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Bisymmetric, Persymmetric Matrices and Its Applications in Eigen-decomposition of Adjacency and Laplacian Matrices

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***Abstract*—In this paper we introduce an efficient solution method for the Eigen-decomposition of bisymmetric and per symmetric matrices of symmetric structures. Here we decompose adjacency and Laplacian matrices of symmetric structures to sub-matrices with low dimension for fast and easy calculation of eigenvalues and eigenvectors. Examples are included to show the efficiency of the method.

***Keywords*—**Graphs theory, Eigensolution, adjacency and Laplacian matrix, Canonical forms, bisymmetric, per symmetric.

I. Introduction

CALCULATION of eigenvalues and eigenvectors of a matrix is important in any engineering problems [1]. Basic and fundamental calculations for stability, vibration and buckling analysis of structural systems require to solving generalized eigenvalue problem [2, 3]. For calculation of eigenvalues and eigenvectors of a matrix the characteristic equation of the matrix should be formed and the corresponding equation of order n should be solved [4]. Recently canonical forms are developed and used for Eigensolution of symmetric structured matrices arising in data analyzing of symmetric and regular structures [5, 6]. There are also classical methods for Eigensolution of structured matrices based on LU decomposition, preconditioning, divide and counter algorithms and other approximate methods [7, 8, 9]. In this paper, a simple and efficient method is presented for computing of the eigenvalues and eigenvectors of bisymmetric matrices. Here Bisymmetric matrices are decomposed into sub-matrices with low dimensions for simple and fast computing of eigenvalues and eigenvectors.

II. Basic Definitions of Graph Theory

A. Definitions from Graph Theory

A *graph* $G(N,E)$ consists of a set of elements, (G), called *nodes* and a set of elements, $E(G)$, called *edges,* together with a relation of incidence which associates two distinct nodes with each edge, known as its *ends*. Two nodes of a graph are called *adjacent* if these nodes are the end nodes of an edge. An edge is called *incident* with a node if it is an end node of the edge.

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The *degree* of a node is the number of edges incident with the node. A *sub-graph* G_i of a graph G is a graph for which $N(G_i) \subseteq N(G)$ and $E(G_i) \subseteq E(G)$, and each edge of G_i has the same ends as in G. A *path graph* P is a simple connected graph with $N(P) = E(P)+1$ that can be drawn in a way that all of its nodes and edges lie on a single straight line. A *cycle graph* C is a simply connected graph with $N(C) = E(C)$ that can be drawn so that all of its nodes and edges lie on a circle. A *path graph* and a *cycle graph* with n nodes are denoted by P_n and C_n, respectively.

B. Matrices Associated with a Graph

Let ***G*** be a graph with *n* nodes. The adjacency matrix ***A*** is an $n \times n$ matrix in which the entry in row *i* and column *j* is 1 if node n_i is adjacent to n_j, and is zero otherwise. This matrix is symmetric and the row sums of **A** are the degrees of nodes of G. The Laplacian matrix of graph *G* is defined as:

$$\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}. \tag{1}$$

Where ***D*** is a diagonal matrix in which the i-th diagonal entry is equal to the degree of node *i* [10].

III. Similarity Transformation of Matrices

A complex scalar λ_i is called an eigenvalue of the square matrix $\mathbf{A}_{n\times n}$ if a nonzero vector $\mathbf{v}_i$ exists such that $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$. The vector $\mathbf{v}_i$ is called an eigenvector of A associated with λ_i. The set of eigenvalues of **A** is called the spectrum of **A**. A scalar λ_i is an eigenvalue of **A** if and only if $\det(\mathbf{A} - \lambda_i \mathbf{I}) = 0$. That is true if and only if λ_i is a root of the characteristic polynomial. Two matrices **A** and **B** are said to be similar if there is a nonsingular matrix U such that:

$$\mathbf{B} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U} \tag{2}$$

The mapping A → B is called a similarity transformation. It can be shown that similarity transformations preserve the eigenvalues of matrices:

$$\mathbf{A}\mathbf{v}_i = \lambda \mathbf{v}_i, \tag{3}$$

$$\mathbf{U}^{-1}\mathbf{A}\mathbf{U}\mathbf{U}^{-1}\mathbf{v}_i = \mathbf{U}^{-1}\lambda \mathbf{v}_i, \tag{4}$$

By substituting $\mathbf{B} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}$ and $\mathbf{y}_i = \mathbf{U}^{-1}\mathbf{v}_i$, we will have:

$$\mathbf{B}\mathbf{y}_i = \lambda \mathbf{y}_i, \tag{5}$$

Equation (5) which is a standard representation of Eigen-problems means that λ_i are also the eigenvalues of the matrix **B** [18].

