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ψ-exponential Stability for Non-linear Impulsive Differential Equations

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Abstract—In this paper, we shall present sufficient conditions for the ψ-exponential stability of a class of nonlinear impulsive differential equations. We use the Lyapunov method with functions that are not necessarily differentiable. In the last section, we give some examples to support our theoretical results.

Keywords—Exponential stability, globally exponential stability, impulsive differential equations, Lyapunov function, ψ-stability.

I. INTRODUCTION

Many evolution processes are characterized by the fact that at certain moments of time, they experience a change of state abruptly. The impulsive system of differential equations are an adequate apparatus for the mathematical simulation of such processes and phenomena studied in biology, economics and technology etc. That is why, in recent years, the study of such systems has been very intensive [3,11]. One of the most investigating problems in stability analysis of such systems is exponential stability, since it has played an important role in many areas such as control theory, designs and applications of neural networks [7,8].

Lyapunov method and Lyapunov-Razumikhin technique have been successfully utilized in the investigation of asymptotic and exponential stability of impulsive differential systems [2,4,5,10].

Akinyele [9] introduced the notion of ψ-stability of degree k with respect to a function $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$, increasing and differentiable on $\mathbb{R}_+$ and such that $\psi(t) \geq 1$ for $t \geq 0$ and $\lim_{t\to\infty} \psi(t) = b, b \in [1, \infty)$. In [6], Morachalo introduced the notions of ψ-stability, ψ-uniform stability and ψ-asymptotic stability of trivial solution of the nonlinear system $x^{'} = f(t, x)$. Diamandescu in [1], proved some sufficient conditions for ψ-stability of the zero solution of a nonlinear Volterra integro-differential system.

The purpose of this paper is to establish sufficient conditions for ψ-exponential stability and ψ-global exponential stability for a class of nonlinear impulsive system of differential equations via proposing a Piecewise Continuous Lyapunov ψ-function. The theoretical result have been supported by some examples in the last section.

II. PRILIMINARIES

Let $\mathbb{R}^n$ denote the Euclidean n-space. Elements of this space are denoted by $x = (x_1, x_2, ..., x_n)^T$ and their norm is given by $\|x\|= \max\{|x_1|, |x_2|, ..., |x_n|\}$. For n×n real matrices, we define the norm $|A| = Sup_{\|x\|\leq 1}\|Ax\|$. Let $\psi_i : \mathbb{R}_+ \to (0, \infty), i = 1, 2, ..., n$, where $\mathbb{R}_+ = [0, \infty)$ be the continuous functions and let $\psi = diag[\psi_1, \psi_2, ..., \psi_n]$.
Consider the impulsive differential system

$$\begin{aligned} \dot{x} &= f(t, x),\ t \neq t_k, \\ \Delta x &= I_k(x), t = t_k,\ k = 1, 2, ..., n, \\ x(t_0 + 0) &= x_0, \end{aligned} \tag{1}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a nonlinear function, $I_k : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are continuous functions, $0 \leq t_0 < t_1 < t_2 < ... < t_n < t$ are fixed moments of impulse effect and $\Delta x = I_k(x) = x(t_k + 0) - x(t_k - 0)$.

Here we assume that functions $f, I_k, k \in N$, satisfy all necessary conditions for the global existence and uniqueness of solution for all $t \geq t_0$.

Definition 2.1: Let $E \subset \mathbb{R}^n$ be an open set containing the origin. A function $V : \mathbb{R}_+ \times E \to \mathbb{R}_+$ is said to belong to class $\mathcal{V}_0$ if

(i) V is continuous in each of the sets $[t_{k-1}, t_k) \times E$.

(ii) $V(t, x)$ is locally Lipschitizian in all $x \in E \subset \mathbb{R}^n$ and for all $t \geq t_0$, $V(t, 0) = 0$.

(iii) For each $x \in E \subset \mathbb{R}^n$ and $t \in [t_{k-1}, t_k), k \in \mathbb{N}, \lim_{(t,y)\to(t_k^-,x)} V(t, y) = V(t_k^-, x)$.

Definition 2.2: Given a function $V : \mathbb{R}_+ \times E \to \mathbb{R}_+$, the upper right hand derivative of V with respect to system (1) is defined by

$$D^+V(t, x) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t + h, x(t + h)) - V(t, x)] \tag{2}$$

for $(t, x) \in \mathbb{R}_+ \times E$.

