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Ruin Probabilities with Dependent Rates of Interest and Autoregressive Moving Average Structures

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Abstract—This paper studies ruin probabilities in two discrete-time risk models with premiums, claims and rates of interest modelled by three autoregressive moving average processes. Generalized Lundberg inequalities for ruin probabilities are derived by using recursive technique. A numerical example is given to illustrate the applications of these probability inequalities.

Keywords—Lundberg inequality, NWUC, Renewal recursive technique, Ruin probability

I. Introduction

For over a century, ruin theory has been of major interest in actuarial science. Since a large portion of the surplus of insurance business comes from investment income, actuaries have been studying ruin problems under risk models with interest force. For example, Sundt and Teugels [5], [6] studied the effects of constant rate on the ruin probability under the compound Poisson risk model. Yang [8] established both exponential and non-exponential upper bounds for ruin probabilities in a risk model with constant interest force and independent premiums and claims. Cai [1] investigated the ruin probabilities in two risk models with independent premiums and claims and used a first-order autoregressive process to model the rates of interest. Cai and Dickson [2] obtained Lundberg inequalities for ruin probabilities in two discrete-time risk processes with a Markov chain interest model and independent premiums and claims.

In classic risk theory, the surplus process of insurance business is usually assumed to have independent and stationary increments. However, because of the increasing complexity of insurance and reinsurance products, actuaries have been paying more and more attention to the modelling of dependent risk. For example, Gerber [3] assumed that the surplus process could be written as an initial surplus plus the annual gains and used a linear model to model the annual gains. Yang and Zhang [9] investigated a discrete-time risk model with constant interest force and adopted first-order autoregressive processes to model both the premiums and claims.

In this paper, we generalize the models considered by Cai [1] to the case that the premiums, claims and rates of interest have autoregressive moving average (ARMA) dependent structures simultaneously. Recursive equations for finite-time ruin probabilities and integral equations for ultimate ruin probabilities are given. Generalized Lundberg inequalities for ruin probabilities are derived. A numerical example is given to illustrate the accuracy of the upper bounds.

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Let $\{Y_n, n = 1, 2, \dots\}$ be a sequence of nonnegative random variables, where Y_n represents the total amount of claims during the nth period, i.e. from time $n-1$ to time n, and satisfies

$$Y_n = \rho_1 Y_{n-1} + W_n + \rho_2 W_{n-1}, \quad 0 \le \rho_1, \rho_2 < 1 \tag{1}$$

with $Y_0 = y_0 \ge 0, W_0 = w_0 \ge 0$ and $\{W_n, n = 1, 2, \dots\}$ being a sequence of independent, identically distributed (i.i.d.) and nonnegative random variables. One possible interpretation of model (1) is the following: the parameter ρ_1 is the proportion of old business, which will remain in the new portfolio; while W_n is the uncertainty to claims occurring in the nth period, and ρ_2 measures the degree of correlation. Model (2) below can be interpreted in a similar way.

Let $\{X_n, n = 1, 2, \dots\}$ be another sequence of nonnegative random variables, where X_n denotes the total amount of premiums during the nth period, and satisfies

$$\begin{aligned} X_n &= a_1 X_{n-1} + \cdots + a_p X_{n-p} \\ &\quad + Z_n + c_1 Z_{n-1} + \cdots + c_q Z_{n-q} \end{aligned} \tag{2}$$

with $0 \le a_1, \dots, a_p, c_1, \dots, c_q < 1$, $X_j = x_j \ge 0 (j = 0, -1, \dots, -p+1)$, $Z_k = z_k \ge 0 (k = 0, -1, \dots, -q+1)$, and $\{Z_n, n = 1, 2, \dots\}$ being a sequence of i.i.d. and nonnegative random variables.

Let $\{I_n, n = 1, 2, \dots\}$ be another sequence of nonnegative random variables, where I_n denotes the rate of interest during the nth period and satisfies

$$\begin{aligned} I_n &= b_1 I_{n-1} + \cdots + b_s I_{n-s} \\ &\quad + R_n + d_1 R_{n-1} + \cdots + d_t R_{n-t} \end{aligned} \tag{3}$$

with $0 \le b_1, \dots, b_s, d_1, \dots, d_t < 1$, $I_j = i_j \ge 0 (j = 0, -1, \dots, -s+1)$, $R_k = r_k \ge 0 (k = 0, -1, \dots, -t+1)$, and $\{R_n, n = 1, 2, \dots\}$ being a sequence of i.i.d. and nonnegative random variables.

Assume the processes $\{W_n, n = 1, 2, \dots\}$, $\{Z_n, n = 1, 2, \dots\}$ and $\{R_n, n = 1, 2, \dots\}$ are mutually independent. Denote $F(w) = \mathbb{P}(W_1 \le w)$, $G(z) = \mathbb{P}(Z_1 \le z)$ and $H(r) = \mathbb{P}(R_1 \le r)$ with $F(0) = 0$.

