World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:4, No:7, 2010

Parallel multisplitting methods for singular linear systems

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Abstract—In this paper, we discuss convergence of the extrapolated iterative methods for linear systems with the coefficient matrices are singular H-matrices. And we present the sufficient and necessary conditions for convergence of the extrapolated iterative methods. Moreover, we apply the results to the GMAOR methods. Finally, we give one numerical example.

Keywords—singular H-matrix, linear systems, extrapolated iterative method, GMAOR method, convergence.

I. Introduction

LET us consider a system of n equations

$$Ax = b, \quad (1)$$

where $A \in C^{n\times n}$ is singular, $b, x \in C^n$ with b known and x unknown. We assume that the system (**??**) is solvable, i.e., it has at least one solution. In order to solve the system (**??**) with parallel multi-splitting iterative methods, we assume that

$$(1)\ A = M_k - N_k,\ k = 1, 2, \cdots, \alpha,$$

where M_k is a nonsingular matrix;

$$(2)\quad \sum_k E_k = I\ (I \in R^{n\times n}),$$

where E_k are diagonal and $E_k \geq 0$.

Then a parallel multi-splitting iterative method for solving (**??**) can be described as follows

$$x^{m+1} = Tx^m + Sb,\ m = 0, 1, 2, \cdots, \quad (2)$$

where $T = \sum_k E_k M_k^{-1} N_k$ is the iteration matrix, $S = \sum_k E_k M_k^{-1}$.

It is well known that for singular systems the iterative method (**??**) is convergent if and only if the associated convergence factor

$$\vartheta(T) \equiv max\{|\mu|, \mu \in (\sigma(T)\backslash\{1\})\} < 1$$

and the elementary divisors associated with $\mu = 1 \in \sigma(T)$ are linear, i.e.,

$$index(I - T) = 1,$$

where $\sigma(T)$ denotes the spectrum of T and $index(B)$ denotes the index of the matrix B, i.e., the smallest nonnegative integer k such that $rank(B^{k+1}) = rank(B^k)$. In this case, T is called a semi-convergent matrix. In the extrapolated case, the method (**??**) can be defined by

$$x^{k+1} = T_\omega x^k + \omega M^{-1}b, k = 0, 1, 2, \ldots, \quad (3)$$

where

$$T_\omega = (1 - \omega)I + \omega T, \quad (4)$$

is the iteration matrix and $\omega \in R$ is called the extrapolated parameter([1]). Clearly, if $\omega = 0$ then $T_0 = I$ and the extrapolated method (**??**) becomes

$$x^{k+1} = x^k,\ k = 0, 1, 2, \ldots.$$

Thus we assume that $\omega \neq 0$ in further considerations.

Now we assume that

$(1) A = D - L_k - U_k,\ k = 1, 2, \ldots, \alpha$, where $diag(D) = diag(A)$, D is a nonsingular matrix, L_k and U_k are matrices with zeros in the diagonal, where $L_k = (l_{ij})_k, U_k = (u_{ij})_k$. In general, we don't assume that L_k and U_k are triangular matrices;

$(2) \sum_k E_k = I$, where E_k are diagonal and $E_k \geq 0$.

Then the collection of triples $(D - L_k, U_k, E_k), k = 1, 2, \ldots, \alpha$, is called a multi-splitting of A.

We introduce the operators F_k by

$$F_k(\gamma, \omega, x) = (D - \gamma L_k)^{-1} \times [(1 - \omega)D + (\omega - \gamma)L_k + \omega U_k]x + \omega b,$$
$$\gamma \geq 0,\ \omega > 0,\ k = 1, 2, \ldots, \alpha.$$

Algorithm: Choose $x^0 \in R^n$ arbitrarily. For $m = 0, 1, 2, \ldots$ until convergence,

$$x^{m+1} = \sum_k E_k F_k(\gamma, \omega, x^m),$$

If we define the matrix

$$\ell_{GMAOR}(\gamma, \omega) = \sum_k E_k (D - \gamma L_k)^{-1} \times [(1 - \omega)D + (\omega - \gamma)L_k + \omega U_k]$$

and the vector

$$b_{GMAOR}(\gamma, \omega) = \sum_k E_k (D - \gamma L_k)^{-1} (\omega b),$$

then from Algorithm we get

$$x^{m+1} = \ell_{GMAOR}(\gamma, \omega)x^m + b_{GMAOR}(\gamma, \omega), m = 0, 1, \ldots.$$

Obviously, if D is diagonal, L_k are strictly lower triangular matrices and U_k are matrices with zeros in the diagonal, then

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the above algorithm will reduce to the well-known MAOR algorithm (parallel multi-splitting AOR algorithm [2]). Hence we call Algorithm a parallel generalized multi-splitting AOR algorithm (GMAOR). Furthermore, we observe that when (γ,ω) is equal to $(\omega,\omega),(1,1),(0,\omega)$ and $(0,1)$ the GMAOR method reduces to the GMSOR, GMGS, GMJOR and GMJ iterative methods, respectively, with the iteration matrices $\ell_{GMSOR}(\gamma),\ell_{GMGS},\ell_{GMJOR}$ and ℓ_{GMJ}.