Definition 2.3: The zero solution of system (1) is ψ-exponentially stable if any solution $x(t, t_0, x_0)$ of (1) satisfies $\|\psi(t)x(t, t_0, x_0)\| \leq \beta(\|x_0\|, t_0)e^{-\delta(t-t_0)}, \forall\ t \in [t_{k-1}, t_k), k = 1, 2, ..., n$ where $\beta(h, t) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a non-negative function increasing in $h \in \mathbb{R}_+$, and δ is a positive constant.

If the function $\beta(.)$ in the above definition does not depend on t_0, the zero solution of (1) is called ψ-uniformly exponentially stable.

Definition 2.4: The zero solution of system (1) is said to be ψ-globally exponentially stable if there exist some constants $\delta > 0$ and $M \geq 1$ such that for any solution $x(t, t_0, x_0)$

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of (1), we have $\|\psi(t)x(t,t_0,x_0)\| \leq Me^{-\delta(t-t_0)}, \quad \forall\ t \in [t_{k-1},t_k), k=1,2,...,n.$

Definition 2.5: A function $V(t,x) \in \mathcal{V}_0$ is called a Piecewise continuous Lyapunov-ψ function for (1) if $V(t,x)$ is continuously differentiable in $[t_{k-1},t_k), k=1,2,...,n$ and there exist positive numbers $\lambda_1,\lambda_2,\lambda_3,L,p,q,r,\delta$ such that

$$\lambda_1\|\psi(t)x(t)\|^p \leq V(t,x) \leq \lambda_2\|\psi(t)x(t)\|^q, \quad \forall t \geq 0, x \in \mathbb{R}^n; \tag{3}$$

$$D^+V(t,x) \leq -\lambda_3\|\psi(t)x(t)\|^r + Le^{-\delta t}, \quad \forall t \in [t_{k-1},t_k), k=1,2,...,n; \tag{4}$$

$$V(t_k,x(t_k)) \leq V(t_k^+,x(t_k^+)). \tag{5}$$

Definition 2.6: A function $V(t,x) \in \mathcal{V}_0$ is called a generalized Piecewise continuous Lyapunov-ψ function for (1) if there exist positive functions $\lambda_1(t),\lambda_2(t),\lambda_3(t)$, where $\lambda_1(t)$ is non-decreasing, and there exist positive numbers L,p,q,r,δ such that

$$\lambda_1\|\psi(t)x(t)\|^p \leq V(t,x) \leq \lambda_2\|\psi(t)x(t)\|^q, \quad \forall t \geq 0, x \in \mathbb{R}^n; \tag{6}$$

$$D^+V(t,x) \leq -\lambda_3\|\psi(t)x(t)\|^r + Le^{-\delta t}, \quad \forall t \in [t_{k-1},t_k), k=1,2,...,n; \tag{7}$$

$$V(t_k,x(t_k)) \leq V(t_k^+,x(t_k^+)). \tag{8}$$

In order to study exponential stability of (1), we need the following comparison principle. Consider a scaler impulsive differential system

$$\begin{aligned} \dot{u} &= g(t,u),\ t \neq t_k, \\ \Delta u &= G_k(u),\ t=t_k,\ k=1,2,...,n, \\ u(t_0+0) &= u_0 \end{aligned}$$

where $g(t,u) \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$ and $g(t,0)=0$.

Lemma 2.1: Let $u(t)$ be a maximal solution of above system. If a piecewise continuous function $\nu(t)$ with $\nu(t)=u_0$ satisfy

$$D^+\nu(t) \leq g(t,u(t)), \quad \forall t \in [t_{k-1},t_k), k=1,2,...,n,$$

then
$\nu(t)-\nu(t_0) \leq \int_{t_0}^t g(s,u(s))ds, \quad \forall t \in [t_{k-1},t_k), k=1,2,...,n.$

III. Main results

In this section, we shall present sufficient conditions for the ψ-exponential stability, ψ-uniformly exponential stability and ψ-globally exponential stability of (1) via proposed Piecewise continuous Lyapunov-ψ function.

