Structural Analysis of Stiffened FGM Thick Walled Cylinders by Application of a New Cylindrical Super Element

S. A. Moeini, M.T.Ahmadian

Abstract—Structural behavior of ring stiffened thick walled cylinders made of functionally graded materials (FGMs) is investigated in this paper. Functionally graded materials are inhomogeneous composites which are usually made from a mixture of metal and ceramic. The gradient compositional variation of the constituents from one surface to the other provides an elegant solution to the problem of high transverse shear stresses that are induced when two dissimilar materials with large differences in material properties are bonded. FGM formation of the cylinder is modeled by power-law exponent and the variation of characteristics is supposed to be in radial direction.

A finite element formulation is derived for the analysis. According to the property variation of the constituent materials in the radial direction of the wall, it is not convenient to use conventional elements to model and analyze the structure of the stiffened FGM cylinders. In this paper a new cylindrical super-element is used to model the finite element formulation and analyze the static and modal behavior of stiffened FGM thick walled cylinders. By using this super-element the number of elements, which are needed for modeling, will reduce significantly and the process time is less in comparison with conventional finite element formulations.

Results for static and modal analysis are evaluated and verified by comparison to finite element formulation with conventional elements. Comparison indicates a good conformity between results.

Keywords—FGMs, Modal analysis, Static analysis, Stiffened cylinders.

I. INTRODUCTION

FUNCTIONALLY graded materials are inhomogeneous composites which are usually made from a mixture of metal and ceramic. The gradient compositional variation of the constituents from one surface to the other provides an elegant solution to the problem of high transverse shear stresses that are induced when two dissimilar materials with large differences in material properties are bonded. This type of materials introduced at first by a group of Japanese researchers to resist in aggressive environment of thermal shock [1],[2]. These materials are anisotropic, inhomogeneous and have variable mechanical properties in one direction which makes them difficult to analyze.

Cylindrical structures stiffened by rings are common parts in industry and they have numerous applications in different fields of engineering such as aerospace, marine vessels, silos, core barrels and pressurized water reactors. Cylindrical structures are of interests of many scientist and numerous papers are dedicated to the analysis of them. Loy et al. [3] obtained the natural frequencies of simply supported FGM cylindrical shells by applying Love's shell theory and Rayleigh-Ritz method. Pradhan et al. [4] analyzed FGM shells under different boundary conditions by Love's theory. Love's theory confines the calculations to the thin walled cylinders under various boundary conditions. Chen et al. [5] evaluated natural frequencies of FGM cylinders by employing a combination of state-space and matrix transfer method. However, this solution is just for simply supported boundary conditions, it makes three dimensional analysis of FG cylinders possible.

Modeling of stiffeners is very important in evaluation of structural behavior of stiffened cylindrical shells and scientists presented many models to achieve better solution prediction of these structures since 1950s. Hoppmann [6] worked on free vibration of these structures by analytical and experimental methods and presented an analytical method for shells stiffened by equal strength rings which are placed closely and equally spaced. Simply supported boundary condition was supposed to be exerted on the ends of the shell. He used an equivalent model of unstiffened shell to investigate stiffened shell analytically. Stiffeners were smeared out through the cylinder surface and displacements were evaluated. Mikulas and McElman [7] investigated the free vibration of eccentrically stiffened simply supported cylindrical shells, they used averaging method and smeared the effects of stiffeners on the surface of the shell and found out that eccentricity has a remarkable effect on the free vibration of stiffened shells and natural frequencies. This method is unable to predict vibration behavior of stiffened shells properly when the space between stiffeners becomes large or the wavelength of vibration becomes smaller than stiffeners' spacing. Thus, other scientists proposed that stiffeners should be treated discretely. Egle and Sewall [8] studied free vibration of an orthogonally stiffened cylindrical shell; they have treated

Seyed Ali Moeini is with the School of Mechanical Engineering of Sharif University of Technology, Azadi Ave, Tehran, Iran, P.B National Institute of Standards and Technology, Boulder, (corresponding author to provide phone: 917-713-8188; e-mail: seyedalimoeini@gmail.com).).

