On Bounds For The Zeros of Univariate Polynomials

Matthias Dehmer[1] and Jürgen Kilian[2]

Abstract—Problems on algebraical polynomials appear in many fields of mathematics and computer science. Especially the task of determining the roots of polynomials has been frequently investigated. Nonetheless, the task of locating the zeros of complex polynomials is still challenging. In this paper we deal with the location of zeros of univariate complex polynomials. We prove some novel upper bounds for the moduli of the zeros of complex polynomials. That means, we provide disks in the complex plane where all zeros of a complex polynomial are situated. Such bounds are extremely useful for obtaining a priori assertions regarding the location of zeros of polynomials. Based on the proven bounds and a test set of polynomials, we present an experimental study to examine which bound is optimal.

Keywords—complex polynomials, zeros, inequalities

I. INTRODUCTION

In mathematics and computer science, polynomials are extremely important objects of investigation and there are various applications in many scientific specialization areas, such as coding theory, cryptography, combinatorics, number theory, mathematical biology and engineering [2], [4], [11], [16]. Especially, the polynomial zeros play an important role, e.g., to solve digital audio signal processing problems [25], control engineering problems [3], and eigenvalue problems in mathematical physics [23]. Historically, the topic of determining the roots of algebraical equations like

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0,$$

has been frequently investigated in pure algebra [8], [20]. Then, in the 20-th century, the investigation of polynomial zeros (or roots of polynomial equations) became a part of applied function theory [22], where the polynomials are considered as holomorphic functions [22]. This specialization area is called the *analytic theory of polynomials* [8], [9], [21] or *geometry of polynomials* [8], [13], [15]. The main area of investigation within the geometry of polynomials is to examine geometric relationships between the zeros and the coefficients of a given polynomial. The geometry of polynomials possesses still challenging and outstanding problems, e.g. SCHÖNBERG's, KATSOPRINAKIS and SENDOV's conjecture [18], [19]. For example, the conjecture of SENDOV [19] deals with the zeros of a polynomial and its derivative: if all zeros of a complex polynomial $f(z)$ lie in the unit circle, then there is always a zero of $f'(z)$ in $|z - a| = 1$, where a is any zero of $f(z)$.

[1]Integrative Bioinformatics, Max F. Perutz Laboratories, Dr. Bohr Gasse 9, A-1030 Vienna, Austria, University of Vienna, Medical University of Vienna and University of Veterinary Medicine Vienna, Austria, e-mail: matthias.dehmer@univie.ac.at. [2]Darmstadt University of Technology, 64289 Darmstadt, Germany, e-mail: kilian@noteserver.org.

Up to now, this conjecture is only solved for certain classes of polynomials [5], [6]. A further important problem of the geometry of polynomials is examining lower or upper bounds for the moduli of the zeros of complex polynomials. In other words, one is mainly interested in finding closed or open disks in the complex plane, which contain either all or $p < n$ zeros of a complex polynomial, e.g., [1], [8], [13], [14], [24]. For example, such bounds are very useful for solving practical problems in numerical analysis, e.g. eigenvalue problems [23], because it is well known that the zeros of a complex polynomial $f(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, a_i \in \mathbb{C}$ can not be computed based on closed terms, if $n \geq 5$. Altogether that means we get *a priori* assertions which provide regions in the complex plane where the zeros of a given polynomial lie. In terms of eigenvalue problems, such bounds are very useful because one is often not interested in computing all eigenvalues precisely. In this paper we throughout deal with complex polynomials, that means the polynomials under consideration have complex valued coefficients. In contrast to this, a deep treatment of methods for examining the zeros of real valued polynomials can be found in [17].

This paper is organized as follows: In Section II we briefly state mathematical preliminaries for examining the zeros of complex polynomials. Section III provides novel upper bounds for the moduli of the zeros of complex polynomials. In order to examine the problem of evaluating the quality of given zero bounds, we perform an experiment in Section IV, based on a set of complex polynomials with randomly chosen coefficients. Section V finishes the paper with a summary and conclusion.

II. MATHEMATICAL PRELIMINARIES

In the following, we state some definitions and assertions for examining the zeros of complex valued polynomials. First, we express the definition of the space of algebraical polynomials with complex coefficients.