Theorem 3.1 The zero solution of system (1) is ψ-exponentially stable if it admits a generalized Piecewise continuous Lyapunov-ψ function and the following two conditions hold:

$$\delta > \inf_{t\in\mathbb{R}_+} \frac{\lambda_3(t)}{[\lambda_2(t)]^{r/q}} > 0,\ t \in [t_{k-1},t_k); \tag{9}$$

$$\exists\, \gamma > 0 \text{ such that } V(t,x) - [V(t,x)]^{r/q} \leq \gamma e^{-\delta t}. \tag{10}$$

Proof. Let $x(t)$ be the solution of (1) with $x(t_0)=x_0$, where $t_0 \geq 0$ is any initial time.
Set

$$Q(t,x(t)) = V(t,x(t))e^{M(t-t_0)}, \quad \forall t \in [t_{k-1},t_k), k=1,2,...,n,$$

where $M = \inf_{t\in\mathbb{R}_+} \frac{\lambda_3(t)}{[\lambda_2(t)]^{r/q}}$.
We see that $M < \delta$ and

$$D^+Q(t,x) = D^+V(t,x)e^{M(t-t_0)} + MV(t,x)e^{M(t-t_0)}.$$

Taking (7) into account, for all $t \geq t_0,\ t \neq t_k$, we get

$$\begin{aligned} D^+Q(t,x) &\leq (-\lambda_3(t)\|\psi(t)x(t)\|^r + Le^{-\delta t})e^{M(t-t_0)} \\ &+ MV(t,x)e^{M(t-t_0)}, \quad \forall t \in [t_{k-1},t_k), k=1,2,...,n. \end{aligned}$$

From (6) and since, by the assumption, $\lambda_2(t) > 0$ for all $t \in [t_{k-1},t_k)$, we have

$$\|\psi(t)x(t)\|^q \geq \frac{V(t,x)}{\lambda_2(t)},$$

equivalently

$$-\|\psi(t)x(t)\|^r \leq -[\frac{V(t,x)}{\lambda_2(t)}]^{r/q}.$$

Therefore, we have

$$\begin{aligned} D^+Q(t,x) &\leq \{-V(t,x)^{r/q}\frac{\lambda_3(t)}{[\lambda_2(t)]^{r/q}} + Le^{-\delta t}\}e^{M(t-t_0)} \\ &+ MV(t,x)e^{M(t-t_0)}. \end{aligned}$$

Since

$$\frac{\lambda_3(t)}{[\lambda_2(t)]^{r/q}} \geq M, \quad \forall t \in [t_{k-1},t_k),$$

and by the condition (10), we obtain

$$\begin{aligned} D^+Q(t,x) &\leq M\{V(t,x) - V(t,x)^{r/q}\}e^{M(t-t_0)} + Le^{-\delta t}e^{M(t-t_0)} \\ &\leq M\gamma e^{-\delta t}e^{M(t-t_0)} + Le^{-\delta t}e^{M(t-t_0)} \\ &= (L+M\gamma)e^{-\delta t}e^{M(t-t_0)} \\ &\leq (L+M\gamma)e^{-\delta(t-t_0)}e^{M(t-t_0)}. \end{aligned}$$

Therefore, $D^+Q(t,x) \leq (L+M\gamma)e^{(M-\delta)(t-t_0)}, \quad \forall\ t \in [t_{k-1},t_k), k=1,2,...,n.$
By Lemma 2.1 to the case

$$\nu(t) = Q(t,x(t)), \quad g(t,u(t)) = (L+M\gamma)e^{(M-\delta)(t-t_0)},$$

we obtain

$$\begin{aligned} Q(t,x(t)) - Q(t_0,x_0) &\leq \int_{t_0}^t (L+M\gamma)e^{(M-\delta)(s-t_0)}ds \quad t \neq t_k \\ &= (L+M\gamma)\frac{1}{M-\delta}\{e^{(M-\delta)(t-t_0)} - 1\}. \end{aligned}$$

Setting $\delta_1 = -(M-\delta)$, by condition (9), we have $\delta_1 > 0$ and

$$\begin{aligned} Q(t,x(t)) &\leq Q(t_0,x_0) + \frac{L+M\gamma}{\delta_1} - \frac{L+M\gamma}{\delta_1}e^{(M-\delta)(t-t_0)} \\ &\leq Q(t_0,x_0) + \frac{L+M\gamma}{\delta_1}. \end{aligned}$$

Since $Q(t_0,x_0) = V(t_0,x_0) \leq \lambda_2(t_0)\|\psi(t_0)x_0\|^q$, we get

$$Q(t,x(t)) \leq \lambda_2(t_0)\|\psi(t_0)x_0\|^q + \frac{L+M\gamma}{\delta_1}.$$

