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Cycle embedding in folded hypercubes with more faulty elements

Wen-Yin Huang, Jia-Jie Liu, and Jou-Ming Chang

Abstract—Faults in a network may take various forms such as hardware/software errors, vertex/edge faults, etc. Folded hypercube is a well-known variation of the hypercube structure and can be constructed from a hypercube by adding a link to every pair of nodes with complementary addresses. Let FF_v (respectively, FF_e) be the set of faulty nodes (respectively, faulty links) in an n-dimensional folded hypercube FQ_n. Hsieh et al. have shown that $FQ_n - FF_v - FF_e$ for $n \geq 3$ contains a fault-free cycle of length at least $2^n - 2|FF_v|$, under the constraints that (1) $|FF_v| + |FF_e| \leq 2n - 4$ and (2) every node in FQ_n is incident to at least two fault-free links. In this paper, we further consider the constraints $|FF_v| + |FF_e| \leq 2n - 3$. We prove that $FQ_n - FF_v - FF_e$ for $n \geq 5$ still has a fault-free cycle of length at least $2^n - 2|FF_v|$, under the constraints : (1) $|FF_v| + |FF_e| \leq 2n - 3$, (2) $|FF_e| \geq n + 2$, and (3) every vertex is still incident with at least two links.

Keywords—Folded hypercubes; Interconnection networks; Cycle embedding; Faulty elements.

I. Introduction

HYPERCUBES are a powerful network that is able to perform various kinds of parallel computations and simulate many other networks [14], [15]. Hypercubes have been widely studied in interconnection networks [6], [7], [8], [20]. A number of other topologies, such as paths, trees, rings, and meshes, can be embedded into a hypercube. There are also many related results in hypercubes with faulty vertices or link [2], [3], [5], [13], [16]. One of the most popular variants is the *folded hypercube*, which is an extension of the hypercube and can be constructed by adding a link to every pair of nodes with complementary address. The folded hypercube has been shown to be able to improve the system's performance over a regular hypercube in many measurements [1], [17].

Since faults may happen on both nodes and edges in a network, it is practically meaningful and important to consider faulty networks. A node is *fault-free* if it is not faulty. A link is *fault-free* if the communication link between end-nodes is not faulty. A path (cycle) is *fault-free* if it contains neither faulty nodes nor faulty links. Previously, the problem of fault-tolerant embedding on an n-dimensional folded hypercube FQ_n has been studied in [9], [10], [17], [18], [19]. Let FF_v (respectively, FF_e) be the set of faulty nodes (respectively, faulty links) in an n-dimensional folded hypercube FQ_n. Hsieh et al. [12] have shown that $FQ_n - FF_v - FF_e$ for $n \geq 3$ contains a fault-free cycle of length at least $2^n - 2|FF_v|$, under the constraints that (1) $|FF_v| + |FF_e| \leq 2n - 4$ and (2) every node in FQ_n is incident to at least two fault-free links. In this paper, we further consider the constraints $|FF_v| + |FF_e| \leq 2n - 3$. We prove that $FQ_n - FF_v - FF_e$ for $n \geq 5$ still has a fault-free cycle of length at least $2^n - 2|FF_v|$, under the constraints : (1) $|FF_v| + |FF_e| \leq 2n - 3$, (2) $|FF_e| \geq n + 2$, and (3) every vertex is still incident with at least two links.

The rest of this paper is organized as follows. In Section 2, we describe some important properties in folded hypercubes. We present our main result in Section 3. Concluding remarks are given in Section 4.