Definition 2.1: Let n be a natural number. Then,

$$\Pi_n := \{f : \mathbb{C} \longrightarrow \mathbb{C} \,|\, f(z) = \sum_{i=0}^{n} a_i z^i, a_i \in \mathbb{C},$$
$$i = 0, 1, \ldots, n\},$$

denotes the space of complex valued polynomials with $\deg(f(z)) \leq n$.

Definition 2.2: Let $z_0 \in \mathbb{C}$ and $r \in \mathbb{R}_+$. Then

$$K(z_0, r) := \{z \in \mathbb{C} \,|\, |z - z_0| \leq r\}$$

is called closed disk with central point z_0 and radius r.

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
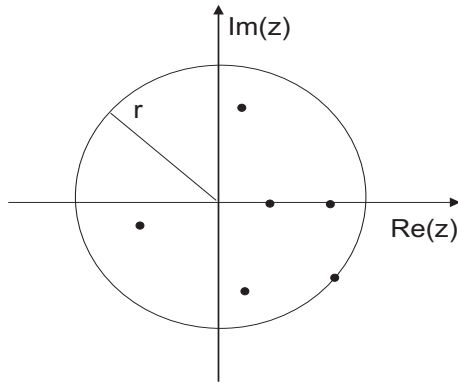Vol:1, No:2, 2007

Fig. 1. Closed disk in the complex plane in which all zeros of a certain complex polynomial $f \in \Pi_6$ are situated.

A well known result in algebra is a famous result due to GAUSS [10] that is called the *fundamental theorem of algebra*, e.g., [8], [10]. Historically, it was the first important result in the specialization area of examining the roots of an algebraical equation.

Theorem 2.1: Let

$$f(z) = \sum_{i=0}^{n} a_i z^i,\, a_n \neq 0,\, a_i \in \mathbb{C},\, i = 0, 1, \ldots, n \quad (1)$$

be a complex polynomial with $\deg(f(z)) = n$. Then, $f(z)$ has exactly n zeros.

The fundamental theorem of algebra states that a complex polynomial with degree n has exactly n zeros, but we do not know their location in the complex plane. For locating the zeros of complex valued polynomials, we will provide in Section III circular regions in the complex plane, where the regions are manifested by the novel bounds. We now state furthermore a well known result about the zero distribution of real valued polynomials which is originally due to DESCARTES. In literature, this result is often called the DESCARTES' Rule of Signs [8], [12], [13].

Theorem 2.2: Let

$$f(z) = \sum_{i=0}^{n} a_i z^i,\, a_i \in \mathbb{R},\, i = 0, 1, \ldots, n$$

be a real valued polynomial. The number of the positive zeros of $f(z)$ is equal to the number of sign changes of the coefficient sequence of $f(z)$, minus a multiple of two.

Based on these preliminaries, we introduce in Section III the problem of finding circular regions in the complex plane which contain all zeros of a complex valued polynomial.

III. NOVEL ZERO BOUNDS

In this section we first introduce the problem of locating the zeros of a complex polynomial in the complex plane. More precisely, we are searching for closed disks

$$K(z_0, r) := \{ z \in \mathbb{C} \,|\, |z - z_0| \leq r \},\, z_0 \in \mathbb{C},\, r \in \mathbb{R}_+,$$

which contain all zeros of a complex valued polynomial. In the following, we assume a complex polynomial $f(z)$. Now,

we want to find a bound $\mathcal{S} = \mathcal{S}(a_0, a_1, \ldots, a_n)$ in such a way that all zeros of $f(z)$ lie in the closed disk

$$K(z_0, \mathcal{S}(a_0, a_1, \ldots, a_n)) :=$$
$$\{ z \in \mathbb{C} \,|\, |z - z_0| \leq \mathcal{S}(a_0, a_1, \ldots, a_n) \}.$$

Without loss of generality we set $z_0 = 0$. That means we are always searching for circular regions with central point zero and radius $\mathcal{S}$. For example, Figure 1 shows a circular region with central point zero and radius r in which all zeros of a complex valued polynomial $f \in \Pi_6$ are situated. In Figure 1, f has two positive zeros and furthermore there is a zero z_i with $z_i = r$. That means the underlying zero bound is sharp. In the following, we will mainly prove some novel zero bounds for the moduli of the zeros of complex valued polynomials. Furthermore we prove two theorems which provide families of zero bounds in a sense that the bounds also depend on a certain parameter $p \in \mathbb{N}$. This leads us to generalizations in such a way that known bounds follow as special cases.