Letting

$$\lambda_2(t_0)\|\psi(t_0)x_0\|^q + \frac{L+M\gamma}{\delta_1} = \beta(\|x_0\|, t_0) > 0,$$

we have

$$Q(t, x(t)) \leq \beta(\|x_0\|, t_0), \quad \forall t \in [t_{k-1}, t_k), k = 1, 2, ..., n. \tag{11}$$

Furthermore, from condition (6), it follows that

$$\lambda_1(t)\|\psi(t)x(t)\|^p \leq V(t, x(t)),$$
$$\|\psi(t)x(t)\| \leq \{\frac{V(t, x(t))}{\lambda_1(t)}\}^{1/p}.$$

Since $\lambda_1(t)$ is non-decreasing, $\lambda_1(t) \geq \lambda_1(t_0)$, we have

$$\|\psi(t)x(t)\| \leq \{\frac{V(t, x(t))}{\lambda_1(t_0)}\}^{1/p}.$$

Substituting

$$V(t, x) = \frac{Q(t, x)}{e^{M(t-t_0)}},$$

into the last inequality, we obtain

$$\|\psi(t)x(t)\| \leq \{\frac{Q(t, x(t))}{e^{M(t-t_0)}\lambda_1(t_0)}\}^{1/p}. \tag{12}$$

Combining (11) and (12),

$$\begin{aligned} \|\psi(t)x(t)\| &\leq \{\frac{\beta(\|x_0\|, t_0)}{e^{M(t-t_0)}\lambda_1(t_0)}\}^{1/p} \\ &= \{\frac{\beta(\|x_0\|, t_0)}{\lambda_1(t_0)}\}^{\frac{1}{p}} e^{-\frac{M}{p}(t-t_0)}, \\ &\qquad \forall t \in [t_{k-1}, t_k), k = 1, 2, ..., n, \end{aligned} \tag{13}$$

which shows that system (1) is ψ-exponentially stable and hence the Theorem.

If we consider ψ as scaler function independent of t, then we get a sufficient condition for ψ-uniformly exponential stability as stated below:

Theorem 3.2 Let ψ be a constant function independent of t. Then the zero solution of system (1) is ψ-uniformly exponentially stable, if it admits a piecewise continuous Lyapunov-ψ function and the following two conditions hold:

$$\delta > \frac{\lambda_3}{[\lambda_2]^{r/q}}, \tag{14}$$

$$\exists\, \gamma > 0 \text{ such that } V(t, x) - V(t, x)^{r/q} \leq \gamma e^{-\delta t}, \tag{15}$$
$$\forall t \in [t_{k-1}, t_k), k = 1, 2, ..., n.$$

Proof. The proof is in the same line of the proof of Theorem 3.1, so omitted.

Theorem 3.3 The zero solution of system (1) is ψ-globally exponentially stable, if it admits a piecewise continuous Lyapunov-ψ function with $p = q = r$ and δ, with $\delta > \frac{\lambda_3}{\lambda_2}$.
Proof. Let

$$Q(t, x) = V(t, x)e^{\lambda_3 t/\lambda_2}, \quad \forall\, t \in [t_{k-1}, t_k), k = 1, 2, ..., n. \tag{16}$$

Then from (3), (4) and (16)

$$\begin{aligned} D^+Q(t, x) &= D^+V(t, x)e^{\lambda_3 t/\lambda_2} + \frac{\lambda_3}{\lambda_2}V(t, x)e^{\lambda_3 t/\lambda_2}, \quad t \neq t_k \\ &\leq (-\lambda_3\|\psi(t)x(t)\|^p + Le^{-\delta t})e^{\lambda_3 t/\lambda_2} + \frac{\lambda_3}{\lambda_2}Q(t, x) \\ &\leq (-\frac{\lambda_3}{\lambda_2}V(t, x) + Le^{-\delta t})e^{\lambda_3 t/\lambda_2} + \frac{\lambda_3}{\lambda_2}Q(t, x) \\ &= Le^{(-\delta + \frac{\lambda_3}{\lambda_2})t} \\ &= Le^{-\beta t}, \quad t \neq t_k, \end{aligned} \tag{17}$$

where $\beta = \delta - \frac{\lambda_3}{\lambda_2}$.
Integrating both sides (17) from t_0 to t, $t \neq t_k$, we get

$$\begin{aligned} Q(t, x) - Q(t_0, x_0) &\leq \beta^{-1}L[e^{-\beta t_0} - e^{-\beta t}] \\ &\leq \beta^{-1}L = L(\delta - \frac{\lambda_3}{\lambda_2})^{-1}. \end{aligned}$$