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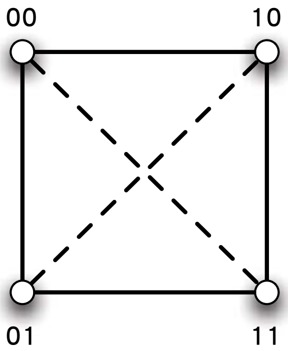
II. Preliminaries

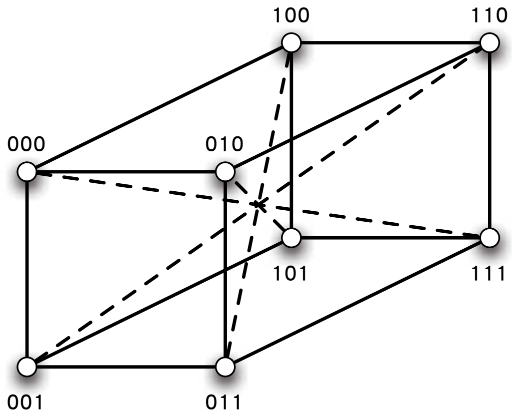
An n-dimensional hypercube Q_n, also called an *n-cube*, can be modeled as a graph with vertex set $V(Q_n)$ and edge set $E(Q_n)$. In Q_n, there are 2^n vertices and $n2^{n-1}$ links. Each vertex u of Q_n can be distinctly labeled by an n-bit string $b_n b_{n-1} \cdots b_2 b_1$. For any i, $1 \leqslant i \leqslant n$, we use $u^{(i)}$ to denote the binary string $b_n b_{n-1} \cdots \bar{b}_i b_{i-1} \cdots b_1$. Thus, if vertices u and v are adjacent, then $u = v^{(i)}$ and $v = u^{(i)}$ for some $1 \leqslant i \leqslant n$ and we call the edge $uu^{(i)}$ an *i-dimensional edge*. We will also refer to the edge $uu^{(i)}$ as $d^i(u)$. Thus, if $v = u^{(i)}$, then $v^{(j)} = (u^{(i)})^{(j)}$ is simplified as $u^{(i)(j)}$. Let $E_i = \{d^i(u) | u \in V(Q_n)\}$, i.e., the set containing all i-dimensional edges of Q_n. It is clear that $|E_i| = 2^{n-1}$ for every $1 \leqslant i \leqslant n$.

An n-dimensional folded hypercube FQ_n can be constructed from an n-dimensional hypercube by adding a link to every pair of nodes with complementary addresses, e.g., node $x = b_n b_{n-1} \cdots b_2 b_1$ and node $\bar{x} = \bar{b}_n \bar{b}_{n-1} \cdots \bar{b}_2 \bar{b}_1$. Thus FQ_n has 2^{n-1} more links than a regular hypercube. We call these extra links *skips* to distinguish them from regular links. Let E_s be the set of skips in FQ_n. Figure 1 illustrates a 2-dimensional and a 3-dimensional folded hypercubes.

A *path* $\mathcal{P}$ of length k from vertex x to vertex y in FQ_n is a sequence of distinct vertices $v_0, v_1, \ldots, v_k$ in which $x = v_0$, $y = v_k$, and $v_i v_{i+1} \in E(FQ_n)$, for $i = 0, 1, \ldots k-1$, where $k \geqslant 1$. We also use $\langle v_0, \mathcal{P}, v_k \rangle$ as another representation of $\mathcal{P}$ in order to indicate the two endpoints v_0 and v_k of $\mathcal{P}$. For consistency, an edge uv can also be represented as a path $\langle u, v \rangle$. For two paths $\langle x, \mathcal{P}, y \rangle$ and $\langle u, \mathcal{Q}, v \rangle$ in which y and u are adjacent, we use $\langle x, \mathcal{P}, y, u, \mathcal{Q}, v \rangle$ to denote the concatenation of paths $\mathcal{P}$ and $\mathcal{Q}$. A *cycle* is also a sequence of distinct vertices $v_0, v_1, \ldots, v_k$ except $v_0 = v_k$. In the following, we introduce some previous results that will be employed later.



(a) FQ_2



(b) FQ_3

Fig. 1. Graphs of FQ_2 and FQ_3, in which complementary links are drawn by dashed lines.

Lemma 1 ([19]). *There is an automorphism δ of FQ_n such that $\delta(E_i) = E_j$ for $i, j \in \{1, 2, \ldots, n\} \cup \{s\}$.*

It directly derives the following corollary.