A. Implicit Zero Bounds

We begin with proving novel *implicit* zero bounds for complex valued polynomials. We call a zero bound implicit if the zero bound is also a zero of a concomitant polynomial obtained from the underlying proof.

Theorem 3.1: Let $f(z)$ be a complex polynomial (see Equation 1). All zeros of $f(z)$ lie in the closed disk $K(0, \max(1, \delta))$, where $\delta \neq 1$ denotes the positive root of the equation

$$z^{n+2} - z^{n+1} - \Delta z^n + \Delta = 0,$$

and

$$\Delta := \max_{1 \leq j \leq n} \left| \frac{a_{n-1}a_{n-j} - a_n a_{n-j-1}}{a_n{}^2} \right|, a_{-1} := 0.$$

Proof: Consider the complex polynomial

$$Q(z) := (a_{n-1} - a_n z) f(z).$$

We obtain

$$Q(z) = -a_n{}^2 z^{n+1} + (a_{n-1}a_{n-1} - a_n a_{n-2}) z^{n-1} + \cdots$$
$$+ (a_{n-1}a_1 - a_n a_0) z + a_{n-1}a_0.$$

Applying (two times) the well known triangle inequality leads us to the inequality

$$|Q(z)| = |a_n|^2 |z|^{n+1} - \{ |a_{n-1}a_{n-1} - a_n a_{n-2}| |z|^{n-1} + \cdots$$
$$+ |a_{n-1}a_1 - a_n a_0| |z| + |a_{n-1}a_0| \}.$$

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:1, No:2, 2007

Furthermore, we conclude

$$
\begin{aligned}
|Q(z)| &= |a_n|^2\Bigg[|z|^{n+1} - \bigg\{\bigg|\frac{a_{n-1}a_{n-1} - a_n a_{n-2}}{a_n{}^2}\bigg||z|^{n-1} \\
&\quad + \cdots + \bigg|\frac{a_{n-1}a_1 - a_n a_0}{a_n{}^2}\bigg||z| + \bigg|\frac{a_{n-1}a_0}{a_n{}^2}\bigg|\bigg\}\Bigg], \\
&\geq |a_n|^2\bigg[|z|^{n+1} - \Delta\sum_{j=0}^{n-1}|z|^j\bigg], \\
&= |a_n|^2\bigg[|z|^{n+1} - \Delta\frac{|z|^n - 1}{|z| - 1}\bigg], \\
&= \frac{|a_n|^2}{|z| - 1}\Big[|z|^{n+2} - |z|^{n+1} - \Delta|z|^n + \Delta\Big],
\end{aligned}
$$

where

$$
\Delta := \max_{1 \leq j \leq n}\bigg|\frac{a_{n-1}a_{n-j} - a_n a_{n-j-1}}{a_n{}^2}\bigg|, \, a_{-1} := 0.
$$

We define $H(z) := z^{n+2} - z^{n+1} - \Delta z^n + \Delta$. Because $H(z)$ is a real valued polynomial, we can apply Theorem 2.2. The application of Theorem 2.2 leads us to the fact that $H(z)$ has either two or no positive zeros. Now, we see that $H(1) = 0$. From this, it follows that $H(z)$ has exactly two positive zeros. In the following, we denote the second positive root as δ. Based on the following observations

$$
\mathrm{sign}\{H(0)\} = 1 \quad \text{and} \quad \mathrm{sign}\{H(+\infty)\} = 1,
$$

we finally conclude

$$
|Q(z)| > 0 \quad \text{for} \quad |z| > \max(1, \delta).
$$

Hence, all zeros of $Q(z)$ lie in $K(0, \max(1, \delta))$. Because of the fact that all zeros of $f(z)$ are zeros of $Q(z)$, the assertion of the theorem holds also for $f(z)$. ∎

The next theorems also provide implicit zero bounds for all zeros of complex valued polynomials. Especially the following theorem is similarly to prove as Theorem 3.1.