Therefore

$$Q(t, x) \leq Q(t_0, x_0) + L(\delta - \frac{\lambda_3}{\lambda_2})^{-1} = a, \tag{18}$$

where $a = Q(t_0, x_0) + L(\delta - \frac{\lambda_3}{\lambda_2})^{-1}$.
From (3), (16) and (18) we have,

$$\begin{aligned} \|\psi(t)x(t)\| &\leq [\frac{1}{\lambda_1}V(t, x)]^{1/p} \\ &= [\frac{1}{\lambda_1}e^{-\lambda_3 t/\lambda_2}Q(t, x)]^{1/p} \\ &\leq [a\frac{1}{\lambda_1}e^{-\lambda_3 t/\lambda_2}]^{1/p} \\ &= Me^{-\eta t} \leq Me^{-\eta(t-t_0)}, \quad \forall t \in [t_{k-1}, t_k), \end{aligned}$$

where $M = (a/\lambda_1)^{1/p}$ and $\eta = \lambda_3/(\lambda_2 p)$.
This completes the proof.

IV. Examples

In this section, we give some examples to support our results in above section.
Example 4.1 Consider an impulsive differential equation

$$\begin{aligned} \dot{x} &= -\frac{1}{6}e^t x^{\frac{3}{5}} + \frac{x}{12} + e^{-3t/2}cost, \quad t \neq k\pi/4,\ k = 1, 2, ..., \\ \Delta x &= -1/2, \quad t = k\pi/4. \end{aligned} \tag{19}$$

Let $\psi(t) = e^t$ and a piecewise continuous Lyapunov-ψ function $V(t, x) \in \mathcal{V}_0$ with $E = \{x : | x | \leq 1\}$ given by

$$V(t, x) = e^{-t/2}x^6.$$

Then (6) holds for $p = q = 6, \lambda_1(t) = e^{-13t/2}, \lambda_2(t) = e^{-6t}$.
Now

$$\begin{aligned} \dot{V}(t, x) &= -\frac{1}{2}e^{-t/2}x^6 - e^{t/2}x^{28/5} + 6e^{-2t}x^5 cost \\ &\leq -e^{t/2}x^{28/5} + 6e^{-2t}, \quad t \neq k\pi/4. \end{aligned}$$

It follows that conditions (7) holds for $\lambda_3(t) = e^{-51t/10}, L = 6, r = \frac{28}{5}, \delta = 2$.
Now $\inf \frac{\lambda_3(t)}{(\lambda_2(t))^{r/q}} = \inf e^{t/2} = 1 < 2 = \delta$
and
$V(t, x) - [V(t, x)]^{r/q} = e^{-t/2}x^6 - e^{-7t/15}x^{28/5} \leq 0 \leq e^{-2t}, \quad t \neq k\pi/4.$

Hence by Theorem 3.1, the zero solution of (19) is ψ-exponentially stable.

Example 4.2 Consider impulsive differential equation

$$\begin{aligned} \dot{x} &= -\frac{1}{3}x^{\frac{5}{2}} + xe^{-2t}, \quad t \neq k,\ k = 1, 2, ..., \\ \Delta x &= -1, \quad t = k. \end{aligned} \qquad (20)$$

Let $\psi(t) = \frac{1}{2}$ and a piecewise continuous Lyapunov-ψ function $V(t,x) \in \mathcal{V}_0$ with $E = \{x : \mid x \mid \leq 1\}$ is

$$V(t,x) = \mid x \mid^3 = \begin{cases} x^3 & \text{for } x \geq 0 \\ -x^3 & \text{for } x < 0 \end{cases}$$

Now

$$\begin{aligned} \dot{V}(t,x) &= \begin{cases} -x^{\frac{9}{2}} + 3x^3e^{-2t} & \text{for } x \geq 0 \\ x^{\frac{9}{2}} - 3x^3e^{-2t} & \text{for } x < 0 \end{cases} \\ &= -\mid x \mid^{\frac{9}{2}} + 3 \mid x \mid^3 e^{-2t} \\ &\leq -\mid x \mid^{\frac{9}{2}} + 3e^{-2t},\ t \neq k. \end{aligned}$$

It follows that conditions (3) and (4) holds for $\lambda_1 = 1, \lambda_2 = 16, \lambda_3 = 2^{9/2}, p = q = 3,$ $L = 3, r = \frac{9}{2}$ and $\delta = 2$.
Now $\frac{\lambda_3}{(\lambda_2)^{r/q}} < 2 = \delta$
and
$V(t,x) - [V(t,x)]^{r/q} = \mid x \mid^{3/2} (\mid x \mid^{3/2} - 1) \leq 0 \leq e^{-2t},$
$t \neq k.$
Hence the zero solution of (20) is ψ-uniformly exponentially stable.