IV. Bisymmetric and Per Symmetric Matrixes

A. Bisymmetric Matrix

In mathematics, a bisymmetric matrix is a square matrix that is symmetric about both of its main diagonals. More precisely, an n×n matrix **M** is bisymmetric if and only if it satisfies $\mathbf{M} = \mathbf{M}^t$ and **M×S = S×M**, where S is the n×n exchange matrix.

$$\mathbf{S} = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{bmatrix}, \tag{6}$$

B. Persymmetric Matrix

In mathematics, persymmetric matrix may refer to a square matrix which is symmetric in the northeast-to-southwest diagonal or a square matrix such that the values on each line perpendicular to the main diagonal are the same for a given line. If B is persymmetric matrix

$$\mathbf{B}^t = \mathbf{SBS} \tag{7}$$

Where, **S** is the exchange matrix.

V. Decomposition of Bisymmetric Matrices

Consider the matrix **M**:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^t & \mathbf{SAS} \end{bmatrix}, \tag{8}$$

If $\mathbf{A}=\mathbf{A}^t$ & $\mathbf{B}^t = \mathbf{SBS}$, then it is obvious that, **M** is bisymmetric. Because:

$$\mathbf{M}=\mathbf{M}^t \ \& \ \mathbf{M}=\mathbf{SMS}, \tag{9}$$

For decomposition of **M**, it is necessary to introduce exchange matrix as:

$$\mathbf{S} = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{bmatrix}, \tag{10}$$

Now we form the matrix P (permutation matrix) as:

$$\mathbf{P} = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{S} & \mathbf{I} \\ \mathbf{I} & -\mathbf{S} \end{bmatrix}, \mathbf{I} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}, \tag{11,12}$$

P is orthogonal matrix so It is obvious that:

$$\mathbf{P}\mathbf{P}^t = \mathbf{I}, \tag{13}$$

So the following multiplying doesn't change the eigenvalues:

$$\mathbf{P}\mathbf{M}\mathbf{P}^t = \begin{bmatrix} \mathbf{A}-\mathbf{BS} & 0 \\ 0 & \mathbf{A}+\mathbf{BS} \end{bmatrix}, \tag{14}$$

This means that we can calculate eigenvalues and eigenvectors of matrix M with sub-matrices with low dimension than M, as:

$$eig(\mathbf{M}) = eig(\mathbf{A}+\mathbf{BS}) \bigcup eig(\mathbf{A}-\mathbf{BS}). \tag{15}$$

VI. Examples

A. Example 1 (Numerical)*: Consider the following sub-matrices:*

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 6 & -3 \\ 2 & -3 & 15 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 9 & 6 & 3 \\ 8 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \ \mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^t & \mathbf{SAS} \end{bmatrix}$$

In this example **A** is symmetric and **B** is persymmetric so we can calculate the eigenvalues of M using present method by eigenvalues of the following sub-matrices:

$$eig(\mathbf{M}) = eig(\mathbf{A}+\mathbf{BS}) \cup eig(\mathbf{A}-\mathbf{BS}).$$

eig(**A**+**BS**)=[0.6833, 9.1077, 30.2089],
eig(**A**-**BS**)=[-13.6225, 7.3721, 16.2504].

So the eigenvalues of matrix **M**:

eig(**M**)=[-13.6225, 0.6833, 7.3721, 9.1077, 16.2504, 30.2089].

B. Example 2 (graph theory):

Consider the graph (**G**) as;

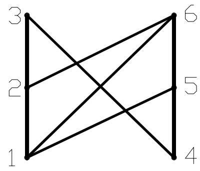


Fig. 1 Graph (**G**)

Adjacency matrix of graph (**G**) M and its sub-matrices **A**, **B** can be formed as:

$$\mathbf{M}=\begin{bmatrix}0&1&&&1&1\\1&0&1&&&1\\&1&0&1&&\\&&1&0&1&\\1&&&1&0&1\\1&1&&&1&0\end{bmatrix},\mathbf{A}=\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix},\mathbf{B}=\begin{bmatrix}&1&1\\&&1\\1&&\end{bmatrix},$$

$$\mathbf{M}=\begin{bmatrix}\mathbf{A}&\mathbf{B}\\\mathbf{B}^t&\mathbf{SAS}\end{bmatrix}.$$