M. T. Ahmadian is with the Center of Excellence in Design, Robotics and Automation(CEDRA), Sharif University of Technology, Azadi Ave, Tehran, Iran (e-mail: ahmadian@sharif.edu).

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stiffeners as discrete elements in their analysis. This study was promoted by consideration of eccentricity in plane and rotary inertia of the stiffeners in Parthan and Johns research [9].

Many scientists used energy methods to predict vibration behavior and natural frequencies of stiffened shells, these methods are capable of being used for structures made of new materials with special properties, such as functionally graded materials. Mustafa and Ali [10] presented an energy method for free vibration analysis of stiffened cylindrical shells. The analysis took into account the flexure and extension of the shell and the flexure, extension and torsion of the stiffeners. Wang et al. [11] used Ritz method to investigate free vibration of stiffened cylindrical shells. In their study stiffeners are made from different materials and dimensions, inplane and rotary inertia of stiffeners were considered. Polynomial functions were employed to expand displacements, whereas it is important to be noted that these functions should satisfy the boundary conditions. Moeini et al. [12] used sander's theory and Ritz method to evaluate natural frequencies of FG cylindrical shells stiffened by uniformly and non-uniformly ring stiffeners.

Analytical methods have their own limitations to predict the solution of FGM stiffened cylinders. Therefore, numerical methods are more suitable for these kinds of problems.

Finite element method has many advantages in solving engineering problems in comparison to the other numerical methods. The FEM provides a mathematically simplified procedure to model and evaluate complex structures. However, this method can be so time consuming when the number of elements, meshing the entire structure increases. By meshing the structures with super-elements the number of elements reduces extensively compared to employing conventional elements and the time consumption decreases. Development and application of super-elements in structural analysis of various mechanical systems have been widely extended in the last decade. F. Ju and Y.S. Choo [13] developed a super-element for modeling a cable passing through multiple pulleys. J. Jiang and M.D. Olson [14] incorporated a super-element to the nonlinear static and dynamic analysis of orthogonally stiffened cylindrical shells.

Application of geometrical super-elements in large deformation of elasto-plastic shells is presented by S.A. Lukasiewicz [15]. Many scientists have used super-elements in static and dynamic analyses of stiffened shells and plates. Ahmadian and Bonakdar [16] introduced a new cylindrical super-element for structural analysis of laminated hollow cylinders and performed static and modal analysis using this super-element.

In this paper the cylindrical super-element is used to evaluate the static and modal behavior of functionally graded thick walled cylinders under different boundary conditions. Deflections and natural frequencies are obtained and compared to conventional finite element models.

II. FORMULATION AND MODELING

A. Element Definition

First of all, we should introduce the concept of cylindrical element formulation. The element's shape and the required notations are illustrated in Fig. 1 [16].

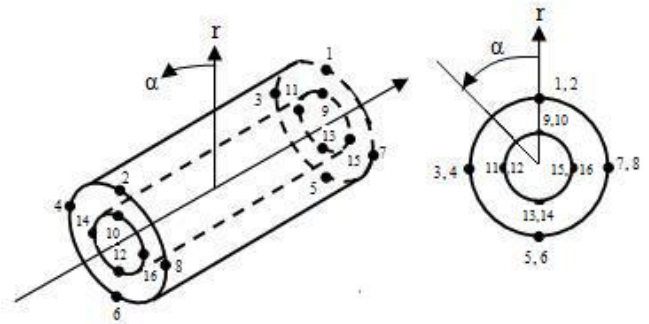L, r_1, r_2 are one half of element length, inner radius and outer radius, respectively