Suppose that the claims are paid at the end of each period and there are two styles of premium collections. On one hand, if the premiums are collected at the beginning of each period, then the surplus process $\{U_n^{(1)}, n = 1, 2, \dots\}$ with initial surplus u is of form

$$U_n^{(1)} = (U_{n-1}^{(1)} + X_n)(1 + I_n) - Y_n, \tag{4}$$

which can be rearranged as

$$U_n^{(1)} = u\prod_{k=1}^{n}(1+I_k) + \sum_{k=1}^{n}\left((X_k(1+I_k)-Y_k)\prod_{j=k+1}^{n}(1+I_j)\right), \quad (5)$$

where $\prod_{j=a}^{b}(1+I_j)=1$ if $a>b$. On the other hand, if the premiums are collected at the end of each period, then the surplus process $\{U_n^{(2)}, n=1,2,\dots\}$ with initial surplus u can be written as

$$U_n^{(2)} = U_{n-1}^{(2)}(1+I_n)+X_n-Y_n, \quad (6)$$

which is equivalent to

$$U_n^{(2)} = u\prod_{k=1}^{n}(1+I_k) + \sum_{k=1}^{n}\left((X_k-Y_k)\prod_{j=k+1}^{n}(1+I_j)\right). \quad (7)$$

Mathematically, models (5) and (7) are the generalizations of surplus processes considered by Cai [1], Yang [8] and Yang and Zhang [9].

For notational convenience, define

$$\begin{aligned} x^{(1)} &= (x_0,\dots,x_{-p+1},z_0,\dots,z_{-q+1}),\\ y^{(1)} &= (y_0,w_0),\\ i^{(1)} &= (i_0,\dots,i_{-s+1},r_0,\dots,r_{-t+1}). \end{aligned}$$

Then, denote the finite-time ruin probability and ultimate ruin probability of model (5) with (1)-(3), respectively, by

$$\Psi_n(u,y^{(1)},x^{(1)},i^{(1)}) = \mathbb{P}\left(\bigcup_{j=1}^{n}\{U_j^{(1)}<0\}\right),$$
$$\Psi(u,y^{(1)},x^{(1)},i^{(1)}) = \mathbb{P}\left(\bigcup_{j=1}^{\infty}\{U_j^{(1)}<0\}\right).$$

And denote the finite-time ruin probability and ultimate ruin probability of model (7) with (1)-(3), respectively, by

$$\Phi_n(u,y^{(1)},x^{(1)},i^{(1)}) = \mathbb{P}\left(\bigcup_{j=1}^{n}\{U_j^{(2)}<0\}\right),$$
$$\Phi(u,y^{(1)},x^{(1)},i^{(1)}) = \mathbb{P}\left(\bigcup_{j=1}^{\infty}\{U_j^{(2)}<0\}\right).$$

It is clear that

$$\lim_{n\to\infty}\Psi_n=\Psi, \quad \lim_{n\to\infty}\Phi_n=\Phi.$$

II. Integral Equations for Ruin Probabilities

Throughout this paper, denote the tail of any distribution function B by $\overline{B}(x)=1-B(x)$. We first give a recursive equation for Ψ_n and an integral equation for Ψ. For notational convenience, define

$$\begin{aligned} \eta_1 &= a_1x_0+\cdots+a_px_{-p+1}+c_1z_0+\cdots+c_qz_{-q+1},\\ \eta_2 &= c_1i_0+\cdots+c_si_{-s+1}+d_1r_0+\cdots+d_tr_{-t+1},\\ \eta_3 &= \rho_1y_0+\rho_2w_0. \end{aligned}$$

Clearly, $X_1=\eta_1+Z_1$, $I_1=\eta_2+R_1$ and $Y_1=\eta_3+W_1$.

Theorem 2.1. *For $n=1,2,\dots$, we have*

$$\begin{aligned} &\Psi_{n+1}(u,y^{(1)},x^{(1)},i^{(1)})\\ &= \int_0^\infty\int_0^\infty \overline{F}(\hbar_{z,r})\mathrm{d}G(z)\mathrm{d}H(r)\\ &\quad+\int_0^\infty\int_0^\infty\int_0^{\hbar_{z,r}}\mathrm{d}F(w)\mathrm{d}G(z)\mathrm{d}H(r)\\ &\qquad\times\Psi_n(\hbar_{z,r}-w,y^{(2)},x^{(2)},i^{(2)}), \end{aligned}$$

and

$$\begin{aligned} &\Psi(u,y^{(1)},x^{(1)},i^{(1)})\\ &= \int_0^\infty\int_0^\infty \overline{F}(\hbar_{z,r})\mathrm{d}G(z)\mathrm{d}H(r)\\ &\quad+\int_0^\infty\int_0^\infty\int_0^{\hbar_{z,r}}\mathrm{d}F(w)\mathrm{d}G(z)\mathrm{d}H(r)\\ &\qquad\times\Psi(\hbar_{z,r}-w,y^{(2)},x^{(2)},i^{(2)}), \end{aligned}$$

where

$$\hbar_{z,r}=(u+x)(1+i)-\eta_3, \quad (8)$$

with

$$x=\eta_1+z, \quad i=\eta_2+r, \quad y=\eta_3+w, \quad (9)$$

and

$$x^{(2)}=(x,x_0,x_{-1}\dots,x_{-p+2},z,z_0,z_{-1}\dots,z_{-q+2}), \quad (10)$$
$$y^{(2)}=(y,w), \quad (11)$$
$$i^{(2)}=(i,i_0,i_{-1}\dots,i_{-s+2},r,r_0,r_{-1}\dots,r_{-t+2}). \quad (12)$$