Theorem 3.2: Let $f(z)$ be a complex polynomial (see Equation 1). Then, all zeros of $f(z)$ lie in the closed disk $K(0, \max(1, \delta))$, where $\delta \neq 1$ denotes the positive root of the equation

$$
\begin{aligned}
|a_n|^2 z^{n+2} - |a_n|^2 z^{n+1} - \Phi z^n + z^{n-1}(\Phi - \tilde{\Delta}|a_n|^2) \\
+ \tilde{\Delta}|a_n|^2 = 0,
\end{aligned}
$$

where

$$
\tilde{\Delta} := \max_{2 \leq j \leq n}\bigg|\frac{a_{n-1}a_{n-j} - a_n a_{n-j-1}}{a_n{}^2}\bigg|, \, a_{-1} := 0,
$$

and $\Phi := |a_{n-1}^2 - a_n a_{n-2}|$.

Theorem 3.3: Let $f(z)$ be a complex polynomial (see Equation 1). Then all zeros of $f(z)$ lie in the closed disk $K(0, \delta)$, where $\Phi - \tilde{\Delta}|a_n|^2 > 0$ and $\delta > 1$ denotes the largest positive root of the equation

$$
|a_n|^2 z^3 - |a_n|^2 z^2 - \Phi z + (\Phi - \tilde{\Delta}|a_n|^2) = 0. \tag{2}
$$

In the case that $\Phi - \tilde{\Delta}|a_n|^2 < 0$, then all zeros of $f(z)$ lie in the closed disk $K(0, \delta)$, where $\delta > 1$ is the unique positive root of Equation 2.

Proof: Starting again from the complex polynomial

$$
Q(z) := (a_{n-1} - a_n z)f(z),
$$

we infer

$$
\begin{aligned}
|Q(z)| &= |a_n|^2\Bigg[|z|^{n+1} - \bigg|\frac{a_{n-1}a_{n-1} - a_n a_{n-2}}{a_n{}^2}\bigg||z|^{n-1} \\
&\quad - \bigg[\bigg|\frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_n{}^2}\bigg||z|^{n-2} \\
&\quad + \cdots + \bigg|\frac{a_{n-1}a_0}{a_n{}^2}\bigg|\bigg]\Bigg], \\
&= |a_n|^2\bigg[|z|^{n+1} - \bigg|\frac{a_{n-1}a_{n-1} - a_n a_{n-2}}{a_n{}^2}\bigg||z|^{n-1} \\
&\quad - \tilde{\Delta}\frac{|z|^{n-1} - 1}{|z| - 1}\bigg].
\end{aligned}
$$

If we now assume $|z| > 1$ and by setting

$$
\Phi := |a_{n-1}^2 - a_n a_{n-2}|,
$$

we further obtain

$$
\begin{aligned}
|Q(z)| &> |a_n|^2\bigg[|z|^{n+1} - \frac{\Phi}{|a_n|^2}|z|^{n-1} - \tilde{\Delta}\frac{|z|^{n-1}}{|z| - 1}\bigg] \\
&= \frac{1}{|z| - 1}\bigg[|a_n|^2|z|^{n+2} - |a_n|^2|z|^{n+1} \\
&\quad - \Phi|z|^n + |z|^{n-1}(\Phi - \Delta|a_n|^2)\bigg], \\
&= \frac{|z|^{n-1}}{|z| - 1}\bigg[|a_n|^2|z|^3 - |a_n|^2|z|^2 - \Phi|z| \\
&\quad + (\Phi - \Delta|a_n|^2)\bigg].
\end{aligned}
$$

We now define

$$
H(z) := |a_n|^2|z|^3 - |a_n|^2|z|^2 - \Phi|z| + (\Phi - \Delta|a_n|^2),
$$

and assume that $\Phi - \Delta|a_n|^2 > 0$. Then, from the definition of $H(z)$ it follows that $H(z)$ has exactly two sign changes in his coefficient sequence. Further, we observe that

$$
H(1) = -\Delta|a_n|^2 < 0, \ H(0) = \Phi - \Delta|a_n|^2 > 0,
$$

and

$$
\lim_{z \to +\infty} H(z) = +\infty.
$$

Hence, we conclude that $H(z)$ possesses exactly two positive zeros. We denote the largest one as δ. From the equations above we finally get $\delta > 1$. Hence, we have proved that all zeros of $Q(z)$ lie in the closed disk $K(0, \delta)$ and δ is the largest positive root of Equation 2. Due to the fact that all zeros of $f(z)$ are zeros of $Q(z)$, the first assertion of the theorem holds also for $f(z)$.