Fig. 1 super-element configuration and coordinate system

Cylindrical coordinate system (r, α, z) is considered to specify the position vector in the super-element. Where r, α, z are radial, tangential and axial coordinates. Dimensionless coordinates are defined as:

$$\xi = \frac{z}{L}, \quad \eta = \frac{2r - (r_2 + r_1)}{r_2 - r_1}, \quad \gamma = \frac{\alpha}{\pi} - 1 \tag{1}$$

Shape functions of this super-element are represented in appendix.
The displacement vector for an internal point of the element is

$$\mathbf{u} = [u_r \ u_\alpha \ u_z]^T \tag{2}$$

which is obtained using the shape functions and the nodal displacement vector, q, according to

$$\mathbf{u} = \mathbf{N}\mathbf{q} \tag{3}$$

where

$$\mathbf{q} = [u_{1r} \ u_{1\alpha} \ u_{1z} \ ... \ u_{16r} \ u_{16\alpha} \ u_{16z}]^T \tag{4}$$

and

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & ... & N_{16} & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & ... & 0 & N_{16} & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & ... & 0 & 0 & N_{16} \end{bmatrix} \tag{5}$$

in which N_i is the shape functions defined in appendix.
In cylindrical coordinate system the strain vector is defined as;

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$$\varepsilon = [\varepsilon_r \ \varepsilon_\alpha \ \varepsilon_z \ ... \ \gamma_{r\alpha} \ \gamma_{zr} \ \gamma_{\alpha z}]^T \qquad (6)$$

Where each of the ε and γ corresponds to normal and shear stresses respectively.

The strain-displacement relations in cylindrical coordinate are;[17]

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \ \varepsilon_\alpha = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\alpha}{\partial \alpha}, \ \varepsilon_z = \frac{\partial u_z}{\partial z}$$

$$\gamma_{r\alpha} = \frac{1}{r}\frac{\partial u_r}{\partial \alpha} + \frac{\partial u_\alpha}{\partial r} - \frac{1}{r}u_\alpha, \ \gamma_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \qquad (7)$$

$$\gamma_{zr} = \frac{\partial u_\alpha}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \alpha}$$

which can be stated in matrix form as;

$$\varepsilon = Lu \qquad (8)$$

where L is the operator matrix;

$$
L = \begin{bmatrix}
\dfrac{\partial}{\partial r} & \dfrac{1}{r} & 0 & \dfrac{\partial}{r\partial \alpha} & \dfrac{\partial}{\partial z} & 0 \\[2mm]
0 & \dfrac{\partial}{r\partial \alpha} & 0 & \dfrac{\partial}{\partial \alpha} - \dfrac{1}{r} & 0 & \dfrac{\partial}{\partial z} \\[2mm]
0 & 0 & \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial r} & \dfrac{\partial}{r\partial z}
\end{bmatrix} \qquad (9)
$$

and u is the displacement vector defined by "(3)".
Using "(3)" and "(8)" the strain vector may be written as;

$$\varepsilon = Bq \qquad (10)$$

where B is the strain-nodal displacement matrix obtained from;

$$B_{6\times 48} = L_{6\times 3} \times N_{3\times 48} \qquad (11)$$

where the subscripts indicate the size of the matrices.

B. FGM Cylindrical Element-Mass and Stiffness Matrices

Various properties of the functionally graded materials change smoothly with location. In the FG cylinders which are considered in this work, the characteristics are only a function of radial coordinate and don't change in circumferential and longitudinal directions. Many relations are available to characterize the varying properties of functionally graded materials [18]. In this work the following relation is considered [3]:

$$\Psi = \Psi^0\left[1 - \left(\frac{r-r_1}{r_2-r_1}\right)^k\right] + \Psi^1\left(\frac{r-r_1}{r_2-r_1}\right)^k \qquad (12)$$

where Ψ^0 and Ψ^1 are specific characteristics for the inner and

outer materials, respectively and Ψ is property of the FGM in the defined radius. The parameter k is known as the gradient index. It is obvious that the FGM becomes a homogenous material when $k=0$.

