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High Resolution Methods Based On Rank Revealing Triangular Factorizations

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Abstract— In this paper, we propose a novel method for subspace estimation used high resolution method without eigendecomposition where the sample Cross-Spectral Matrix (CSM) is replaced by upper triangular matrix obtained from LU factorization. This novel method decreases the computational complexity. The method relies on a recently published result on Rank-Revealing LU (RRLU) factorization. Simulation results demonstrates that the new algorithm outperform the Householder rank-revealing QR (RRQR) factorization method and the MUSIC in the low Signal to Noise Ratio (SNR) scenarios.

Keywords— Factorization, Localization, Matrix, Signal subspace.

I. Introduction

The high resolution methods and spectrum analysis are search subjects motivated several works in various domains e.g. telecommunications, underwater acoustics, geophysics, speech processing or the ancillary medical domain. The first technical applications of array processing are in radar [1] and sonar [2] to localize sources and to delete the interferences. Recently, numeric communications were the object of all the attentions. There are also applications in seismology [3], in biomedical imagery, etc. The most popular method is MUSIC [4], it requires singular values decomposition (SVD) or the eigenvalues decomposition (EVD) to estimate the subspaces, it is complex to implement in certain applications; e.g. in underwater acoustics, where it can have many sensors. The minimum-norm algorithm is proposed by Kumaresan and al. in [5], this algorithm is a subspace based on the DOA estimation method and allows estimation of the DOA of signals in known number, from signals sources received on the array of the sensors, by minimizing the projection of the directional vector on the vector belonging to the noise subspace. In recent years, several methods which obtain the subspaces without EVD or without SVD are proposed in the literature [4], [6]. The implemented methods based on the subspaces definition require essentially two calculation stages. First step, relaying on the decomposition of the observation space into two projection subspaces. The second calculation step consists in extracting the angles of arrival of sources or the signals frequencies from the estimators built with signal subspace or noise subspace. The remainder of the paper is as follows. Section 2 briefly presents the structure of signal subspace problems. Section 3 presents our algorithm and how a RRLU factorization can be used in the signal subspace approach. In section 4, a class of DOA estimators are briefly recalled and we evaluate the numerical complexities. The efficiency of the new algorithm corresponding to the RRLU factorization method is studied using the simulated data in the presence of the spatially correlated noise with limited length or banded CSM noise. Finally this paper provides a conclusion and summary of results.

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II. Problem Formulation

Consider an array of N sensors receiving the wavefield generated by P $(P < N)$ narrow-band sources in the presence of an additive noise. The received signal vector $\mathbf{x}(f)$ is sampled and the FFT algorithm is used to transform the data into the frequency domain. Without loss generality we will omit, in all the continuation of this paper, the frequency *f*. We present these samples by

$$\mathbf{x} = \mathbf{A}\,\mathbf{s} + \mathbf{b} \qquad (1)$$

where $\mathbf{s} \in \mathbf{C}^{P\times 1}$ is the vector of the complex envelopes of the source signals, $\mathbf{b} \in \mathbf{C}^{N\times 1}$ is an additive noise, $\mathbf{A} = [\mathbf{a}(\theta_1),..., \mathbf{a}(\theta_P)] \in \mathbf{C}^{N\times P}$ is the matrix of the steering vector $\mathbf{a}(\theta_p)$, and θ_p, $p = 1,..., P$ is the direction of arrival (DOA) of signal p measured with respect to the normal of the array. The CSM of received signals is:

$$\mathbf{\Gamma} = \mathbf{A}\mathbf{\Gamma}_s(f)\mathbf{A}^H + \mathbf{\Gamma}_b \qquad (2)$$