Directly calculation of the eigenvalues of **M** yields:
eig(**M**)= (-1.7912, -1.6180, -1.0000, 0.6180, 1.0000, 2.7912)
Now we can decompose **M** to (**A**+**BS**) and (**A-BS**) so eigenvalues of **M**:

$$eig(\mathbf{M})=eig(\mathbf{A}+\mathbf{BS})\cup eig(\mathbf{A}-\mathbf{BS}),$$

$$\mathbf{A}+\mathbf{BS}=\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}+\begin{bmatrix}0&1&1\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}&&1\\&1&\\1&&\end{bmatrix}=$$

$$\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}+\begin{bmatrix}1&1&0\\1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&2&0\\2&0&1\\0&1&1\end{bmatrix},$$

eig(**A+BS**)=(-1.7912, 1.0000, 2.7912),

$$\text{A - BS}=\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}-\begin{bmatrix}0&1&1\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}&&1\\&1&\\1&&\end{bmatrix}=$$

$$\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}-\begin{bmatrix}1&1&0\\1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}-1&0&0\\0&0&1\\0&1&-1\end{bmatrix},$$

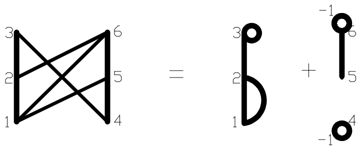
eig(**A-BS**)=(-1.61803, -1.0000, 0.61803).

Finally eigenvalues of **M** can be formed as:

$$eig(\mathbf{M})=eig(\mathbf{A}+\mathbf{BS})\cup eig(\mathbf{A}-\mathbf{BS}),$$

eig(**M**)= (-1.7912, -1.6180, -1.0000, 0.6180, 1.0000, 2.7912).

According the above calculation, we can decompose the graph **G** to sub-graph $\mathbf{G_1}$ and $\mathbf{G_2}$ in the following form:



$$(G)=(G_1\cup G_2)$$

Fig. 2 Graph (**G**) and its decomposition and healed form

*C. Example 3 (*structural mechanics*):*

Consider the truss models $\mathbf{G_1}$, $\mathbf{G_2}$, $\mathbf{G_3}$, $\mathbf{G_4}$ and their adjacency and Laplacian matrices of the truss model as:

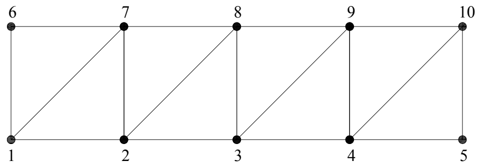


Fig. 3 Graph model of truss $\mathbf{G_1}$

$$Adj(G_1)=\begin{bmatrix}&1&&&&1&1&&&\\1&&1&&&&1&1&&\\&1&&1&&&&1&1&\\&&1&&1&&&&1&1\\&&&1&&&&&&1\\1&&&&&&1&&&\\1&1&&&&1&&1&&\\&1&1&&&&1&&1&\\&&1&1&&&&1&&1\\&&&1&1&&&&1&\end{bmatrix},$$

$$Lap(G_1)=\begin{bmatrix}3&-1&&&&-1&-1&&&\\-1&4&-1&&&&-1&-1&&\\&-1&4&-1&&&&-1&-1&\\&&-1&4&-1&&&&-1&-1\\&&&-1&2&&&&&-1\\-1&&&&&2&-1&&&\\-1&-1&&&&-1&4&-1&&\\&-1&-1&&&&-1&4&-1&\\&&-1&-1&&&&-1&4&-1\\&&&-1&-1&&&&-1&3\end{bmatrix}$$

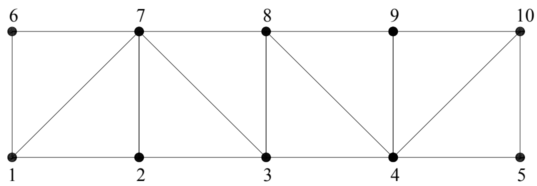


Fig. 4 Graph model of truss $\mathbf{G_2}$

$$Adj(G_3)=\begin{bmatrix}&1&&1&1&1&&\\1&&1&&&1&&\\&1&&1&&1&1&1\\1&&1&&&&&1\\1&&&&&1&&1\\1&1&1&&1&&1&\\&&1&&&1&&1\\&&1&1&1&&1&\end{bmatrix},$$

$$Lap(G_3)=\begin{bmatrix}4&-1&&-1&-1&-1&&\\-1&3&-1&&&-1&&\\&-1&5&-1&&-1&-1&-1\\-1&&-1&3&&&&-1\\-1&&&&3&-1&&-1\\-1&-1&-1&&-1&5&-1&\\&&-1&&&-1&3&-1\\&&-1&-1&-1&&-1&4\end{bmatrix},$$