The stress vector in cylindrical coordinate system is defined as;

$$\sigma = [\sigma_r \ \sigma_\alpha \ \sigma_z \ \tau_{r\alpha} \ \tau_{zr} \ \tau_{\alpha z}] \qquad (13)$$

which is related to the strain vector according to the following equation; [19]

$$\sigma = D\varepsilon \qquad (14)$$

where D is the elasticity matrix of the FGM calculated by substituting the elasticity of each material into "(12)"

$$D = D^0\left[1 - \left(\frac{r-r_1}{r_2-r_1}\right)^k\right] + D^1\left(\frac{r-r_1}{r_2-r_1}\right)^k \qquad (15)$$

where D^i is the symmetric material property matrix for each of constituents. The stiffness matrix is calculated from: [19]

$$K^e = \int_{V^e} B^T D B \, dv \qquad (16)$$

The above integral is carried over the cylindrical element volume. To calculate the integral it is useful to express it in terms of local coordinates.

$$K^e = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} B^T D B \, dv \qquad (17)$$

The mass matrix of the element is: [19]

$$M^e = \int_{V^e} N^T \rho N \, dv \qquad (18)$$

$$= \int_{V^e} N^T \left\{\rho^0\left[1 - \left(\frac{r-r_1}{r_2-r_1}\right)^k\right] + \rho^1\left(\frac{r-r_1}{r_2-r_1}\right)^k N\right\} dv$$

in which the FGM density is substituted by the aid of "(12)".

III. STATIC ANALYSIS

Consider an FGM hollow cylinder with length $L_{cy}=4.8$, inner radius $r_{cy}=0.2$, wall thickness $h_{cy}=0.2$ stiffened with four ring stiffeners with length $L_{stf}=0.2$, inner radius $r_{stf}=0.4$, thickness $h_{stf}=0.05$. Stiffeners are placed equally spaced from each other and the ends of the cylinder. Material specifications for the cylinder and the rings are expressed in Table I. It is considered to use $k=3$ as FGM power law exponent. Fig. 2 represents the modulus of elasticity of FGM material property "E", versus the cylinder radius " r".

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TABLE I
MATERIAL PROPERTIES

	E(Gpa)	P(Kg/m³)	v
Inner material (Cu)	110	8960	0.34
Outer material (W)	400	19300	0.28
Stiffener material (St)	200	7800	0.3

As follows, the cylinder is subjected to various static loads and the results are compared to results from conventional finite element models.

The conventional finite element result is gained with sufficient number of brick elements to assure a mesh-independent result. To use brick element for modeling of FG cylinder, a procedure is invoked here that assumes several constant property coaxial layers in which the material properties are changed gradually from the inner to the outer layer according to the specific curve in Fig. 2.
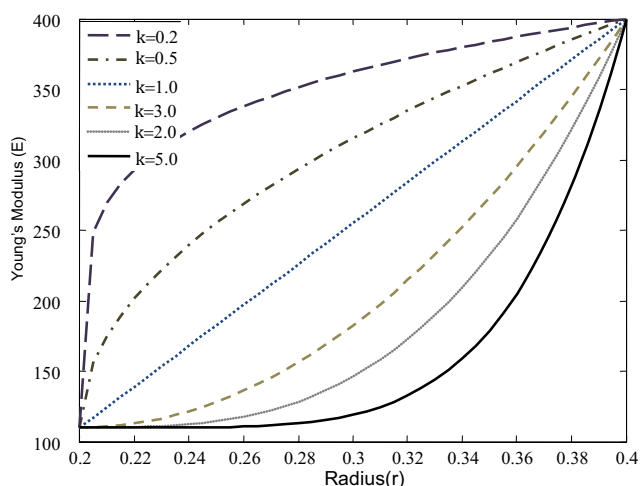

Fig. 2 Modulus of elasticity of FGM cylinder in radial direction

In the first loading condition one end of the cylinder is fixed and the free end is subjected to an axial load of magnitude $F=377(N)$ such that the applied load is distributed equally among the nodes on the free end. Table II represents the maximum elongation of the cylinder for various numbers of super-elements used, compared with brick elements result.