where $\mathbf{\Gamma} = E[\mathbf{x}\mathbf{x}^H]$, $\mathbf{\Gamma}_s = [\mathbf{s}\mathbf{s}^H]$ and $\mathbf{\Gamma}_b = E[\mathbf{b}\mathbf{b}^H]$ is the noise spectral matrix which depends on the estimation method and used assumptions. $(\cdot)^H$ is the Hermitian transposition of $(\cdot)$. The performances of the high standard resolution algorithms degrade when the noise is spatially correlated. Several techniques have been proposed in order to improve them. Note that in [7] the DOA θ are computed from the roots of

$$f(\theta) = \sqrt{\mathbf{a}^H(\theta)\mathbf{V}\mathbf{V}^H\mathbf{a}(\theta)} \qquad (3)$$

where $\mathbf{V}$ is a basis for the nullspace of $\mathbf{\Gamma} - \mathbf{\Gamma}_b$. This algorithm needs the knowledge of the noise spectral matrix. Rank Revealing Triangular Factorizations

The LU or QR factorization algorithms are the most important and popular methods used for estimating the eigenvalues and the eigenvectors. The upper triangular matrix can be obtained either from the LU factorization or the QR factorization[8, 9, 10], which allows extraction from the CSM, needed information to estimate the noise subspace. The rank of CSM is related to its eigenvalues $\sigma(\mathbf{\Gamma})=\{\lambda_1,\lambda_2,...,\lambda_N\}$ since $rank(\mathbf{\Gamma})=P$ if $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_P > \lambda_{P+1} = ... = \lambda_N = 0$.

A. Rank-Revealing QR factorization

In this section, the algorithm requires an initial triangularization of the matrix, which can be carried out by means of the QR factorization. This method is based upon a RRQR factorization [8], [10]-[11] which allows extraction from the CSM, necessary information to estimate the noise subspace or the signal subspace. One can identify the $N-P$ smallest singular values of $\mathbf{\Gamma}$ and if $\lambda_{P+1}(\mathbf{\Gamma}) \leq \|\mathbf{R}\|_{22}$ its RRQR factorization is [8], [10], [12]:

$$\mathbf{\Gamma} = \mathbf{Q}\begin{pmatrix} \overset{P}{\mathbf{R}_{11}} & \overset{N\text{-}P}{\mathbf{R}_{12}} \\ \mathbf{0} & \mathbf{R}_{22} \end{pmatrix}\begin{matrix} P \\ N-P \end{matrix} \quad (4)$$

where the $N \times N$ matrix $\mathbf{Q}$ has orthonormal columns. If we have such a factorization, then [7]

$$\mathbf{Y}_N = \begin{bmatrix}\mathbf{R}_{11}^{-1}\mathbf{R}_{12} & -\mathbf{I}\end{bmatrix}^T \mathbf{F} \quad (5)$$

s a basis of noise subspace, where $\mathbf{F}$ is a diagonal scaling natrix chosen such that the columns of $\mathbf{Y}_N$ have unit norm.

B. Rank-Revealing LU factorization

In [8], [10]-[11] two theoretical approximations for :omputing the numerical rank of a triangular matrix are ntroduced. This triangular matrix can be obtained by means of the LU factorization. Our implementation incorporates several improvements over the QR algorithm. Specifically, an ncremental condition estimator is employed to reduce the mplementation cost. The principle is based on a RRLU actorization [8], [10]-[11] which allows extraction from the CSM, necessary information to estimate the subspace noise. Assume $\mathbf{\Gamma}$ has numerical RRLU factorization. Then the actorization

$$\mathbf{\Gamma} = \mathbf{L}\begin{pmatrix} \overset{P}{\mathbf{U}_{11}} & \overset{N\text{-}P}{\mathbf{U}_{12}} \\ \mathbf{0} & \mathbf{U}_{22} \end{pmatrix}\begin{matrix} P \\ N-P \end{matrix} \quad (6)$$