Corollary 2. *$FQ_n - E_i$ is isomorphic to Q_n for $i \in \{1, 2, \ldots, n\} \cup \{s\}$.*

In an n-dimensional faulty hypercube Q_n, let F_v and F_e be the sets of faulty nodes and faulty links of Q_n, respectively. On the problem of finding the lower bound of longest fault-free cycle in Q_n, Du et al. [4] have shown the result as Lemma 3.

Lemma 3 ([4]). *$Q_n - F_v - F_e$ for $n \geq 3$ contains a fault-free cycle of length at least $2^n - 2|F_v|$ if (1) $|F_v| + |F_e| \leq 2n - 4$ and $|F_e| \leq 2n - 5$ and (2) every node in Q_n is incident to at least two fault-free links.*

Lemma 4 ([11]). *Every edge of $Q_n - F_v - F_e$ lies on a cycle of every length from 4 to $2^n - 2|F_v|$ even if $|F_v| + |F_e| \leq n - 2$, where $n \geq 3$.*

III. Fault-free cycle in the faulty folded hypercubes

In this section, we present our main result on considering the constraints that (1) $|FF_v| + |FF_e| \leq 2n - 3$, (2) $|FF_e| \geq n + 2$, and (3) every vertex in FQ_n is incident with at least two links, as shown in Theorem 6. In an n-dimensional faulty folded hypercube FQ_n, we call a non-faulty node *k-free* if it is incident to at most k fault-free links.

Lemma 5. *If $|FF_v| + |FF_e| \leq 2n - 3$, there are at most two 2-free nodes contained in FQ_n.*

Proof. By the definition of k-free node, a 2-free nodes is adjacent to at least $n - 1$ faulty elements, included faulty links and faulty nodes. Since $|FF_v| + |FF_e| \leq 2n - 3$, there is at most two 2-free nodes contained in FQ_n and these two nodes are adjacent with a common faulty link, say (u, v) (see Figure 3 as an example). ■

Theorem 6. *$FQ_n - FF_v - FF_e$, for $n \geq 5$ contains a fault-free cycle of length at least $2^n - 2|FF_v|$ if (1) $|FF_v| + |FF_e| \leq 2n - 3$, (2) $|FF_e| \geq n + 2$, and (3) every vertex is incident with at least two links.*

Proof. We consider the following three cases according to the number of 2-free nodes:

Case 1: FQ_n contains no 2-free node.

Since $|FF_e| \geq n + 2$, there exists a dimension i such that $F(E_i) \geq 2$, for $i \in \{1, 2, \ldots, n\} \cup \{s\}$. By Corollary 2, $FQ_n - E_i$ is isomorphic to Q_n. Thus, $|FF_v| + |FF_e| \leq 2n - 5$ in Q_n. Since every node in FQ_n is k-free for some $k \geq 3$, every node in Q_n is incident at least two fault-free links. By Lemma 3, there exists a fault-free cycle of length $2^n - 2|F_v|$ (=$2^n - 2|FF_v|$) in $Q_n - F_v - F_e$ since $|F_v| + |F_e| \leq 2n - 4$, $|F_e| \leq 2n - 5$, and every node in Q_n is incident to at least two fault-free links. Therefore, we obtain that $FQ_n - FF_v - FF_e$ contains a fault-free cycle of length at least $2^n - 2|FF_v|$.

Case 2: There is a unique 2-free node u in FQ_n and every node in $FQ_n - \{u\}$ is k-free for some $k \geq 3$.

Assume without loss of generality that $d^1(u)$ and $d^2(u)$ are two non-faulty links and either $d^i(u)$ is faulty link or $u^{(i)}$ is a faulty node, for $i \in \{3, 4, \ldots, n\} \cup \{s\}$. Since $|FF_e| \geq n + 2$, there exists a dimension j such that $F(E_j) \geq 2$, for $j \in \{1, 2, \ldots, n\} \cup \{s\}$. If $j \notin \{1, 2\}$, $FQ_n - E_j$ is isomorphic to Q_n. With the same arguments as Case 1, we have that $FQ_n - E_j$ also satisfies the constraints in Lemma 3. It derives that $FQ_n - FF_v - FF_e$ contains a fault-free cycle of length at least $2^n - 2|FF_v|$.