Proof: Given $W_1=w, Z_1=z$ and $R_1=r$, from (5), we have

$$\begin{aligned} U_1^{(1)} &= (u+X_1)(1+I_1)-Y_1\\ &= (u+\eta_1+z)(1+\eta_2+r)-\eta_3-w=\hbar_{z,r}-w. \end{aligned}$$

Thus, if $w>\hbar_{z,r}$, then

$$\mathbb{P}(U_1^{(1)}<0 \mid W_1=w,Z_1=z,R_1=r)=1,$$

which implies that for $w>\hbar_{z,r}$,

$$\mathbb{P}\left(\bigcup_{k=1}^{n+1}\{U_k^{(1)}<0\} \;\middle|\; W_1=w,Z_1=z,R_1=r\right)=1;$$

while if $0\le w\le\hbar_{z,r}$, then

$$\mathbb{P}(U_1^{(1)}<0 \mid W_1=w,Z_1=z,R_1=r)=0. \quad (13)$$

Let $\{\widetilde{W}_n, n = 1,2,\dots\}$, $\{\widetilde{Z}_n, n = 1,2,\dots\}$ and $\{\widetilde{R}_n, n = 1,2,\dots\}$ be independent copies of $\{W_n, n = 1,2,\dots\}$, $\{Z_n, n = 1,2,\dots\}$ and $\{R_n, n = 1,2,\dots\}$, respectively. Given $W_1 = w$, consider process $\{\widetilde{Y}_n, n = 1,2,\dots\}$ which satisfies

$$\widetilde{Y}_n = \rho_1 \widetilde{Y}_{n-1} + \widetilde{W}_n + \rho_2 \widetilde{W}_{n-1}$$

with initial values $\widetilde{Y}_0 = \eta_3 + w = y$ and $\widetilde{W}_0 = w$. Apparently, $\{\widetilde{Y}_n, n = 1,2,\dots\}$ has a similar structure to that of $\{Y_n, n = 1,2,\dots\}$ but with different initial values. Given $Z_1 = z$, consider process $\{\widetilde{X}_n, n = 1,2,\dots\}$ which satisfies

$$\begin{aligned}\widetilde{X}_n &= a_1 \widetilde{X}_{n-1} + \dots + a_p \widetilde{X}_{n-p} \\ &\quad + \widetilde{Z}_n + c_1 \widetilde{Z}_{n-1} + \dots + c_q \widetilde{Z}_{n-q}\end{aligned}$$

with initial values $\widetilde{X}_0 = \eta_1 + z = x$, $\widetilde{X}_{j-1} = x_j (j = 0, -1, -2, \dots, -p+2)$, $\widetilde{Z}_0 = z$, $\widetilde{Z}_{k-1} = z_k (k = 0, -1, -2, \dots, -q+2)$. Clearly, $\{\widetilde{X}_n, n = 1,2,\dots\}$ has a similar structure to that of $\{X_n, n = 1,2,\dots\}$, but with different initial values. Similarly, given $R_1 = r$, consider process $\{\widetilde{I}_n, n = 1,2,\dots\}$ which satisfies

$$\begin{aligned}\widetilde{I}_n &= b_1 \widetilde{I}_{n-1} + \dots + b_s \widetilde{I}_{n-s} \\ &\quad + \widetilde{R}_n + d_1 \widetilde{R}_{n-1} + \dots + d_t \widetilde{R}_{n-t}\end{aligned}$$

with initial values $\widetilde{I}_0 = \eta_2 + r = i$, $\widetilde{I}_{j-1} = i_j (j = 0, -1, -2, \dots, -s+2)$, $\widetilde{R}_0 = r$, $\widetilde{R}_{k-1} = r_k (k = 0, -1, -2, \dots, -t+2)$. Obviously, $\{\widetilde{I}_n, n = 1,2,\dots\}$ has a similar structure to that of $\{I_n, n = 1,2,\dots\}$, but with different initial values. Thus, (13) and (5) imply that for $0 \le w \le \hbar_{z,r}$,

$$\begin{aligned}
&\mathbb{P}\left(\bigcup_{k=1}^{n+1} \{U_k^{(1)} < 0\} \,\Big|\, W_1 = w, Z_1 = z, R_1 = r\right) \\
&= \mathbb{P}\left(\bigcup_{k=2}^{n+1} \{U_k^{(1)} < 0\} \,\Big|\, W_1 = w, Z_1 = z, R_1 = r\right) \\
&= \mathbb{P}\left(\bigcup_{k=2}^{n+1} \left\{(\hbar_{z,r} - w) \prod_{j=2}^{k} (1 + I_j) \right.\right. \\
&\qquad \left.\left. + \sum_{j=2}^{k} (X_j(1 + I_j) - Y_j) \prod_{t=j+1}^{k} (1 + I_t) < 0 \right\}\right) \\
&= \mathbb{P}\left(\bigcup_{k=1}^{n} \left\{(\hbar_{z,r} - w) \prod_{j=1}^{k} (1 + \widetilde{I}_j) \right.\right. \\
&\qquad \left.\left. + \sum_{j=1}^{k} (\widetilde{X}_j(1 + \widetilde{I}_j) - \widetilde{Y}_j) \prod_{t=j+1}^{k} (1 + \widetilde{I}_t) < 0 \right\}\right) \\
&= \Psi_n(\hbar_{z,r} - w, y^{(2)}, x^{(2)}, i^{(2)}).
\end{aligned}$$