Example 4.3 Consider impulsive differential equation

$$\begin{aligned} \dot{x} &= -x^{\frac{1}{2}} + x^2e^{-3t}, \quad t \neq k, k = 1, 2, ..., \\ \Delta x &= -2, \quad t = k. \end{aligned} \qquad (21)$$

Let $\psi(t) = 1$ and a piecewise continuous Lyapunov ψ-function $V(t,x) \in \mathcal{V}_0$ with $D = \{x : \mid x \mid \leq 1\}$ given by

$$V(t,x) = \mid x \mid^{1/2} = \begin{cases} x^{1/2} & \text{for } x \geq 0 \\ -x^{1/2} & \text{for } x < 0 \end{cases}$$

Now

$$\begin{aligned} \dot{V}(t,x) &= \begin{cases} -\frac{1}{2}x^0 + \frac{1}{2}x^{3/2}e^{-3t} & \text{for } x \geq 0 \\ \frac{1}{2}x^0 - \frac{1}{2}x^{3/2}e^{-3t} & \text{for } x < 0 \end{cases} \\ &= -\frac{1}{2} \mid x \mid^0 + \frac{1}{2} \mid x \mid^{3/2} e^{-3t} \\ &\leq -\frac{1}{2} \mid x \mid^{\frac{1}{2}} + \frac{1}{2}e^{-3t}, \quad t \neq k. \end{aligned}$$

It follows that conditions (3) and (4) holds for $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1/2, p = q = r = 1/2, L = 1/2, \delta = 3$.
Now $\frac{\lambda_3}{\lambda_2} = 1/2 < 3 = \delta$.
Hence the zero solution of (21) is ψ-globally exponentially stable.

References

[1] A. Dimandescu; *On the ψ-stability of nonlinear voltera integro-differential systems*, Electronic Journal of differential equations, 56: 1-14 (2005).
[2] B. Liu, X. Z. Liu, K. Teo, Q. Wang, *Razumikhin-type theorems on exponential stability of impulsive delay systems*, IMA J. Appl. Math. 71: 47 - 61 (2006).
[3] D. D. Bainov , P. S. Simeonov , *Systems with impulse effect : Stability, Theory and Applications*, Ellis Horwood, Chichester, UK, 1989.
[4] I. M. Stamova, G. T. Stamov, *LyapunovRazumikhin method for impulsive functional equations and applications to the population dynamics*, J. Comput. Appl. Math. 130: 163-171 (2001).
[5] J. Shen, J. Yan, *Razumikhin type stability theorems for impulsive functional differential equations*, Nonlinear Anal. 33: 519-531 (1998).
[6] J. Morchalo; *On ($\psi - L_p$)-stability of nonlinear systems of differential equations*, Analele Stiintifice ale Universitatii Al. I. Cuza Iasi, Tomul XXXVI, s. I-a, Matematica, f.4: 353-360 (1990).
[7] L. Berezansky, L. Idels, *Exponential stability of some scalar impulsive delay differential equation*, Commun. Appl. Math. Anal. 2: 301-309 (1998).
[8] N. M. Linh and V. N. Phat, *Exponential stability of nonlinear time-varying differential equations and applications*, Electronic Journal of differential equations 34: 1-13(2001).
[9] O. Akinyele; *On partial stability and boundedness of degree k*, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.,(8), 65: 259-264(1978).
[10] Q. Wang, X. Z. Liu, *Exponential stability for impulsive delay differential equations by Razumikhin method*, J. Math. Anal. Appl. 309: 462-473 (2005).
[11] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov,"*Theory of Impulsive Differential Equations*." World Scientific, Singepore / New Jersey /London , 1989.
[12] X. Z. Liu, G. Ballinger, *Uniform asymptotic stability of impulsive delay differential equations*, Comput. Math. Appl. 41: 903-915 (2001).