vhere $\mathbf{L}$ is unit lower triangular of dimension $N \times N$. The lecomposition (6) is called to be an RRQR factorization i.e. he separation of the eigenvalue spectrum of $\mathbf{\Gamma}$ into groups of "large" and "small" eigenvalues shows up in $\mathbf{U}$ as a small trailing block. The RRQR factorization reveals the numerical rank of $\mathbf{\Gamma}$ by having a well-conditioned leading submatrix $\mathbf{U}_{11}$, and a trailing submatrix $\mathbf{U}_{22}$ of small norm. According to [8], [10], $\lambda_{P+1}(\mathbf{\Gamma}) \leq \|\mathbf{U}_{22}\|_2$. With such factorization, we verify that

$$\begin{bmatrix}\mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{0} & \mathbf{U}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{U}_{11}^{-1}\mathbf{U}_{12} \\ -\mathbf{I}\end{bmatrix} = \begin{bmatrix}\mathbf{0} \\ -\mathbf{U}_{22}\end{bmatrix} \quad (7)$$

We deduct that $\mathbf{Z}_N = \begin{bmatrix}\mathbf{U}_{11}^{-1}\mathbf{U}_{12} & -\mathbf{I}\end{bmatrix}^T \mathbf{E}$ is a basis of noise subspace, where $\mathbf{E}$ is a matrix chosen such that the columns of $\mathbf{Z}_N$ have unit norm. So, the matrix $\mathbf{E}$ is $\mathbf{E} = \mathbf{U}_{12}^H \mathbf{U}_{11}^{-H} \mathbf{U}_{11}^{-1} \mathbf{U}_{12} + \mathbf{I}$, where $\mathbf{E}$ is a diagonal scaling matrix chosen such that the columns of $\mathbf{Z}_N$ have unit norm. Note that $\mathbf{Y}_N$ and $\mathbf{Z}_N$ are not an orthonormal basis, as was provided by the SVD method. One could obtain an orthonormal basis from $\mathbf{Y}_N$ by the Gram-Schmidt orthogonalization process [7]. However, in general, this is not necessary since the roots of $f(\theta)$ from Eq. (3) are identical for all bases $\mathbf{Y}_N$ and all bases $\mathbf{Z}_N$ of the noise subspace. Thus, by using a rank-revealing factorization, we can obtain the relevant subspace information without resorting to the more expensive SVD or eigenvalue decompositions. An important case where the conditions of application of the previous theorems (Eqs. (4) and (6)) are verified is that where the matrix $\mathbf{\Gamma}$ is *symmetric defined positive*.

III. Localization Methods

In this section a class of DOA estimators are briefly recalled. A relationship between the different estimators is established. Let $\mathbf{U}_N = [\mathbf{u}_{P+1}, \mathbf{u}_{P+2}, ..., \mathbf{u}_N]$ be the $N \times (N-P)$ matrix consisting of the eigenvectors associated to the $(N-P)$ smallest $(N-P)$ eigenvalues of the CSM $\mathbf{\Gamma}$.

A. Brief recall of the Minimum-norm method

The norm-minimum algorithm allows estimation of the DOA of signals in known number *P*, from signals measured on *N* sensors, by minimizing the spectral estimation:

$$F_{MN}(\theta) = \left[\mathbf{a}^H(\theta)\mathbf{U}_N\mathbf{U}_N^H\mathbf{e}_1\right]^{-1} \quad (8)$$

where $\mathbf{e}_1$ denotes an $(N \times 1)$ vector with all zero elements except the first one, equal to unity $\mathbf{e}_1 = [1,0,...,0]^T$. The minimum-norm algorithm estimates the DOA as the location of *P*, the highest peaks of $F_{MN}(\theta)$.

B. Brief recall of the MUSIC method

MUSIC relies on the decomposition of the observation space into two orthogonal subspaces, the signal subspace and the noise subspace. The method for source bearing estimation then consists in finding the DOA's θ as the

arguments of the maxima of the function $F_{MUSIC}(\theta)$ defined by:

$$F_{MUSIC}(\theta) = \left[\mathbf{a}^H(\theta)\mathbf{U}_N\mathbf{U}_N^H\mathbf{a}(\theta)\right]^{-1} \quad (9)$$

It has been shown that $F_{MUSIC}(\theta)$ has maximum points at round θ in $\{\theta_1,...,\theta_P\}$. Therefore we can estimate the DOA by taking the local maximal points of $F_{MUSIC}(\theta)$.

