t) \quad (14)$$

Where  $\Phi_i$  are material shape functions of particles,  $u_i$  are nodal generalized displacements,  $\mathbf{X}$  is material coordinate and  $N$  is total number of particles. For imposition of essential boundary conditions by transformation method the corrected shape functions  $\hat{\Phi}_i$  are defined as following:

$$\hat{\Phi} = \Phi \cdot \mathbf{T}^{-1} \quad (15)$$

where  $\mathbf{T}$  is transformation matrix. Indeed the nodal displacements  $\hat{u}_i$  related to generalized nodal displacement by the transformation matrix as following:

$$\hat{u}_{iI}^h = \sum_{J=1}^N \Phi_J(x_I) u_{iJ} \quad (16)$$

For one dimensional case (16) is rewritten as:

$$\begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_N \end{bmatrix} = \begin{bmatrix} \Phi_1(x_1) & \Phi_2(x_1) & \cdots & \Phi_N(x_1) \\ \Phi_1(x_2) & \Phi_2(x_2) & \cdots & \Phi_N(x_2) \\ \vdots & \vdots & & \vdots \\ \Phi_1(x_N) & \Phi_2(x_N) & \cdots & \Phi_N(x_N) \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \quad (17)$$

Therefore (14) can be written as:

$$u_i^h(\mathbf{X}, t) = \sum_{I=1}^N \hat{\Phi}_I(\mathbf{X}) \hat{u}_{iI}(t) = \sum_{I=1}^N \Phi_I(\mathbf{X}) u_{iI}(t) \quad (18)$$

Using approximation (18) in the weak form (13) leads to following set of nonlinear equations:

$$\hat{\mathbf{f}}_I^{\text{int}} = \hat{\mathbf{f}}_I^{\text{ext}} \quad (19)$$

Where:

$$\hat{\mathbf{f}}_I^{\text{int}} = \int_{\Omega_0} \mathbf{B}_I^M \mathbf{T}^T \{ \mathbf{S} \} d\Omega_0 \quad (20)$$

$$\hat{\mathbf{f}}_I^{\text{ext}} = \int_{\Omega_0} \hat{\Phi}_I \mathbf{b} d\Omega_0 + \int_{\Gamma_0} \hat{\Phi}_I \bar{\mathbf{t}} d\Gamma_0 \quad (21)$$

In the above equations the stress vector  $\{ \mathbf{S} \}$  and material gradient matrix  $\mathbf{B}_I^M$  are defined as following:

$$\{\mathbf{S}\}^T = [S_{11}, S_{22}, S_{21}] \quad (22)$$

$$\mathbf{B}_I^M = \begin{bmatrix} F_{11} \frac{\partial \hat{\Phi}_I}{\partial X_1} & F_{21} \frac{\partial \hat{\Phi}_I}{\partial X_1} \\ F_{12} \frac{\partial \hat{\Phi}_I}{\partial X_2} & F_{22} \frac{\partial \hat{\Phi}_I}{\partial X_2} \\ F_{12} \frac{\partial \hat{\Phi}_I}{\partial X_1} + F_{11} \frac{\partial \hat{\Phi}_I}{\partial X_2} & F_{22} \frac{\partial \hat{\Phi}_I}{\partial X_1} + F_{21} \frac{\partial \hat{\Phi}_I}{\partial X_2} \end{bmatrix} \quad (23)$$

Set of nonlinear equations (19) is linearized for solution. The incremental form of the mentioned equations at step  $n+1$  and iteration  $v$  is:

$$\hat{\mathbf{f}}^{\text{int}}(\hat{\mathbf{u}}^v) + \Delta \hat{\mathbf{f}}^{\text{int}}(\hat{\mathbf{u}}^v) = \hat{\mathbf{f}}^{\text{ext}}(\hat{\mathbf{u}}) \quad (24)$$