TABLE II
STIFFENED CYLINDER UNDER AXIAL LOADING

	Super-element		Brick Element
Number of Elements	24	48	30000
Maximum Elongation	2.423e-7	2.452 e-7	2.49e-7
Difference With Brick Element Method (%)	2.69	1.61	

The second loading is concerned with a transverse load of magnitude $F=-10(KN)$ applied to the free end of the cylinder with the other end being fixed. The maximum deflection of the cylinder in this loading condition is given in Tab. III.

TABLE III
STIFFENED CYLINDER UNDER TRANSVERSE LOADING

	Super-element			Brick Element
Number of Elements	24	48	120	30000
Maximum Elongation	8.2965e-5	8.4815e-5	8.65e-5	8.73e-5
Difference With Brick Element Method (%)	4.9	2.8	0.9	

It should be noted that according to the nonlinear shape of the bended cylinder, more number of super-elements are required to analyze, which is attributed to the linear characteristic of the shape functions in the axial direction of the super-element.

The last loading condition deals with torsion of the cylinder. As shown in Fig. 3 eight forces in the α-direction constitute the torque applied to the end of the cylinder. Each force is of magnitude of $F=1000(N)$, the total axial torque becomes $T=1200(N.m)$. The maximum angles of rotation obtained by super-element and brick elements are given in Tab. 4.

TABLE IV
STIFFENED CYLINDER UNDER AXIAL TORQUE

	Super-element		Brick Element
Number of Elements	24	120	30000
Maximum Elongation	1.338e-6	1.339e-6	1.376e-7
Difference With Brick Element Method (%)	2.7	2.6	

As it is shown a huge number of brick elements are required to establish a mesh independent result for the FG cylinders. This amount is reduced greatly using the new super-element.
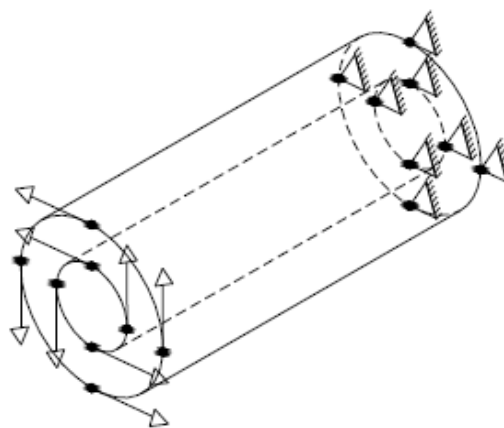

Fig. 3 torsion loading: clamped-free boundary conditions

IV. MODAL ANALYSIS

The equation of motion for a multi degree of freedom udamped system is expressed as

$$M\ddot{Q} + KQ = 0 \tag{19}$$

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Where Q is the degree of freedom vector and M and K are respectively, the system mass and stiffness matrices. When vibration in one of the mode shapes, φ_i , all the points in the system undergo simple harmonic motion with the corresponding natural frequency ω_i, which can be stated as;

$$Q = A_i \sin(\omega_i t) \tag{20}$$

in which A_i is the amplitude vector with each component corresponding to the specific degree of freedom. Substituting (20) into (19) yields;

$$(-M\omega_i^2 + K)A_i = 0 \tag{21}$$

To avoid a nontrivial solution for the above equation, it follows that the determinant of the coefficient matrix, -$M\omega^2+K$, should vanish.