Therefore, by conditioning on W_1, Z_1 and R_1, we can get

$$\begin{aligned}
&\Psi_{n+1}(u, y^{(1)}, x^{(1)}, i^{(1)}) \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \mathrm{d}F(w)\mathrm{d}G(z)\mathrm{d}H(r) \\
&\quad \times \mathbb{P}\left(\bigcup_{k=1}^{n+1} \{U_k^{(1)} < 0\} \Big| W_1 = w, Z_1 = z, R_1 = r\right) \\
&= \int_0^\infty \int_0^\infty \int_{\hbar_{z,r}}^\infty \mathrm{d}F(w)\mathrm{d}G(z)\mathrm{d}H(r) \\
&\quad + \int_0^\infty \int_0^\infty \int_0^{\hbar_{z,r}} \mathrm{d}F(w)\mathrm{d}G(z)\mathrm{d}H(r) \\
&\qquad \times \Psi_n(\hbar_{z,r} - w, y^{(2)}, x^{(2)}, i^{(2)}) \\
&= \int_0^\infty \int_0^\infty \overline{F}(\hbar_{z,r})\mathrm{d}G(z)\mathrm{d}H(r) \\
&\quad + \int_0^\infty \int_0^\infty \int_0^{\hbar_{z,r}} \mathrm{d}F(w)\mathrm{d}G(z)\mathrm{d}H(r) \\
&\qquad \times \Psi_n(\hbar_{z,r} - w, y^{(2)}, x^{(2)}, i^{(2)}).
\end{aligned} \tag{14}$$

Thus, from the dominated convergence theorem, the integral equation for Ψ in Theorem 2.1 follows immediately by letting $n \to \infty$ in (14). ■

Similarly, the following recursive equation for Φ_n and integral equation for Φ hold.

Theorem 2.2. *For* $n = 1,2,\dots,$

$$\begin{aligned}
&\Phi_{n+1}(u, y^{(1)}, x^{(1)}, i^{(1)}) \\
&= \int_0^\infty \int_0^\infty \overline{F}(\ell_{z,r})\mathrm{d}G(z)\mathrm{d}H(r) \\
&\quad + \int_0^\infty \int_0^\infty \int_0^{\ell_{z,r}} \mathrm{d}F(w)\mathrm{d}G(z)\mathrm{d}H(r) \\
&\qquad \times \Phi_n(\ell_{z,r} - w, y^{(2)}, x^{(2)}, i^{(2)})
\end{aligned}$$

and

$$\begin{aligned}
&\Phi(u, y^{(1)}, x^{(1)}, i^{(1)}) \\
&= \int_0^\infty \int_0^\infty \overline{F}(\ell_{z,r})\mathrm{d}G(z)\mathrm{d}H(r) \\
&\quad + \int_0^\infty \int_0^\infty \int_0^{\ell_{z,r}} \mathrm{d}F(w)\mathrm{d}G(z)\mathrm{d}H(r) \\
&\qquad \times \Phi(\ell_{z,r} - w, y^{(2)}, x^{(2)}, i^{(2)})
\end{aligned}$$

with y, x, i, $y^{(2)}$, $x^{(2)}$ *and* $i^{(2)}$ *specified in* (9)-(12) *and* $\ell_{z,r} = u(1+i) + x - \eta_3$.

III. Probability Inequalities for Ruin Probabilities

Using the recursive equations for Ψ_n and Φ_n, we can derive probability inequalities for Ψ and Φ by an inductive approach. We first give the probability inequality for Ψ.

Theorem 3.1. *Suppose that there exists some constant* $\gamma_1 > 0$ *satisfying*

$$\mathbb{E}\mathrm{e}^{\gamma_1[(1+\rho_1+\rho_2)W_1 - Z_1(1+R_1) + \eta_3]} = 1. \tag{15}$$

Then, for any $\eta_1 \geq \eta_3$,

$$\Psi(u, y^{(1)}, x^{(1)}, i^{(1)}) \tag{16}$$
$$\leq \beta_1 \mathbb{E}e^{\gamma_1(1+\rho_1+\rho_2)W_1} \mathbb{E}e^{-\gamma_1[(u+X_1)(1+I_1)-\eta_3]} \tag{17}$$

with

$$\beta_1^{-1} = \inf_{t \geq 0} \frac{\int_t^\infty e^{\gamma_1(1+\rho_1+\rho_2)w} dF(w)}{e^{\gamma_1(1+\rho_1+\rho_2)t}\overline{F}(t)}. \tag{18}$$

Proof: For any $t \geq 0$, we have

$$\overline{F}(t) = \left(\frac{\int_t^\infty e^{\gamma_1(1+\rho_1+\rho_2)w} dF(w)}{e^{\gamma_1(1+\rho_1+\rho_2)t}\overline{F}(t)}\right)^{-1} e^{-\gamma_1(1+\rho_1+\rho_2)t}$$
$$\times \int_t^\infty e^{\gamma_1(1+\rho_1+\rho_2)w} dF(w)$$
$$\leq \beta_1 e^{-\gamma_1 t} \int_t^\infty e^{\gamma_1(1+\rho_1+\rho_2)w} dF(w) \tag{19}$$
$$\leq \beta_1 e^{-\gamma_1 t} \mathbb{E}e^{\gamma_1(1+\rho_1+\rho_2)W_1}. \tag{20}$$