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:1, No:2, 2007

Now, we assume that $\Phi - \Delta |a_n|^2 < 0$. Then, we observe that $H(z)$ has exactly one sign change in his coefficient sequence. Applying Theorem 2.2 again, we conclude that $H(z)$ has a unique positive zero δ. Further, it holds

$$H(1) = -\Delta |a_n|^2 < 0 \quad \text{and} \quad H(0) = \Phi - \Delta |a_n|^2 < 0.$$

Finally, it follows that $\delta > 1$. Hence, the second assertion of the theorem holds for the zeros of $Q(z)$ and finally also for the zeros of $f(z)$. This finalizes the proof. ∎

The next theorem is more general in a sense that it provides a family of zero bounds for all zeros of a given complex polynomial.

Theorem 3.4: Let $f(z)$ be a complex polynomial (see Equation 1) and $\mathbb{N} \ni p \geq 1$. We further define

$$\Lambda := \max_{0 \leq j \leq n+p-1} \frac{M_j}{|a_n|}.$$

Then all zeros of $f(z)$ lie in the closed disk $K(0, \max(1, \delta))$, where $\delta \neq 1$ denotes the positive root of the equation

$$z^{n+p+1} - z^{n+p}(\Lambda + 1) + \Lambda = 0. \tag{3}$$

Proof:
We define the complex polynomial

$$P(z) := (1-z)^p f(z) = \sum_{j=0}^{p} \binom{p}{j} (-1)^j z^j f(z).$$

By simple algebraical rearrangements, we obtain

$$P(z) = \binom{p}{p}(-1)^p a_n z^{n+p}$$
$$+ z^{n+p-1}\left[\binom{p}{p}(-1)^p a_{n-1} + \binom{p}{p-1}(-1)^{p-1} a_n\right]$$
$$+ z^{n+p-2}\left[\binom{p}{p}(-1)^p a_{n-2} + \binom{p}{p-1}(-1)^{p-1} a_{n-1}\right.$$
$$\left. + \binom{p}{p-2}(-1)^{p-2} a_n\right] + \cdots$$
$$+ z^n\left[\binom{p}{p}(-1)^p a_{n-p} + \binom{p}{p-1}(-1)^{p-1} a_{n-p+1}\right.$$
$$\left. + \cdots + \binom{p}{0}(-1)^0 a_n\right] + \cdots$$
$$+ z\left[\binom{p}{1}(-1)^1 a_0 + \binom{p}{0}(-1)^0 a_1\right] + \binom{p}{0}(-1)^0 a_0.$$

Based on the definition ($p \leq n$ and $j \leq p$)

$$M_{n+p-j} := \sum_{i=0}^{j} a_{n-j+i} \binom{p}{p-i} (-1)^{p-i},$$

we get for the modulus of $P(z)$

$$\begin{aligned} |P(z)| &\geq |a_n||z|^{n+p} - \{M_{n+p-1}|z|^{n+p-1} \\ &\quad + \cdots + M_1|z| + M_0\}, \\ &\geq |a_n|\left[|z|^{n+p} - \Lambda\left(\frac{|z|^{n+p}-1}{|z|-1}\right)\right], \\ &= \frac{|a_n|}{|z|-1}\left[|z|^{n+p+1} - |z|^{n+p}(\Lambda+1) + \Lambda\right], \end{aligned}$$

where $\Lambda = \max\left(\frac{M_0}{|a_n|}, \frac{M_1}{|a_n|}, \ldots, \frac{M_{n+p-1}}{|a_n|}\right)$. If we now define $H(z) := |z|^{n+p+1} - |z|^{n+p}(\Lambda+1) + \Lambda$, we see that $H(1) = 0$. By using Theorem 2.2, we conclude that $H(z)$ has exactly two positive zeros. It further holds

$$\lim_{z \to +\infty} H(z) = +\infty.$$

Just as in Theorem 3.1 we finally infer that all zeros of $P(z)$ lie in $K(0, \max(1, \delta))$. But once again, all zeros of $f(z)$ are zeros of $P(z)$ and hence the theorem is completely proved. ∎

Now, we see immediately that a special choice of p leads us to special bounds. As a corollary, we get

Corollary 3.5: If we set $p = 0$ in Equation 3, we obtain a known bound which has been originally proven by DEHMER [7].