The above equation is rearranged to set of linear equations as following:

$$[\hat{\mathbf{K}}(\hat{\mathbf{u}}_{n+1}^v)] \cdot \{\Delta \hat{\mathbf{u}}\}^v = \{\hat{\mathbf{f}}^{\text{ext}}\}_{n+1} - \{\hat{\mathbf{f}}^{\text{int}}(\hat{\mathbf{u}}_{n+1}^v)\} \quad (25)$$

The stiffness matrix  $\hat{\mathbf{K}}$  constitutes from two parts called material stiffness matrix and geometric stiffness matrix. These two parts are defined as following:

$$\hat{\mathbf{K}}^M = \int_{\Omega_0} \mathbf{B}^{M^T} [\mathbf{C}] \mathbf{B}^M d\Omega_0 \quad \text{or} \quad \hat{\mathbf{K}}_{IJ}^M = \int_{\Omega_0} \mathbf{B}_I^{M^T} \mathbf{C} [\mathbf{B}]_J^M d\Omega_0 \quad (26)$$

$$\hat{\mathbf{K}}_{IJ}^G = \int_{\Omega_0} \mathbf{B}_I^{G^T} [\mathbf{D}] \mathbf{B}_J^G d\Omega_0 \quad (27)$$

The fourth order tensor  $\mathbf{C}$  is material response tensor and for hyperelastic materials obtained from strain energy density by:

$$C_{ijkl} = \frac{\partial^2 W}{\partial E_{ij} \partial E_{kl}} \quad (28)$$

For two dimensional problem matrices  $\mathbf{C}$ ,  $\mathbf{B}_I^G$  and  $\mathbf{D}$  used in (26) and (27) are defined as following:

$$\mathbf{C} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1112} \\ & C_{2222} & C_{2212} \\ \text{sym.} & & C_{1212} \end{bmatrix} \quad (29)$$

$$\mathbf{B}_I^G = \begin{bmatrix} \frac{\partial \hat{\Phi}_I}{\partial X_1} & 0 \\ 0 & \frac{\partial \hat{\Phi}_I}{\partial X_2} \\ \frac{\partial \hat{\Phi}_I}{\partial X_2} & 0 \\ 0 & \frac{\partial \hat{\Phi}_I}{\partial X_1} \end{bmatrix} \quad (30)$$

$$\mathbf{D} = \begin{bmatrix} S_{11} & 0 & S_{12} & 0 \\ 0 & S_{22} & 0 & S_{21} \\ S_{21} & 0 & S_{22} & 0 \\ 0 & S_{12} & 0 & S_{11} \end{bmatrix} \quad (31)$$

In incremental method for calculation of  $\hat{\mathbf{u}}_{n+1}$ , using initial guess  $\hat{\mathbf{u}}_{n+1}^0$  stiffness matrix  $\hat{\mathbf{K}}$ , internal force vector  $\hat{\mathbf{f}}^{\text{int}}$  and external force vector  $\hat{\mathbf{f}}^{\text{ext}}$  are computed. Then  $\Delta \hat{\mathbf{u}}_{n+1}^0$  is computed from (25). Consequently the new value of  $\hat{\mathbf{u}}_{n+1}$  is computed. Iterations are continued until convergence condition is satisfied.

### III. NUMERICAL EXAMPLES

#### A. Simple Tension of a Rubber Strip

In this example deformation of a long rubber strip subjected to simple tension is simulated. The constants of strain energy density function of hyper elastic material are assumed to be as  $A_{10} = 0.373 \text{ Mpa}$ ,  $A_{20} = -0.031 \text{ Mpa}$ ,  $A_{30} = 0.005 \text{ Mpa}$ ,  $k = 10^5 \text{ Mpa}$ .

For RKPM approximation a linear base vector and a cubic spline material kernel with a rectangular support  $\rho = 1.4 \times \Delta \mathbf{x}$  are used. A mesh with  $N_y \times N_x = 4 \times 9$  arrangement and a  $1 \times 4$  mesh for back ground cells with 5 Gauss points in each direction in each cell are employed in simulation. In Fig. 1 arrangement of grids, back ground cells and gauss points are shown. Deformation of strip in four steps is shown in Fig. 2. As shown in this figure elongation is continued up to 1000 percent stretch. In Fig. 3 the reduced stress defined as  $\sigma_x / (\lambda^2 - \lambda^{-2})$  at central Gauss point is plotted verses stretch ratio  $\lambda$ . In Fig. 4 results given in [6] for this simulation are shown. A good agreement is seen between results obtained in Fig. 3 and results shown in Fig. 4. Notice that Fig. 4 contains both of the simple compression and simple tension.