Clearly, $(u+X_1)(1+I_1)-\eta_3 \geq 0$ when $\eta_1 \geq \eta_3$. Then, from (20) we have

$$\begin{aligned}
&\Psi_1(u, y^{(1)}, x^{(1)}, i^{(1)}) \\
&= \mathbb{P}(W_1 > (u+\eta_1+Z_1)(1+\eta_2+R_1)-\eta_3) \\
&= \int_0^\infty\int_0^\infty \overline{F}((u+\eta_1+z)(1+\eta_2+r)-\eta_3) dG(z) dH(r) \\
&\leq \beta_1 \mathbb{E}e^{\gamma_1(1+\rho_1+\rho_2)W_1} \\
&\quad \times \int_0^\infty\int_0^\infty e^{-\gamma_1[(u+\eta_1+z)(1+\eta_2+r)-\eta_3]} dG(z) dH(r) \\
&= \beta_1 \mathbb{E}e^{\gamma_1(1+\rho_1+\rho_2)W_1} \mathbb{E}e^{-\gamma_1[(u+\eta_1+Z_1)(1+\eta_2+R_1)-\eta_3]} \\
&= \beta_1 \mathbb{E}e^{\gamma_1(1+\rho_1+\rho_2)W_1} \mathbb{E}e^{-\gamma_1[(u+X_1)(1+I_1)-\eta_3]}.
\end{aligned}$$

Under an inductive hypothesis, we assume that for any $y_0, w_0, x_0, \ldots, x_{-p+1}, z_0, \ldots, z_{-q+1}, i_0, \ldots, i_{-s+1}, r_0, \ldots, r_{-t+1} \geq 0$ and $\eta_1 \geq \eta_3$,

$$\Psi_n(u, y^{(1)}, x^{(1)}, i^{(1)})$$
$$\leq \beta_1 \mathbb{E}e^{\gamma_1(1+\rho_1+\rho_2)W_1} \mathbb{E}e^{-\gamma_1[(u+X_1)(1+I_1)-\eta_3]} \tag{21}$$
$$\leq \beta_1 \mathbb{E}e^{\gamma_1(1+\rho_1+\rho_2)W_1} \mathbb{E}e^{-\gamma_1[(u+Z_1)(1+R_1)-\eta_3]}. \tag{22}$$

Take y, x, i, $y^{(2)}$, $x^{(2)}$, $i^{(2)}$ and $\hbar_{z,r}$ as in (9)-(12). Then, for $0 \leq w \leq \hbar_{z,r}$, by (15) and (22) we have

$$\begin{aligned}
&\Psi_n(\hbar_{z,r}-w, y^{(2)}, x^{(2)}, i^{(2)}) \\
&\leq \beta_1 \mathbb{E}e^{\gamma_1(1+\rho_1+\rho_2)W_1} \\
&\quad \times \mathbb{E}e^{-\gamma_1[(\hbar_{z,r}-w+Z_1)(1+R_1)-\rho_1(w+\eta_3)-\rho_2 w]} \\
&= \beta_1 \mathbb{E}e^{\gamma_1(1+\rho_1+\rho_2)W_1} \\
&\quad \times \mathbb{E}[e^{-\gamma_1[Z_1(1+R_1)-\rho_1\eta_3]} e^{-\gamma_1[(\hbar_{z,r}-w)(1+R_1)-(\rho_1+\rho_2)w]}] \\
&\leq \beta_1 \mathbb{E}e^{\gamma_1(1+\rho_1+\rho_2)W_1} \\
&\quad \times \mathbb{E}e^{-\gamma_1[Z_1(1+R_1)-\eta_3]} e^{-\gamma_1[\hbar_{z,r}-(1+\rho_1+\rho_2)w]} \\
&= \beta_1 e^{-\gamma_1[(u+x)(1+i)-\eta_3-(1+\rho_1+\rho_2)w]}.
\end{aligned} \tag{23}$$

Thus, by Theorem 2.1, (19) and (23), we get

$$\begin{aligned}
&\Psi_{n+1}(u, y^{(1)}, x^{(1)}, i^{(1)}) \\
&\leq \beta_1 \int_0^\infty\int_0^\infty e^{-\gamma_1[(u+x)(1+i)-\eta_3]} dG(z) dH(r) \\
&\qquad \times \int_{\hbar_{z,r}}^\infty e^{\gamma_1(1+\rho_1+\rho_2)w} dF(w) \\
&+ \beta_1 \int_0^\infty\int_0^\infty e^{-\gamma_1[(u+x)(1+i)-\eta_3]} dG(z) dH(r) \\
&\qquad \times \int_0^{\hbar_{z,r}} e^{\gamma_1(1+\rho_1+\rho_2)w} dF(w) \\
&= \beta_1 \int_0^\infty\int_0^\infty e^{-\gamma_1[(u+x)(1+i)-\eta_3]} dG(z) dH(r) \\
&\qquad \times \int_0^\infty e^{\gamma_1(1+\rho_1+\rho_2)w} dF(w) \\
&= \beta_1 \mathbb{E}e^{\gamma_1(1+\rho_1+\rho_2)W_1} \mathbb{E}e^{-\gamma_1[(u+X_1)(1+I_1)-\eta_3]}.
\end{aligned}$$

Thus, for all $n = 1, 2, \ldots$, (21) holds. Therefore, (17) follows by letting $n \to \infty$ in (21). ■

Similarly, we can obtain the following probability inequality for Φ.