B. Explicit Zero Bounds

In order to finalize our theoretical Section III, we state a theorem that is similar to prove as Theorem 3.4. Thereby, Theorem 3.6 is a *explicit* zero bound. We call a zero bound *explicit* if the bound can be directly computed based on the polynomial coefficients.

Theorem 3.6: Let $f(z)$ be a complex polynomial (see Equation 1) and $\mathbb{N} \ni p \geq 1$. Just as in Theorem 3.4, we define

$$\Lambda := \max_{0 \leq j \leq n+p-1} \frac{M_j}{|a_n|}.$$

All zeros of $f(z)$ lie in the closed disk

$$K(0, 1 + \Lambda). \tag{4}$$

As a direct consequence of Theorem 3.6, we find

Corollary 3.7: If we set $p = 0$ in the Closed Disk 4, we obtain a classical result of CAUCHY [8], [13].

IV. EXPERIMENTAL RESULTS

In this section we briefly describe our experiment for comparing the zero bounds. We performed our experiments based on a set S_f of 1000 complex valued polynomials with $\deg(f(z)) = 19 \, (\forall f(z) \in S_f)$. The coefficients were randomly chosen in such a way that their absolute values follow an uniform distribution and for all polynomials it holds the relation $0 \leq |a_i| \leq 30, i = 0, 1, \ldots, \deg(f(z))$. We want to mention that the main goal of this experimental study was to determine which bounds gives us optimal values, based on the chosen set S_f. In order to achieve this, we computed the ratios of the corresponding zero bounds. For example, Figure 2 depicts the bound ratios of Theorem 3.2 and Theorem 3.4. We see

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
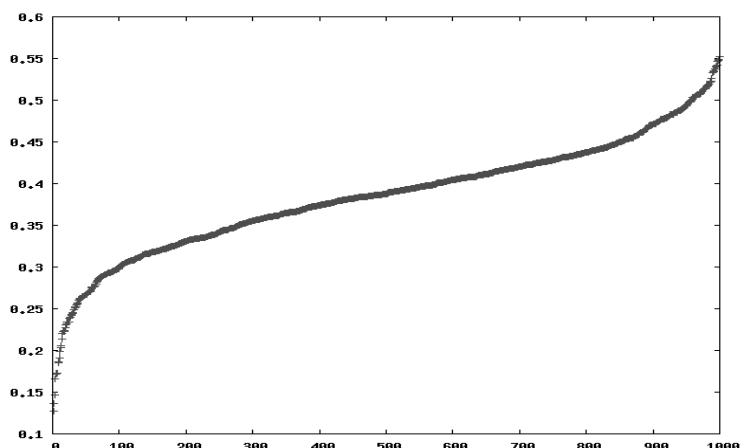Vol:1, No:2, 2007

Fig. 2.   The x-axis shows the polynomial index, sorted by ascending y-axis values. The y-axis shows the ratios of the bounds of Theorem 3.2 and Theorem 3.4. Further it holds that the minimum value of the ratio of the computed bounds is in this case $\min_{3.2/3.4} \doteq 0.1277$. For the maximum value, it holds $\max_{3.2/3.4} \doteq 0.5527$. The symbol $\doteq$ denotes truncated float numbers.

Bound ratio	$\bar{m}$ (Th. i/Th. j)	σ
3.2/3.1	0.9270	0.0524
3.2/3.3	0.9999	$1.1719 \cdot 10^{-5}$
3.2/3.4 ($p = 2$)	0.3849	0.0676
3.2/3.6 ($p = 2$)	0.3849	0.0676
3.4/3.6 ($p = 2$)	0.9999	$2.1522 \cdot 10^{-12}$

Fig. 3.   Experimental comparison of zero bounds based on the set S_f. $\bar{m}$ (Th. i/Th. j) denotes the mean of all ratios of the bounds of Theorem i and Theorem j. The symbol σ denotes the associated standard deviation.