$$\left| -M\omega_i^2 + K \right| = 0 \tag{22}$$

By pre-multiplying the above equation with M^{-1} we obtain;

$$\left| D - I\lambda_i \right| = 0 \tag{23}$$

where $D = M^{-1} \times K$ is the dynamic matrix and $\lambda_i = \omega_i^2$. Solving "(23)" actually leads the eigenvalues, λ_i, $i=1\ldots$ S, of the dynamic matrix, where S is the size of the mass or stiffness matrices which equals the number of degrees of freedom of the entire system. In fact the eigenvalue problem in "(23)" leads the square of natural frequencies and the mode shapes as the eigenvalues and eigenvectors of D, respectively. Cylinders are categorized into two main groups according to their wall thickness. The ratio of the wall thickness to radius is the main criterion. According to the Love theory those cylinders with *h/r≤0.05* are referred as shells and are analyzed based on shell theorems or numerical methods while for thick walled cylinders mostly the numerical methods such as finite element is implemented. In this section thick walled FG cylinders are analyzed and natural frequencies are evaluated for different values of *h/r*. The finite element method using conventional methods (brick element) is used to verify the super-element capability in modeling of FG cylinders.
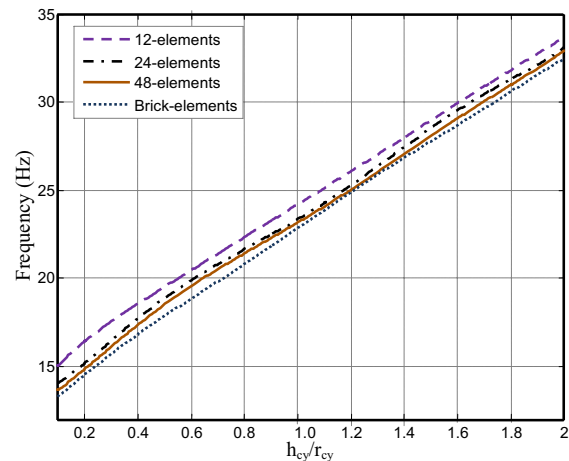

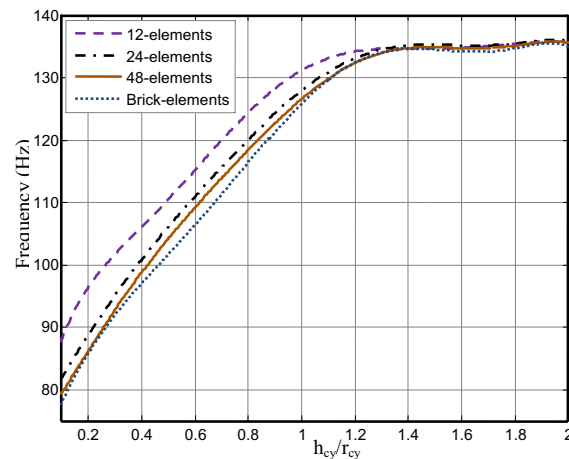Fig. 4 First bending mode frequency: clamped-free boundary conditions

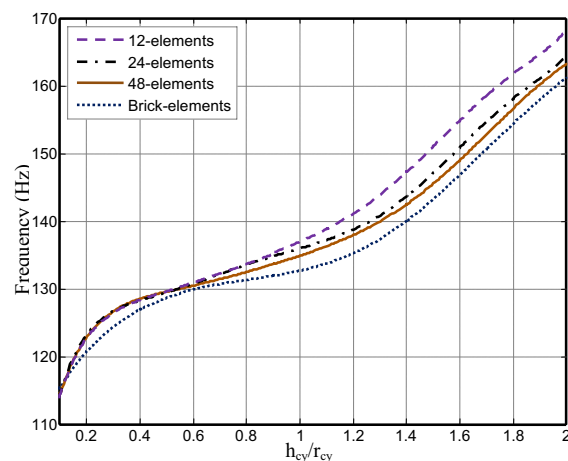Fig. 5 First torsion mode frequency: clamped-free boundary conditions