Now, we consider the case that $j \in \{1, 2\}$. There are two subcases to consider.

Subcase 2.1: There exists a faulty link $d^a(u)$ such that $d^1(u^{(a)})$ is an non-faulty link and $u^{(a)}$ and $u^{(a)(1)}$ are non-faulty nodes, where $a \in \{3, 4, \ldots, n\} \cup \{s\}$ (see Figure 2(a)).

Hence, $FQ_n - E_k$ is isomorphic to Q_n, where $k \in \{3, 4, \ldots, n\} \cup \{s\} - \{a\}$. Furthermore, Q_n can be decomposed to Q^L_{n-1} and Q^R_{n-1} at dimension 1 and $u \in Q^L_{n-1}$. Assume that $d^a(u)$ is an non-faulty link. Let F^L_v and F^L_e (respectively, F^R_v and F^R_e) denote the set of faulty nodes and faulty links in Q^L_{n-1} (respectively, Q^R_{n-1}), respectively. Since u is a 2-free node, $F(E_1) \geq 2$, $F(E_k) \geq 1$, and $d^a(u)$ is an non-faulty link, $|F^L_v| + |F^L_e| \geq n - 1$ and $|F^L_v| + |F^L_e| \leq 2n - 3 - 4 = 2n - 7$. Let $F^L(w)$ denote the set of faulty elements adjacent to node w, where $w \in Q^L_{n-1}$. Since $F^L(u) = n - 3$ and $|F^L_v| + |F^L_e| \leq 2n - 7$, $F^L(j) \leq n - 3$ for all $j \in Q^L_{n-1}$ except u. Thus, every node in Q^L_{n-1} is incident at least two fault-free links. By Lemma 3, there exists a fault-free cycle $\mathcal{C}^L$ of length $2^{n-1}-2|F_v^L|$ in $Q_{n-1}^L - F_v^L - F_e^L$ since $|F_v^L|+|F_e^L| \leq 2n-6$, $|F_e^L| \leq 2n-7$, and every node in Q_{n-1}^L is incident to at least two fault-free links.

If $u, u^{(a)} \in \mathcal{C}^L$, then we denote $u^{(1)}$ and $u^{(a)(1)}$ by x and y, respectively; otherwise, we choice any link $(p,q) \in \mathcal{C}^L$ such that $d^1(p)$ and $d^1(q)$ are two non-faulty links and denote $p^{(1)}$ and $q^{(1)}$ by x and y, respectively. Since u is a 2-free node, $F(E_1) \geq 2$, and $|FF_v|+|FF_e| \leq 2n-3$, $|F_v^R|+|F_e^R| \leq n-4$. By Lemma 4, edge $d^a(x)$ lies on a fault-free cycle $\mathcal{C}^R$ of length $2^{n-1} - 2|F_v^R|$ in $Q_{n-1}^R - F_v^R - F_e^R$ since $|F_v^R| + |F_e^R| \leq (n-1) - 2$. Therefore, we can obtain a fault-free cycle $\langle u, \mathcal{C}^L, u^{(a)}, y, \mathcal{C}^R, x, u \rangle$ (respectively, $\langle p, \mathcal{C}^L, q, y, \mathcal{C}^R, x, p \rangle$) of length $2^{n-1}-2|F_v^L|-1+2^{n-1}-2|F_v^R|-1+2=2^n-2|FF_v|$.