that the distribution of ratios follows approximately a normal distribution. As shown in Figure 4, this is not the case between the bound ratios of Theorem 3.2 and Theorem 3.1. Table 3 shows the mean and standard deviation of the comparison of the novel zero bounds presented as described above. As a main significant characteristic for comparing the novel zero bounds, we chose the ratios of the zero bounds (of the corresponding theorems). The observed numbers and the calculated statistical values show that the bounds of Theorem 3.2 are optimal in the very most cases. The main reason for this is related with the number of terms in the concomitant polynomial of Theorem 3.2 and the constant $\tilde{\Delta}$. Generally, in case if the leading coefficient of a given polynomial is very large, then $\tilde{\Delta}$ is mostly smaller than $M := \max_{0 \leq j \leq n-1} \left| \frac{a_j}{a_n} \right|$, where M often appears in classical results, see [13], [8]. For example, in terms of the bound ratio of Theorem 3.2 and Theorem 3.1 (see Figure 4) we observe that the mean value is close to 1 and the corresponding σ is relative small. Further, that means that based on the observed mean the chance is very small (correlates with σ) for finding samples such that the bound of Theorem 3.1 is better than the bound of Theorem 3.2. We further conclude that the bounds of Theorem 3.4 and Theorem 3.6 give us the worst bounds. Another observation from Table 3 is that the bound values of Theorem 3.4 and Theorem 3.6 are almost equal. This is mainly indicated by the corresponding mean and standard deviation value. In order to

show this mathematically, we take a closer look at Equation 3 and assume that $\delta \neq 1$ is a positive root of Equation 3. Generally, we get

$$\delta^{n+p+1} - \delta^{n+p}(\Lambda + 1) + \Lambda = 0,$$

and

$$\delta + \frac{\Lambda}{\delta^{n+p}} = \Lambda + 1.$$

From the last equation it follows immediately that if $n + p$ is sufficiently large, then we infer $\delta \approx \Lambda + 1$. But $\Lambda + 1$ is exactly the direct bound of Theorem 3.6. The observation described above holds for all $p \in \mathbb{N}$. In terms of the bound ratio of Theorem 3.2 and Theorem 3.3, we can argue in a similar way.

V. CONCLUSION

In this work we presented some novel zero bounds for locating the zeros of complex valued polynomials. We showed that there are basically two types of zero bounds: implicit and explicit zero bounds. Here, we called a zero bound implicit if the zero bound is also a zero of a concomitant polynomial obtained from the underlying proof. A zero bound is called explicit if the bound can directly be determined from the polynomial coefficients. Most of our proven results are implicit bounds. As a special theoretical result we noticed that Theorem 3.4 and Theorem 3.6 provide families of bounds (generalizations), depending on the parameter $p \in \mathbb{N}$. If we set $p = 0$, we obtain an already known [7] and also a classical result of CAUCHY [8], [13]. The main result of the experimental section is that based on the chosen set S_f, we found that the bound of Theorem 3.2 is optimal, compared to the remaining ones. Starting from the algebraical definitions of zero bounds, we generally do not see (a priory) which bound is the best one (without numerically computing the bound values). We conclude that it is difficult to obtain general assertions for describing the quality of zero bounds. In fact, the process of deriving quality assertions of zero bounds must be associated with a certain set of polynomials. Finally, we want to emphasize that for our experiments we used

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
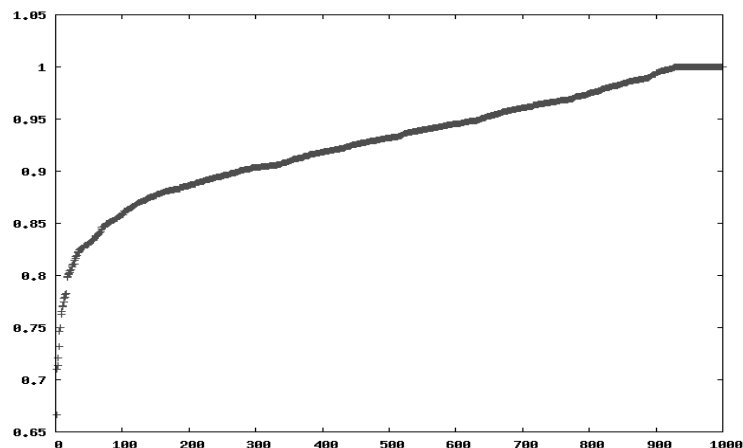Vol:1, No:2, 2007

Fig. 4. The axis designation is equal as in Figure (2). This picture shows the ratios of the bounds of Theorem 3.2 and Theorem 3.1. Further it holds $\min_{3.2/3.1} \doteq 0.6664$ and $\max_{3.2/3.1} \doteq 1.000$.

polynomials with a certain degree and a certain value range for the moduli of the coefficients. Thus our experimental results can yet not be generalized for arbitrary cases. As future work, we therefore like to analyze the dependencies between the observer statistical numbers ($\bar{m}$, σ) and the parameters of the evaluated polynomials ($\deg(f(z))$, value range for the moduli of the coefficients) in more detail. We are also interested in analyzing the distribution type of the ratios between the different bounds in depth. Here, we want to find the reason why, for example, the ratio between Theorem 3.2 and Theorem 3.6 follows a normal distribution while the ratio of Theorem 3.2 and Theorem 3.1 follows a different distribution.