Theorem 3.2. *Suppose that there exists some constant* $\gamma_2 > 0$ *satisfying*

$$\mathbb{E}e^{\gamma_2[(1+\rho_1+\rho_2)W_1 - Z_1 + \eta_3]} = 1. \tag{24}$$

Then, for any $\eta_1 \geq \eta_3$,

$$\Phi(u, y^{(1)}, x^{(1)}, i^{(1)})$$
$$\leq \beta_2 \mathbb{E}e^{\gamma_2(1+\rho_1+\rho_2)W_1} \mathbb{E}e^{-\gamma_2[u(1+I_1)+X_1-\eta_3]} \tag{25}$$

with

$$\beta_2^{-1} = \inf_{t \geq 0} \frac{\int_t^\infty e^{\gamma_2(1+\rho_1+\rho_2)w} dF(w)}{e^{\gamma_2(1+\rho_1+\rho_2)t}\overline{F}(t)}. \tag{26}$$

Refinements of upper bounds in Theorem 3.1 and Theorem 3.2 can be obtained when F is new worse than used in convex ordering (NWUC). A lifetime distribution B is said to be NWUC if for all $x \geq 0$, $y \geq 0$,

$$\int_{x+y}^\infty \overline{B}(t) dt \geq \overline{B}(x) \int_y^\infty \overline{B}(t) dt.$$

The class of NWUC distributions is larger than the class of decreasing failure rate (DFR) distributions. See Shaked and Shanthikumar [4] for properties of NWUC and other classes of lifetime distributions.

Corollary 3.1. *Under the conditions of Theorem 3.1 and Theorem 3.2, if F is NWUC and* $\eta_1 \geq \eta_3$, *then,*

$$\Psi(u, y^{(1)}, x^{(1)}, i^{(1)}) \leq \mathbb{E}e^{-\gamma_1[(u+X_1)(1+I_1)-\eta_3]}, \tag{27}$$

and

$$\Phi(u, y^{(1)}, x^{(1)}, i^{(1)}) \leq \mathbb{E}e^{-\gamma_2[u(1+I_1)+X_1-\eta_3]}. \tag{28}$$

Proof: From Proposition 6.1.1 of Willmot and Lin [7], we can get that if F is NWUC, then $\beta_1^{-1} = \mathbb{E}e^{\gamma_1(1+\rho_1+\rho_2)W_1}$ and $\beta_2^{-1} = \mathbb{E}e^{\gamma_2(1+\rho_1+\rho_2)W_1}$. Thus, by Theorem 3.1 and Theorem 3.2, we can conclude the proof. ■

The constants γ_1 defined in (15) and γ_2 defined in (24) are called adjustment coefficients. And the following remark gives sufficient conditions of the existences of γ_1 and γ_2.

Remark 3.1. If $\mathbb{E}[(1+\rho_1+\rho_2)W_1 - Z_1(1+R_1)+\eta_3] < 0$ and $\mathbb{P}((1+\rho_1+\rho_2)W_1 - Z_1(1+R_1)+\eta_3 > 0) > 0$ hold simultaneously, then, there exists a unique positive constant γ_1 satisfying (15).

If $\mathbb{E}[(1+\rho_1+\rho_2)W_1 - Z_1+\eta_3] < 0$ and $\mathbb{P}((1+\rho_1+\rho_2)W_1 - Z_1+\eta_3 > 0) > 0$ hold simultaneously, then, there exists a unique positive constant γ_2 satisfying (24).

Proof of Remark 3.1: Define

$$f(r) = \mathbb{E}\mathrm{e}^{r[(1+\rho_1+\rho_2)W_1 - Z_1(1+R_1)+\eta_3]} - 1. \tag{29}$$

Then,

$$f''(r) = \mathbb{E}\left\{[(1+\rho_1+\rho_2)W_1 - Z_1(1+R_1)+\eta_3]^2 \times \mathrm{e}^{r[(1+\rho_1+\rho_2)W_1 - Z_1(1+R_1)+\eta_3]}\right\} \geq 0,$$

which implies that $f(r)$ is a convex function with $f(0) = 0$ and

$$f'(0) = \mathbb{E}[(1+\rho_1+\rho_2)W_1 - Z_1(1+R_1)+\eta_3] < 0.$$

By $\mathbb{P}((1+\rho_1+\rho_2)W_1 - Z_1(1+R_1)+\eta_3 > 0) > 0$, we can find some constant $\delta > 0$ such that

$$\mathbb{P}\left((1+\rho_1+\rho_2)W_1 - Z_1(1+R_1)+\eta_3 > \delta\right) > 0.$$

Then, we can get that

$$\begin{aligned} f(r) &= \mathbb{E}\mathrm{e}^{r[(1+\rho_1+\rho_2)W_1 - Z_1(1+R_1)+\eta_3]} - 1 \\ &\geq \mathbb{E}\left[\mathrm{e}^{r[(1+\rho_1+\rho_2)W_1 - Z_1(1+R_1)+\eta_3]} \times \mathbb{I}_{[(1+\rho_1+\rho_2)W_1 - Z_1(1+R_1)+\eta_3>\delta]}\right] - 1 \\ &\geq \mathrm{e}^{\delta r}\mathbb{P}\left((1+\rho_1+\rho_2) - Z_1(1+R_1)+\eta_3 > \delta\right) - 1 \\ &\to \infty, \text{ as } r \to \infty. \end{aligned}$$

Therefore, there exists some (unique) constant $\gamma_1 > 0$ satisfying (15).