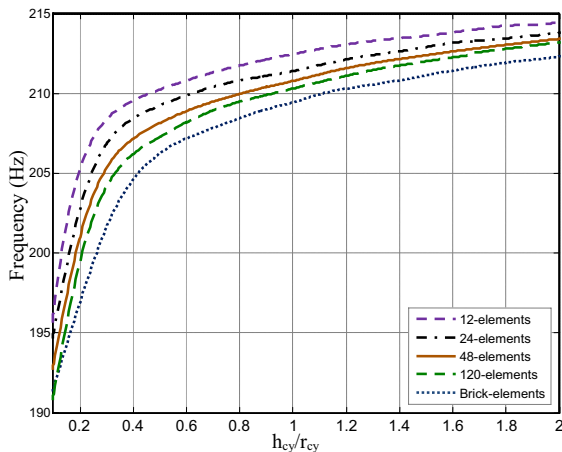Fig. 6 Second bending mode frequency: clamped-free boundary conditions

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Fig. 7 First longitudinal mode frequency: clamped-free boundary conditions

APPENDIX

$$N_1(\xi,\eta,\gamma) = \tfrac{1}{8}(Cos^2\pi\gamma - Cos\pi\gamma)(1+\xi)(1+\eta)$$

$$N_2(\xi,\eta,\gamma) = \tfrac{1}{8}(Cos^2\pi\gamma - Cos\pi\gamma)(1-\xi)(1+\eta)$$

$$N_3(\xi,\eta,\gamma) = \tfrac{1}{8}(Sin^2\pi\gamma - Sin\pi\gamma)(1+\xi)(1+\eta)$$

$$N_4(\xi,\eta,\gamma) = \tfrac{1}{8}(Sin^2\pi\gamma - Sin\pi\gamma)(1-\xi)(1+\eta)$$

$$N_5(\xi,\eta,\gamma) = \tfrac{1}{8}(Cos^2\pi\gamma + Cos\pi\gamma)(1+\xi)(1+\eta)$$

$$N_6(\xi,\eta,\gamma) = \tfrac{1}{8}(Cos^2\pi\gamma + Cos\pi\gamma)(1-\xi)(1+\eta)$$

$$N_7(\xi,\eta,\gamma) = \tfrac{1}{8}(Sin^2\pi\gamma + Sin\pi\gamma)(1+\xi)(1+\eta)$$

$$N_8(\xi,\eta,\gamma) = \tfrac{1}{8}(Sin^2\pi\gamma + Sin\pi\gamma)(1-\xi)(1+\eta)$$

$$N_9(\xi,\eta,\gamma) = \tfrac{1}{8}(Cos^2\pi\gamma - Cos\pi\gamma)(1+\xi)(1-\eta)$$

$$N_{10}(\xi,\eta,\gamma) = \tfrac{1}{8}(Cos^2\pi\gamma - Cos\pi\gamma)(1-\xi)(1-\eta)$$

$$N_{11}(\xi,\eta,\gamma) = \tfrac{1}{8}(Sin^2\pi\gamma - Sin\pi\gamma)(1+\xi)(1-\eta)$$

$$N_{12}(\xi,\eta,\gamma) = \tfrac{1}{8}(Sin^2\pi\gamma - Sin\pi\gamma)(1-\xi)(1-\eta)$$

$$N_{13}(\xi,\eta,\gamma) = \tfrac{1}{8}(Cos^2\pi\gamma + Cos\pi\gamma)(1+\xi)(1-\eta)$$

$$N_{14}(\xi,\eta,\gamma) = \tfrac{1}{8}(Cos^2\pi\gamma + Cos\pi\gamma)(1-\xi)(1-\eta)$$

$$N_{15}(\xi,\eta,\gamma) = \tfrac{1}{8}(Sin^2\pi\gamma + Sin\pi\gamma)(1+\xi)(1-\eta)$$

$$N_{16}(\xi,\eta,\gamma) = \tfrac{1}{8}(Sin^2\pi\gamma + Sin\pi\gamma)(1-\xi)(1-\eta)$$

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