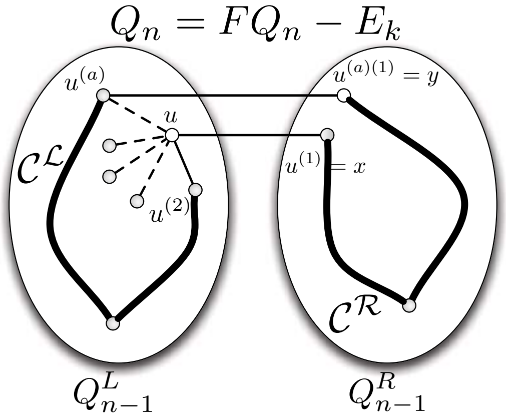
Subcase 2.2: If $d^a(u)$ is a faulty link, then $d^1(u^{(a)})$ is also a faulty link, for $a \in \{3,4,\ldots,n\} \cup \{s\}$ (see Figure 2(b)).

Since $|FF_v| + |FF_e| \leq 2n-3$ and $|FF_e| \geq n+2$, $|FF_v| \leq n-5$. If $|FF_v| = 0$, then $|FF_e| \geq 2n-2$ since every faulty link $d^a(u)$ is adjacent to another faulty link $d^1(u^{(a)})$, for $a \in \{3,4,\ldots,n\} \cup \{s\}$. Therefore, $|FF_v| > 0$. Since u is a 2-free node in FQ_n and $|FF_v| \leq n-5$, there exists at least four faulty links, say $d^3(u)$, $d^4(u)$, $d^5(u)$, and $d^6(u)$, such that $d^1(u^{(3)})$, $d^1(u^{(4)})$, $d^1(u^{(5)})$, and $d^1(u^{(6)})$ are also fault. Hence, $FQ_n - E_3$ is isomorphic to Q_n and Q_n can be decomposed to Q_{n-1}^L and Q_{n-1}^R at dimension 4 and $u \in Q_{n-1}^L$. Note that, $d^1(u^{(3)})$, $d^1(u^{(5)})$, and $d^1(u^{(6)})$ are in Q_{n-1}^L while $d^1(u^{(4)})$ is in Q_{n-1}^R. Thus, $|F_v^L|+|F_e^L| \leq 2n-3-3 = 2n-6$. Since $|FF_v| > 0$ and $|F_v^L| + |F_e^L| \leq 2n-6$, $|F_e^L| \leq 2n-7$. Since $F^L(u) = n-3$, $|F_v^L| + |F_e^L| \leq 2n-6$, and $d^1(u^{(3)})$, $d^1(u^{(5)})$ and $d^1(u^{(6)})$ are in Q_{n-1}^L, $F^L(j) \leq n-4$ for all $j \in Q_{n-1}^L$ except u. Thus, every node in Q_{n-1}^L is incident at least two fault-free links. By Lemma 3, there exists a fault-free cycle $\mathcal{C}^L$ of length $2^{n-1} - 2|F_v^L|$ in $Q_{n-1}^L - F_v^L - F_e^L$ since $|F_v^L| + |F_e^L| \leq 2n-6$, $|F_e^L| \leq 2n-7$, and every node in Q_{n-1}^L is incident to at least two fault-free links.

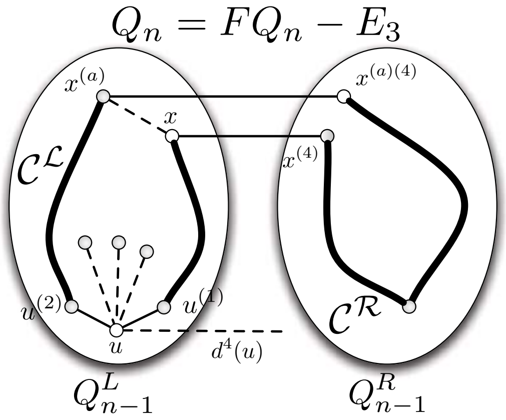
Choose any link, say $d^a(x)$, in $\mathcal{C}^L$ such that $x^{(4)}$ and $x^{(a)(4)}$ are non-faulty nodes in Q_{n-1}^R and $d^4(x)$ and $d^4(x^{(a)})$ are non-faulty links. Since u is a 2-free node, both $d^1(u^{(3)})$ and $d^1(u^{(5)})$ are in Q_{n-1}^L, and $|FF_v| + |FF_e| \leq 2n-3$, $|F_v^R| + |F_e^R| \leq n-5$. By Lemma 4 again, edge $d^a(x^{(4)})$ lies on a fault-free cycle $\mathcal{C}^R$ of length $2^{n-1}-2|F_v^R|$ in $Q_{n-1}^R - F_v^R - F_e^R$ since $|F_v^R| + |F_e^R| \leq (n-1) - 2$. Therefore, we can obtain a fault-free cycle $\langle x, \mathcal{C}^L, x^{(a)}, x^{(a)(4)}, \mathcal{C}^R, x^{(4)}, x \rangle$ of length $2^{n-1} - 2|F_v^L| - 1 + 2^{n-1} - 2|F_v^R| - 1 + 2 = 2^n - 2|FF_v|$.