By the same approach, we can prove the existence of $\gamma_2 > 0$. ■

Now, we consider the relationship between γ_1 and γ_2.

Proposition 3.1. *Suppose that* $\mathbb{E}[(1+\rho_1+\rho_2)W_1 - Z_1+\eta_3] < 0$. *If there exists some constant* $\gamma_1 > 0$ *satisfying* (15) *and there exists some constant* $\gamma_2 > 0$ *satisfying* (24)*, then,* $\gamma_1 \geq \gamma_2$.

Proof: Recall that $f(r)$ is a convex function with $f(0) = 0$ and

$$\begin{aligned} f'(0) &= \mathbb{E}[(1+\rho_1+\rho_2)W_1 - Z_1(1+R_1)+\eta_3] \\ &\leq \mathbb{E}[(1+\rho_1+\rho_2)W_1 - Z_1+\eta_3] < 0. \end{aligned}$$

Then, γ_1 is the unique positive roots of equation $f(r) = 0$ on $(0,\infty)$. Furthermore, if $r > 0$ and $f(r) \leq 0$, then $\gamma_1 \geq r$. From (24), we have

$$\begin{aligned} f(\gamma_2) &= \mathbb{E}\mathrm{e}^{\gamma_2[(1+\rho_1+\rho_2)W_1 - Z_1(1+R_1)+\eta_3]} - 1 \\ &\leq \mathbb{E}\mathrm{e}^{\gamma_2[(1+\rho_1+\rho_2)W_1 - Z_1+\eta_3]} - 1 = 0. \end{aligned}$$

Thus, $\gamma_1 \geq \gamma_2$. ■

It is easy to see that for any $\eta_1 \geq \eta_3$,

$$\Psi(u, y^{(1)}, x^{(1)}, i^{(1)}) \leq \Phi(u, y^{(1)}, x^{(1)}, i^{(1)}). \tag{30}$$

This shows the impact of timing of premium payments on the ruin probabilities Ψ and Φ. It is natural to think of the relationship between upper bounds since (30) holds.

Denote the upper bounds in Theorem 3.1 and Theorem 3.2 respectively by $\Lambda_1(u, y^{(1)}, x^{(1)}, i^{(1)})$ and $\Lambda_2(u, y^{(1)}, x^{(1)}, i^{(1)})$. Then, from (15) and (24), we have

$$\Lambda_1(u, y^{(1)}, x^{(1)}, i^{(1)}) = \beta_1\mathbb{E}\mathrm{e}^{-\gamma_1[(u+\eta_1)(1+I_1)+\eta_2 Z_1]}, \tag{31}$$

and

$$\Lambda_2(u, y^{(1)}, x^{(1)}, i^{(1)}) = \beta_2\mathbb{E}\mathrm{e}^{-\gamma_2[u(1+I_1)+\eta_1]}. \tag{32}$$

From Proposition 3.1, we can get that

$$\begin{aligned} &\frac{\int_t^\infty \mathrm{e}^{\gamma_1(1+\rho_1+\rho_2)w}\mathrm{d}F(w)}{\mathrm{e}^{\gamma_1(1+\rho_1+\rho_2)t}\overline{F}(t)} \\ &= \frac{\int_t^\infty \mathrm{e}^{\gamma_1(1+\rho_1+\rho_2)(w-t)}\mathrm{d}F(w)}{\overline{F}(t)} \\ &\geq \frac{\int_t^\infty \mathrm{e}^{\gamma_2(1+\rho_1+\rho_2)(w-t)}\mathrm{d}F(w)}{\overline{F}(t)} \\ &= \frac{\int_t^\infty \mathrm{e}^{\gamma_2(1+\rho_1+\rho_2)w}\mathrm{d}F(w)}{\mathrm{e}^{\gamma_2(1+\rho_1+\rho_2)t}\overline{F}(t)}, \end{aligned}$$

which, using (18) and (26), implies that

$$\beta_1^{-1} \geq \beta_2^{-1}, \quad \text{or} \quad \beta_1 \leq \beta_2. \tag{33}$$

Thus, from (31)-(32) and (33), we have the following proposition.

Proposition 3.2. *Under the conditions of Proposition 3.1, we have, for any* $\eta_1 \geq \eta_3$,

$$\Lambda_1(u, y^{(1)}, x^{(1)}, i^{(1)}) \leq \Lambda_2(u, y^{(1)}, x^{(1)}, i^{(1)}). \tag{34}$$

IV. Numerical Example

In this section, we give a numerical example to illustrate the tightness of the upper bounds derived in the section above. We use 2500 time intervals so that the true ruin probability could be a little larger than its simulated result. The calculations are obtained by Maple software and R programming language.