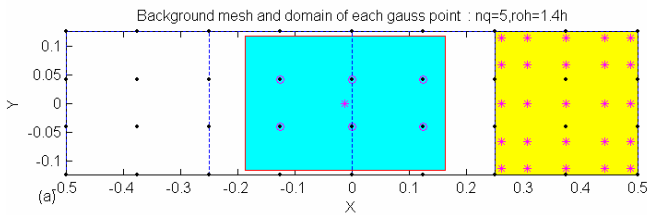


Fig. 1 Grid arrangement, back ground cells and Gauss points

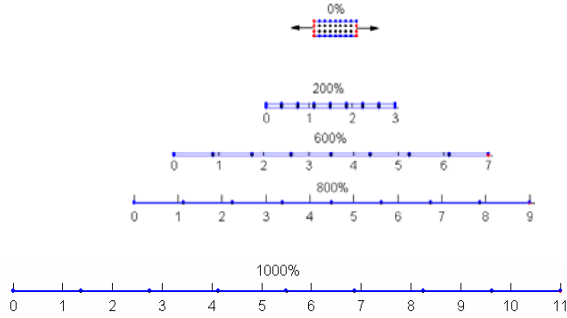


Fig. 2 Deformation stages of rubber strip under simple tension

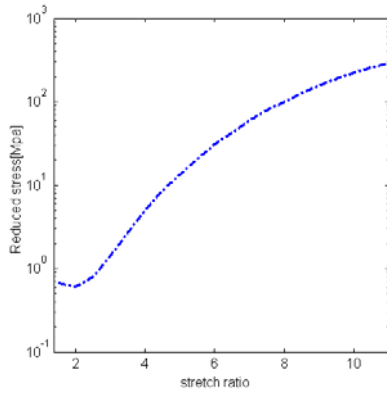


Fig. 3 Reduced stress versus stretch ratio

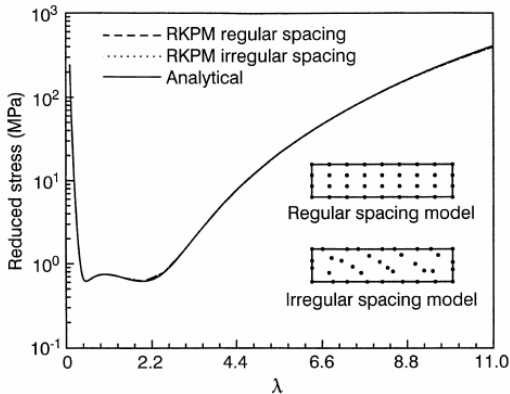


Fig. 4 Reduced stress versus stretch ratio [6]

### B. Constrained Tension of a Rubber Strip

In this example a rubber strip with the same material property as previous example stretched between two rigid plates along the width of strip. Particles on the upper and lower edges of strip are prevented from moving on the  $x$  direction. Due to this boundary condition displacement field is nonlinear. In this simulation a  $N_y \times N_x = 11 \times 11$  regular mesh of grids and a  $10 \times 10$  back ground cell with 4 Gauss points in each direction are used. For RKPM approximation a linear base function and a cubic spline kernel function with rectangular support  $\rho = 1.2 \times \Delta x$  are employed. In this simulation strip is stretched up to 200 percent in  $y$  direction. In Fig. 5 the initial configuration of grids and three steps of deformed configurations of grids are shown. Due to geometric nonlinearity, simulation of this problem by the FEM is impossible without remeshing. By using material shape functions in RKPM approximations readjustment of stretch parameter is not needed in last two examples that include large deformations.