It should be noted that, if $\gamma\neq 0$, the GMAOR method is an extrapolated method of the GMSOR method with the relaxation factor γ and the extrapolated parameter $\frac{\omega}{\gamma}$, namely

$$\ell_{GMAOR}(\gamma,\omega)=(1-\frac{\omega}{\gamma})I+\frac{\omega}{\gamma}\ell_{GMSOR}(\gamma).$$

In this paper, we discuss convergence of the extrapolated iterative methods for solving singular linear systems with the coefficient matrices are singular H-matrices. In Section 2 some sufficient and necessary conditions for convergence of the extrapolated iterative methods are presented. In Section 3 we apply the results of Section 2 to the GMAOR method, which are the extrapolated methods of the GMSOR method.

Definition 1.1([3]) A matrix $A=(a_{ij})\in R^{n\times n}$ is called a singular M-matrix if A can be expressed in the form

$$A=sI-B,\ s>0,B\geq 0, \tag{5}$$

and

$$s=\rho(B).$$

Definition 1.2([4]) A matrix $A=(a_{ij})\in C^{n\times n}$ is called a singular H-matrix if its comparison matrix $M(A)=(\tilde{a}_{ij})$ is a singular M-matrix, where

$$\tilde{a}_{ij}=\begin{cases} \mid a_{ii}\mid, & i=j\\ -\mid a_{ij}\mid, & i\neq j\end{cases}$$

Definition 1.3([5]) Let $A\in R^{n\times n}$. $A=M-N(M,N\in R^{n\times n})$ is called as an H-splitting if $M(M)-|N|$ is an M-matrix. If $M(A)=M(M)-|N|$, then $A=M-N$ is called as an H-compatible splitting.

II. Sufficient and necessary conditions for convergence

Lemma 2.1([6]) The extrapolated method

$$x^{(m+1)}=[(1-\omega)I+\omega M^{-1}N]x^{(m)}+\omega M^{-1}b,$$

$m=0,1,2,\cdots$, is convergent if and only if index$(I-T)=1$ and one of the following conditions is satisfied.

(1) $Re\mu<1$, for all $\mu\in\sigma(T)\backslash\{1\}$, and

$$0<\omega<\min_{\mu\in\sigma(T)\backslash\{1\}}\frac{2(1-Re\mu)}{1-2Re\mu+|\mu|^2};$$

(2) $Re\mu>1$, for all $\mu\in\sigma(T)\backslash\{1\}$, and

$$0>\omega>\max_{\mu\in\sigma(T)\backslash\{1\}}\frac{2(1-Re\mu)}{1-2Re\mu+|\mu|^2}.$$

Theorem 2.1 The extrapolated method (**??**) is convergent if and only if index$(I-T)=1$ and one of the following conditions is satisfied.

(1) $Re\mu<1$, for all $\mu\in\sigma(T)\backslash\{1\}$, and

$$0<\omega<\min_{\mu\in\sigma(T)\backslash\{1\}}\frac{2(1-Re\mu)}{1-2Re\mu+|\mu|^2};$$

(2) $Re\mu>1$, for all $\mu\in\sigma(T)\backslash\{1\}$, and

$$0>\omega>\max_{\mu\in\sigma(T)\backslash\{1\}}\frac{2(1-Re\mu)}{1-2Re\mu+|\mu|^2}.$$

Proof: By Lemma 1 we know that Theorem 1 holds obviously.

Now we denote

$$\tau(T)=\min_{\mu\in\sigma(T)\backslash\{1\}}\frac{2(1-Re\mu)}{1-2Re\mu+|\mu|^2}.$$

Corollary 2.1 If $\rho(T)=1$, then the following statements are true.

(1) The extrapolated iterative method is convergent if and only if index$(I-T)=1$ and $0<\omega<\tau(T)$.

(2) The inequalities

$$\tau(T)\geq\frac{2}{1+\vartheta(T)}\geq 1$$

hold.

Proof: (1) Since $\rho(T)=1$, we have $Re\mu<1$ for $\mu\in\sigma(T)\backslash\{1\}$. Thus by Theorem 1 it follows that (**??**) is convergent if and only if index$(I-T)=1$ and $0<\omega<\tau(T)$.