Example 4.1. Let the claims be modelled by (1) with initial values $y_0 = w_0 = 0.1$ and coefficients $\rho_1 = \rho_2 = 0.1$. In addition, let $\{W_n, n = 1, 2, \dots\}$ have a common gamma density

$$f(w) = \frac{w^{\alpha-1}}{\lambda^\alpha\Gamma(\alpha)}\mathrm{e}^{-w/\lambda}, \quad w \geq 0$$

with shape $\alpha = 0.5$ and scale $\lambda = 1$. Here, $\Gamma(\cdot)$ denotes the gamma function.

Assume the premiums are modelled by an ARMA(3,3) process, namely, for all $n = 1, 2, \dots,$

$$\begin{aligned} X_n &= a_1X_{n-1} + a_2X_{n-2} + a_3X_{n-3} \\ &\quad + Z_n + c_1Z_{n-1} + c_2Z_{n-2} + c_3Z_{n-3}, \end{aligned} \tag{35}$$

TABLE I
UPPER BOUNDS AND RUIN FREQUENCIES (RF) OF EXAMPLE 4.1.

u	RF of (5)	RF of (7)	(27) for Ψ	(28) for Φ
0.5	0.2022	0.2122	0.5328	0.5440
1.5	0.0955	0.0988	0.3485	0.3594
2.5	0.0441	0.0490	0.2279	0.2375
3.5	0.0182	0.0237	0.1491	0.1569
4.5	0.0073	0.0116	0.0975	0.1036

with initial values $x_0 = x_{-1} = x_{-2} = 0.5$, $z_0 = z_{-1} = z_{-2} = 0.5$, coefficients $a_1 = c_1 = 0.1$, $a_2 = c_2 = 0.05$, $a_3 = c_3 = 0.01$. In addition, let $\{Z_n, n = 1, 2, \dots\}$ have a common Weibull density

$$g(z) = \frac{\eta}{\theta}\left(\frac{z}{\theta}\right)^{\eta-1} \mathrm{e}^{-(z/\theta)^{\eta}}, \quad z \geq 0$$

with shape $\eta = 2$ and scale $\theta = 1$.

Let the rates of interest follow an ARMA(3,3) process, i.e. for all $n = 1, 2, \dots,$

$$\begin{aligned} I_n &= b_1 I_{n-1} + b_2 I_{n-2} + b_3 I_{n-3} \\ &\quad + R_n + d_1 R_{n-1} + d_2 R_{n-2} + d_3 R_{n-3} \end{aligned} \tag{36}$$

with initial values $i_0 = i_{-1} = i_{-2} = 0.014$, $r_0 = r_{-1} = r_{-2} = 0.012$, coefficients $b_1 = d_1 = 0.1$, $b_2 = d_2 = 0.05$ and $b_3 = d_3 = 0.01$. In addition, suppose $\{R_n, n = 1, 2, \dots\}$ have a common uniform distribution on $[0.01, 0.014]$.

We can get that $\gamma_1 = 0.41782$, $\gamma_2 = 0.40794$, which supports Proposition 3.1. Since $0 < \alpha < 1$, the distribution function F of W_1 is DFR and hence NWUC. Notice that $\eta_1 = 0.16 > \eta_3 = 0.02$. Then, (27) applies to Ψ and (28) applies to Φ. The simulated results and upper bounds are given in Table I.

From Table I, we can see that the upper bounds are about two to ten times their ruin frequencies, respectively. However, it is not easy to obtain the true ruin probability in general. The upper bounds, like the ones in this paper, are very easy to obtain, and in most of the practical problems, we only need the upper bound for the ruin probability. It is evident that both the ruin frequency and upper bound decrease as the initial surplus u increases. Table I demonstrate exactly the same relationship between the upper bounds as shown in Proposition 3.2.

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REFERENCES

[1] Cai, J. Ruin probabilities with dependent rates of interest. J. Appl. Prob., 2002, 39(2): 312-323.
[2] Cai, J., Dickson, D. Ruin probabilities with a Markov chain interest model. Insurance Math. Econom., 2004, 35(3): 513-525.
[3] Gerber, H. Ruin theory in the linear model. Insurance Math. Econom., 1982, 1(3): 213-217.
[4] Shaked, M., Shanthikumar, J. Stochastic Orders and their Applications. Academic Press, San Diego, 1994.
[5] Sundt, B., Teugels, J. Ruin estimates under interest force. Insurance Math. Econom., 1995, 16(1): 7-22.
[6] Sundt, B., Teugels, J. The adjustment function in ruin estimates under interest force. Insurance Math. Econom., 1997, 19(2): 85-94.
[7] Willmot, G. E., Lin, X. S. Lundberg Approximations for Compound Distributions with Insurance Applications. Springer-Verlag, New York, 2001.
[8] Yang, H. Non-exponential bounds for ruin probability with interest effect included. Scand. Actuarial J., 1999, 1999(1): 66-79.
[9] Yang, H., Zhang, L. Martingale method for ruin probability in an autoregressive model with constant interest rate. Prob. Eng. Inf. Sci., 2003, 17(2): 183-198.