VI. ACKNOWLEDGMENTS

REFERENCES

[1] R. E Allerdice, On a limit of the roots of an equation that is independent of all but two of the coefficients. Bull. Amer. Math. Soc. Vol. 13, 1906/07, 443–447.
[2] H. Anai, K. Horimoto, Algebraic Biology 2005. Proceedings of the First International Conference on Algebraic Biology. Computer Algebra in Biology, 2005, Tokyo/Japan.
[3] C. Bissel, Control Engineering. 2nd Edition, CRC Press, 1996.
[4] P. Borwein, T. Erdelyi, Polynomials and Polynomial Inequalities. Springer, 1995.
[5] J. E. Brown, G. Xiang, Proof of the Sendov conjecture for polynomials of degree at most eight. Journal of Mathematical Analysis and Applications, Vol. 232, No2, 1999, 272–292.
[6] A. Byrne, Some results for the Sendov conjecture. Journal of Mathematical Analysis and Applications, Vol. 199, No3, 1996, 754–768.
[7] M. Dehmer, On the location of zeros of complex polynomials. Journal of Inequalities in Pure and Applied Mathematics, Vol. 7, Issue 1, 2006.
[8] M. Dehmer, Die analytische Theorie der Polynome. Nullstellenschranken für komplexwertige Polynome. Weissensee-Verlag, Berlin, 2004.
[9] J. Dieudonné, La théorie des polynômes d'une variable. Mémor. Sci. Math., Vol. 93, 1938.
[10] B. Fine, G. Rosenberger, The Fundamental Theorem of Algebra. Springer, 1997.
[11] W. Heitzinger W. I. Troch, G. Valentin, Praxis nichtlinearer Gleichungen. Carl Hanser Verlag, München-Wien, 1985.
[12] P. Henrici, Applied and computational complex analysis. Volume I. Wiley Classics Library Edition, 1988.
[13] M. Marden, Geometry of polynomials. Mathematical Surveys. Amer. Math. Soc., Rhode Island, Vol. 3, 196.
[14] M. Mignotte, D. Stefanescu, Estimates for polynomial roots. Appl. Alg. Eng. Comm. Comp., Vol. 12, 2001, 437–453.
[15] M. Mignotte, D. Stefanescu, Polynomials: An Algorithmic Approach. Springer Series in Discrete Mathematics and Theoretical Computer Science, 1999
[16] G. V. Milovanovic, D. S. Mitrinovic, T. M. Rassias, Topics in Polynomials. Extremal Problems, Inequalities, Zeros. World Scientific, 1994
[17] N. Obreschkoff, Verteilung und Berechnung der Nullstellen reeller Polynome. Hochschulbücher für Mathematik. Vol. 55. Berlin, VEB Deutscher Verlag der Wissenschaften, 196.
[18] R. Pereira, Differentiators and the geometry of polynomials. J. Math. Anal. Appl. 285, Vol. 1, 2003, 336–348.
[19] R. Pereira, Trace Vectors in Matrix Analysis. PhD thesis, University of Toronto, 2003.
[20] V. V. Prasolov, Polynomials. Springer, 2004.
[21] Q. I. Rahman, G. Schmeisser, Analytic Theory of Polynomials. Critical Points, Zeros and Extremal Properties. Clarendon Press, 2002.
[22] F. Rühs, Funktionentheorie. VEB Deutscher Verlag der Wissenschaften Berlin, 1971.
[23] H. Sagan, Boundary and Eigenvalue Problems in Mathematical Physics. Dover Publications, 1989.
[24] W. Specht, Die Lage der Nullstellen eines Polynoms. Mathematische Nachrichten, Vol. 15, 1956, 353–374.
[25] U. Zölzer, Digital Audio Signal Processing. John Wiley & Sons, 1997.