(2) For $\mu\in\sigma(T)\backslash\{1\}$, we have

$$\frac{(1-Re\mu)}{1-2Re\mu+|\mu|^2}\geq\frac{1}{1+|\mu|},$$

if $|\mu|<1$. And if $|\mu|=1$ then $Re\mu<1$, hence

$$\frac{(1-Re\mu)}{1-2Re\mu+|\mu|^2}=\frac{1}{1+|\mu|}=\frac{1}{2}.$$

Correspondingly, we have

$$\tau(T)\geq\min_{\mu\in\sigma(T)\backslash\{1\}}\frac{2}{1+|\mu|}=\frac{2}{1+\vartheta(T)},$$

thus (2) follows immediately.

Lemma 2.2([7]) Let $A\in R^{n\times n}$ be a irreducible singular H-matrix. Further, assume that the splitting $A=M_k-N_k(k=1,2,\ldots,\alpha)$ is an H-compatible splitting, then $\rho(T)=1$ and index$(I-T)=1$.

Lemma 2.3([7]) Let $A\in R^{n\times n}$ be a singular H-matrix. Further, assume that the splitting $A=M_k-N_k(k=1,2,\ldots,\alpha)$ is an H-compatible splitting and $ind_0(A)=inf\{k:ker(A^k)=ker(A^{k+1})\}=1$, then $\rho(T)=1$ and index$(I-T)=1$, where $kerA$ is the kernel of the linear transformation A.

Theorem 2.2 Let $A\in R^{n\times n}$ be a irreducible singular H-matrix. Further, assume that the splitting $A=M_k-N_k,k=1,2,\ldots,\alpha$ is an H-compatible splitting. Then the extrapolated method (**??**) is convergent if and only if $0<\omega<\tau(T)$.

Proof: From Lemma 2 we know that $\rho(T)=1$ and index$(I-T)=1$. From Corollary 1 we know that the extrapolated method (**??**) is convergent if and only if $0<\omega<\tau(T)$.

Theorem 2.3 Let $A\in R^{n\times n}$ be a singular H-matrix. Further, assume that the splitting $A=M_k-N_k,k=1,2,\ldots,\alpha$ is an

H-compatible splitting and $ind_0(A) = 1$. Then the extrapolated method (**??**) is convergent if and only if $0 < \omega < \tau(T)$.

Proof: From Lemma 3 we know that $\rho(T) = 1$ and index$(I - T) = 1$. From Corollary 1 we know that the extrapolated method (**??**) is convergent if and only if $0 < \omega < \tau(T)$.

III. Applications

Theorem 3.1 Let $A \in R^{n\times n}$ be a irreducible singular H-matrix, $d_{ij} - \gamma(l_{ij})_k \geq 0, (1-\gamma)d_{ij} + \gamma(u_{ij})_k \geq 0$ or $d_{ij} - \gamma(l_{ij})_k \leq 0, (1-\gamma)d_{ij} + \gamma(u_{ij})_k \leq 0$. If $0 < \gamma \leq 1$, then the GMAOR method is convergent if and only if $0 < \frac{\omega}{\gamma} < \tau(\ell_{GMSOR}(\gamma))$.

Proof: Let $M_k = \frac{1}{\gamma}(D - \gamma L_k)$ and $N_k = \frac{1}{\gamma}[(1-\gamma)D + \gamma U_k]$, so the iterative matrix of GMSOR method is $\ell_{GMSOR}(\gamma) = \sum\limits_k E_k M_k^{-1} N_k$.

From hypothesis we have

$$M(A) = M(\frac{1}{\gamma}(D - \gamma L_k)) - |\frac{1}{\gamma}[(1-\gamma)D + \gamma U_k]|,$$

so $A = \frac{1}{\gamma}(D - \gamma L_k) - \frac{1}{\gamma}[(1-\gamma)D + \gamma U_k]$ is an H-compatible splitting. Hence Theorem 3.1 follows by Theorem 2.2 immediately.

Corollary 3.1 Let $A \in R^{n\times n}$ be a irreducible singular H-matrix, $(l_{ij})_k \geq 0, (u_{ij})_k \geq 0$ or $(l_{ij})_k \leq 0, (u_{ij})_k \leq 0$. If $0 < \gamma \leq 1$, then the MAOR method is convergent if and only if $0 < \frac{\omega}{\gamma} < \tau(\ell_{MSOR}(\gamma))$.

Theorem 3.2 Let $A \in R^{n\times n}$ be a singular H-matrix, $d_{ij} - \gamma(l_{ij})_k \geq 0, (1-\gamma)d_{ij} + \gamma(u_{ij})_k \geq 0$ or $d_{ij} - \gamma(l_{ij})_k \leq 0, (1-\gamma)d_{ij} + \gamma(u_{ij})_k \leq 0$. Further, assume that $ind_0(A) = 1$, $0 < \gamma \leq 1$, then the GMAOR method is convergent if and only if $0 < \frac{\omega}{\gamma} < \tau(\ell_{GMSOR}(\gamma))$.