Case 3: There are two 2-free nodes u and v in FQ_n.

Since $|FF_v| + |FF_e| \leq 2n-3$ and there are two 2-free nodes u and v in FQ_n, u and v are adjacent, $|FF_v|+|FF_e| = 2n-3$. Assume without loss of generality that link $(u,v) = d^1(u) = d^1(v)$. Assume that $d^a(u)$, $d^b(u)$, $d^c(v)$, and $d^d(v)$ are four non-fautly links with respect to u and v, where $a \neq b$, $c \neq d$, and $a,b,c,d \in \{2,3,\ldots,n\} \cup \{s\}$. Since $n \geq 5$ and $|FF_e| \geq n+2$, there exists a dimension k such that $d^k(u)$ and $d^k(v)$ are faulty links, where $k \in \{2,3,\ldots,n\} \cup \{s\} - \{a,b,c,d\}$. By Corollary 2, $FQ_n - E_k$ is isomorphic to Q_n. Thus, $|FF_v| + |FF_e| \leq 2n-5$ in Q_n and every node in Q_n is incident at least two fault-free links. By Lemma 3, there exists a fault-free cycle of length $2^n - 2|F_v|$ (=$2^n - 2|FF_v|$) in $Q_n - F_v - F_e$ since $|F_v| + |F_e| \leq 2n-4$, $|F_e| \leq 2n-5$, and every node in Q_n is incident to at least two fault-free links. Therefore, we obtain that $FQ_n - FF_v - FF_e$ contains a fault-free cycle of length at least $2^n - 2|FF_v|$.



(a) Constraint (1)



(b) Constraint (2)

Fig. 2. An illustration of Constraints (1) and (2).

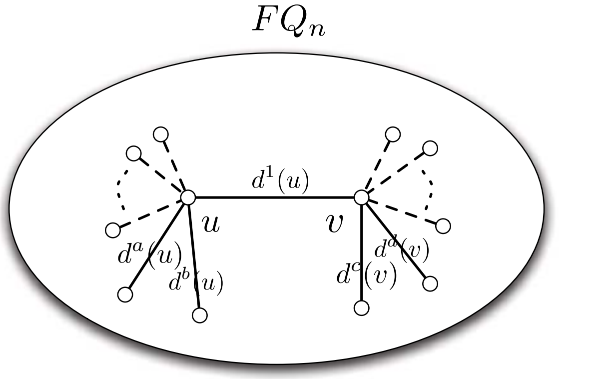


Fig. 3. There are two 2-free nodes u and v in FQ_n.

■

IV. Conclusion

In this paper, we consider the n-dimensional folded hypercube with some faulty elements with the constraints that (1)

$|FF_v| + |FF_e| \leq 2n - 3$, (2) $|FF_e| \geq n + 2$, and (3) every vertex is still incident with at least two links. We proved that $FQ_n - FF_v - FF_e$ for $n \geq 5$ has a fault-free cycle of length at least $2^n - 2|FF_v|$. In the further work, we interest to consider whether $FQ_n - FF_v - FF_e$ for $n \geq 5$ still has a fault-free cycle of length at least $2^n - 2|FF_v|$ under the constraints : (1) $|FF_v| + |FF_e| \leq 2n - 3$, (2) $|FF_e| < n + 2$, and (3) every vertex is still incident with at least two links.

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