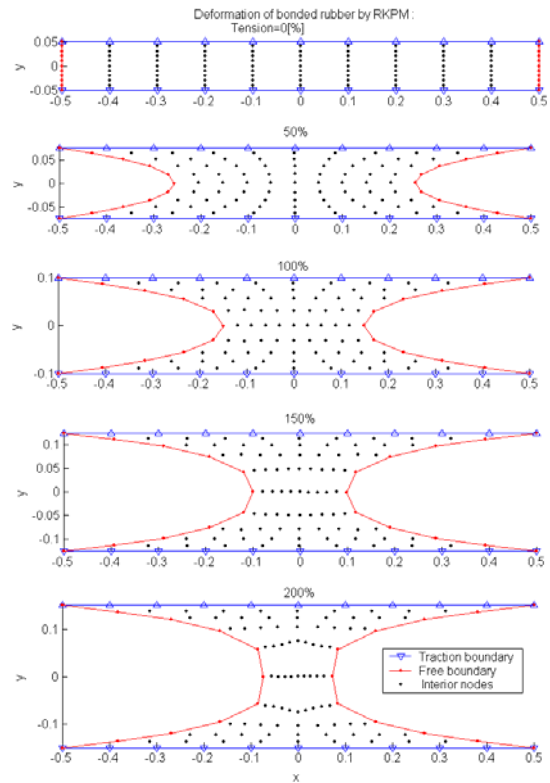


Fig. 5 Initial and deformed configurations of rubber strip under constrained tension

### IV. CONCLUSION

In this paper RKPM method is employed for simulation of linear and nonlinear problems. Transformation method is used for imposition of essential boundary conditions. By using this method, CPU time consumption is decreased considerably in

comparison with other methods such as Lagrange multipliers. For simulation of deformation of hyperelastic materials which contain both of material and geometric nonlinearity, material shape functions are used. By using these shape functions particles covered by the influence domain of a certain particle are not changed during large deformations. Hence readjustment of stretch factor is not needed. Nonlinear simulations presented in this paper include large deformations that make them impossible for the FEM without remeshing.

#### REFERENCES

- [1] T. Belytschko, Y.Y.Lu, L.Gu "Element-free Galerking methods", *Internat. J. Numer. Methods in Eng.*, vol.37, 1994, pp. 229 -256.
- [2] P. Lancaster, K. Salkauskas "Surface Generated by Moving Least Squares Methods", *Math. Comput.*, vol. 37, 1981, pp.141-158.
- [3] W. K. Liu, S. Li, T. Belytschko, "Moving Least Square Reproducing Kernel Methods (I) Methodology and Convergence" *Comp. Methods. Appl. Mech. Eng.*, 1996.
- [4] W. K. Liu, S. Jun, S. Li, J. Adee, T. Belytschko, "Reproducing Kernel Particle Methods for Structural Dynamics", *Internat.J.Numer.Methods Engrg.*, vol.38, 1995, pp.1655-1679.
- [5] W. K. Liu, S. Jun, Y. F. Zhang, "Reproducing Kernel Particle Methods", *Internat. J. Numer. Methods Fluids*, vol.20, 1995, pp.1081-1106.
- [6] J. S. Chen, C.Pan, C.T.Wu, W.K.Liu "Reproducing Kernel Particle Methodes Large Deformation analysis of non-linear structures" *Comp.Methods.Appl. Mech Eng.*, vol. 139, 1996, pp.195-227.
- [7] J.S. Chen, H. P. Wang "New Boundary Condition Treatment in Mesh-free Computation of Contact Problems", *Comput.Methods Appl Mech. Engrg.*, vol.187, 2000, pp.441-468.
- [8] J.S. Chen, C. Pan, C. T. Wu, "Large Deformation Analysis of Rubber based on a Reproducing Kernel Particle Method", *Comput. Mech.*, vol.19, 1997, pp.211-227.
- [9] W.R.Penn "Volume changes accompanying the extension of rubber" *Trans. Soc. Rheo.* vol. 14, 1970, pp.509-517