Proof: Let $M_k = \frac{1}{\gamma}(D - \gamma L_k)$ and $N_k = \frac{1}{\gamma}[(1-\gamma)D + \gamma U_k]$, so the iterative matrix of GMSOR method is $\ell_{GMSOR}(\gamma) = \sum\limits_k E_k M_k^{-1} N_k$.

From hypothesis we have

$$M(A) = M(\frac{1}{\gamma}(D - \gamma L_k)) - |\frac{1}{\gamma}[(1-\gamma)D + \gamma U_k]|,$$

so $A = \frac{1}{\gamma}(D - \gamma L_k) - \frac{1}{\gamma}[(1-\gamma)D + \gamma U_k]$ is an H-compatible splitting. Hence Theorem 3.2 follows by Theorem 2.3 immediately.

Corollary 3.2 Let $A \in R^{n\times n}$ be a singular H-matrix, $(l_{ij})_k \geq 0, (u_{ij})_k \geq 0$ or $(l_{ij})_k \leq 0, (u_{ij})_k \leq 0$. Further, assume that $ind_0(A) = 1$. If $0 < \gamma \leq 1$, then the MAOR method is convergent if and only if $0 < \frac{\omega}{\gamma} < \tau(\ell_{MSOR}(\gamma))$.

IV. Numerical Example

Consider $Ax = b$, where

$$A = \begin{bmatrix} 3 & -1 & 0 & -1 & 0 & -1 \\ 0 & 2 & -1 & 0 & -1 & 0 \\ -1 & 0 & 3 & -1 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ -1 & 0 & -1 & 0 & 3 & -1 \\ 0 & -1 & 0 & -1 & 0 & 2 \end{bmatrix}$$

We choose $D = diag(A)$, $n = 6, p = 3, \gamma = 0.5$,

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$A = D - L_i - U_i,\ i = 1, 2, 3.$

It's easy to know that A is an irreducible H-matrix and D, L_i, U_i satisfy the conditions of Theorem 1.

We choose $E_1 = diag(1, 1, 0, 0, 0, 0)$, $E_2 = diag(0, 0, 1, 1, 0, 0)$, $E_3 = diag(0, 0, 0, 0, 1, 1)$, then

$$\ell_{GMSOR}(0.5) = \begin{bmatrix} 1/2 & 1/6 & 0 & 1/6 & 0 & 1/6 \\ 0 & 1/2 & 1/4 & 0 & 1/4 & 0 \\ 1/12 & \frac{23}{768} & \frac{2309}{4608} & \frac{113}{576} & \frac{1}{512} & 3/16 \\ 0 & 1/8 & 1/16 & 1/2 & \frac{5}{16} & 0 \\ \frac{7}{72} & \frac{931}{27648} & \frac{4643}{55296} & \frac{47}{768} & \frac{3079}{6144} & \frac{385}{1728} \\ 0 & \frac{5}{32} & \frac{5}{64} & 1/8 & \frac{9}{64} & 1/2 \end{bmatrix}$$

$$\tau(\ell_{GMSOR}(0.5)) = \frac{5902206365182033}{2251799813685248}.$$

From Theorem 1 we know that the GMAOR method is convergent when

$$0 < \omega < 0.5\tau(\ell_{GMSOR}(0.5)) = \frac{5902206365182033}{4503599627370496}.$$

For example we choose $\omega = 1.25 < \frac{5902206365182033}{4503599627370496}$, then

$\ell_{GMAOR}(0.5, 1.25) =$

$$\begin{bmatrix} -1/4 & \frac{5}{12} & 0 & \frac{5}{12} & 0 & \frac{5}{12} \\ 0 & -1/4 & 5/8 & 0 & 5/8 & 0 \\ \frac{5}{24} & \frac{115}{1536} & -\frac{2279}{9216} & \frac{565}{1152} & \frac{5}{1024} & \frac{15}{32} \\ 0 & \frac{5}{16} & \frac{5}{32} & -1/4 & \frac{25}{32} & 0 \\ \frac{35}{144} & \frac{4655}{55296} & 0.2099 & \frac{235}{1536} & -\frac{3037}{12288} & \frac{1925}{3456} \\ 0 & \frac{25}{64} & \frac{25}{128} & \frac{5}{16} & \frac{45}{128} & -1/4 \end{bmatrix}$$

$\vartheta(\ell_{GMAOR}(0.5, 1.25)) = \frac{4136045821846107}{4503599627370496} < 1$, $index(I - \ell_{GMAOR}(0.5, 1.25)) = 1$. That's the GMAOR method is convergent.

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