$$Adj(G_2)=\begin{bmatrix} & 1 & & & & 1 & 1 & & & \\ 1 & & 1 & & & & 1 & & & \\ & 1 & & 1 & & & 1 & 1 & & \\ & & 1 & & 1 & & & 1 & 1 & 1 \\ & & & 1 & & & & & & 1 \\ 1 & & & & & & 1 & & & \\ 1 & 1 & 1 & & & 1 & & 1 & & \\ & & 1 & 1 & & & 1 & & 1 & \\ & & & 1 & & & & 1 & & 1 \\ & & & 1 & 1 & & & & 1 & \end{bmatrix},$$

$$Lap(G_2)=\begin{bmatrix} 3 & -1 & & & & -1 & -1 & & & \\ -1 & 3 & -1 & & & & -1 & & & \\ & -1 & 4 & -1 & & & -1 & -1 & & \\ & & -1 & 5 & -1 & & & -1 & -1 & -1 \\ & & & -1 & 2 & & & & & -1 \\ -1 & & & & & 2 & -1 & & & \\ -1 & -1 & -1 & & & -1 & 5 & -1 & & \\ & & -1 & -1 & & & -1 & 4 & -1 & \\ & & & -1 & & & & -1 & 3 & -1 \\ & & & -1 & -1 & & & & -1 & 3 \end{bmatrix},$$

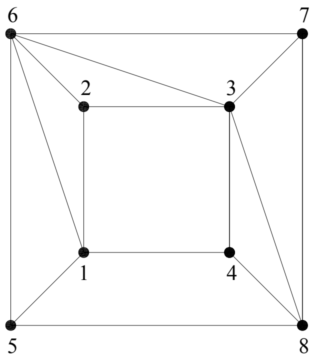


Fig. 5 raph model of truss $\mathbf{G_3}$

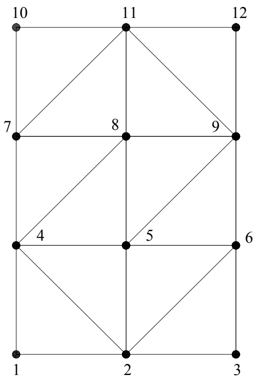


Fig. 6 Graph model of truss $\mathbf{G_4}$

$$Adj(\mathbf{G}_4)=\begin{bmatrix} & 1 & & 1 & & & & & & & & \\ 1 & & 1 & 1 & 1 & 1 & & & & & & \\ & 1 & & & & 1 & & & & & & \\ 1 & 1 & & & 1 & & 1 & 1 & & & & \\ & 1 & & 1 & & 1 & & 1 & 1 & & & \\ & 1 & 1 & & 1 & & & & 1 & & & \\ & & & 1 & & & & 1 & & 1 & 1 & \\ & & & 1 & 1 & & 1 & & 1 & & 1 & \\ & & & & 1 & 1 & & 1 & & & 1 & 1 \\ & & & & & & 1 & & & & 1 & \\ & & & & & & 1 & 1 & 1 & 1 & & 1 \\ & & & & & & & & 1 & & 1 & \end{bmatrix},$$

$$Lap(\mathbf{G}_4)=\begin{bmatrix} 2 & -1 & & -1 & & & & & & & & \\ -1 & 5 & -1 & -1 & -1 & -1 & & & & & & \\ & -1 & 2 & & & -1 & & & & & & \\ -1 & -1 & & 5 & -1 & & -1 & -1 & & & & \\ & -1 & & -1 & 5 & -1 & & -1 & -1 & & & \\ & -1 & -1 & & -1 & 4 & & & -1 & & & \\ & & & -1 & & & 4 & -1 & & -1 & -1 & \\ & & & -1 & -1 & & -1 & 5 & -1 & & -1 & \\ & & & & -1 & -1 & & -1 & 5 & & -1 & -1 \\ & & & & & & -1 & & & 2 & -1 & \\ & & & & & & -1 & -1 & -1 & -1 & 5 & -1 \\ & & & & & & & & -1 & & -1 & 2 \end{bmatrix},$$

In all of these examples adjacency and Laplacian matrices are persymmetric and sub-matrices hold in the defined conditions so we can decompose to smaller sub-matrices for easy and fast computing of their eigenvalues.

VII. Concluding Remarks

In this paper, a simple method is presented for calculating the eigenvalues of adjacency and Laplacian matrices of bisymmetric and persymmetric matrices of structural and graph theory models.

Examples studied here show that the results obtained by the present method are exact solution method for the problem. The calculated eigenvalues are exact values, and can efficiently be used for solution of the models whose structural matrices are or can be transformed into the presented form. The present method can be used in combinatorial optimization problems such as the ordering and partitioning of structural models.

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