C. Localisation methods based on RRQR or RRLU factorizations

The DOA of signal sources are given by the minima of one of the following functions:

$$g(\theta) = \left[\mathbf{a}^H(\theta)\mathbf{Y}_N\mathbf{Y}_N^H\mathbf{a}(\theta)\right]^{-1/2} \quad \text{or} \quad h(\theta) = \left[\mathbf{a}^H(\theta)\mathbf{Z}_N\mathbf{Z}_N^H\mathbf{a}(\theta)\right]^{-1/2} \quad (10)$$

where $(\mathbf{Y}_N \mathbf{Y}_N^H)$ and $(\mathbf{Z}_N \mathbf{Z}_N^H)$ are the orthogonal projector upon the noise subspace.

D. Comparison of algorithms complexities

To evaluate the numerical complexity of the LU factorization algorithm, the Householder algorithm for the QR method and the MUSIC method, one counts the number of operations needed. We present in this section some examples of simulation to illustrate the performances of algorithms presented in the previous sections. It is clear that a comparative study detailed by these algorithms has to take into account the speed of convergence, performances in permanent state, precision of estimations and robustness. To compare the algorithms of the signal subspace, the same procedure of search for the DOA is used for all the algorithms. One supposes that errors concerning this search for the DOA show themselves in the same way on various compared algorithms. The CSM requires N^3 operations to obtain subspaces by means of a classic EDV. This subspaces estimation technique is not very reliable when sources are not very spaced out. The of the RRQR algorithm factorization is successful. It is difficult to estimate the exact complexity necessary for the implanting of the algorithm, given that it appeals to several external subroutines. By comparing the performances of the RRLU factorization algorithm with MUSIC and RRQR factorisation algorithms, we notice that algorithm of RRLU factorization turns out to be more reliable. This algorithm allows reduce of its complexity to $O\left(\frac{1}{3}N^3\right)$ operations from $O\left(\frac{2}{3}N^3\right)$ [9]. The complexities required by $\mathbf{U}_N$ in MUSIC is $O\left(N^3\right)$ [13]. So the LU method is approximately tow times faster than the QR method.

TABLE I
ALGORITHMS COMPLEXITIES FOR THE SUBSPACES ESTIMATION

MUSIC	RRQR factorization	RRLU factorization
N^3	$\frac{2}{3}N^3$	$\frac{1}{3}N^3$

IV. SIMULATION RESULTS

In this section, we describe experiments for localization of the sources using the QR factorization algorithm and the proposed algorithm introduced in section 3. In our simulations, we assumed a uniformly spaced linear array of seven sensors separated by half a wavelength and also assumed that the additive noise was stationary, uncorrelated with the signals sources, Gaussian with zero-mean. The number of snapshots taken was 100 and the number of observations was equal to 100. We considered two uncorrelated sources with equal power and SNR, impinging from 12° and 16°. We assume that the estimate of the number of sources is correct (equal to 2).

A. Experiment 1: uncorrelated noise

In practice, when the noise is spatially not white, its CSM deviates appreciably from the diagonal scaling matrix.

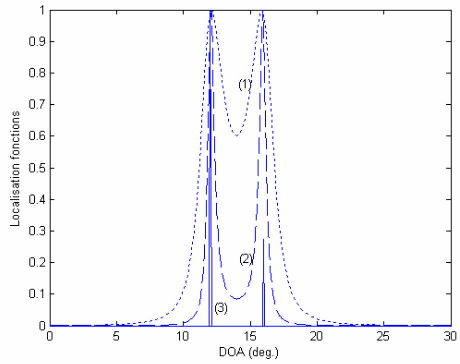

FIG. 1 Localisation method with: (1) - RRLU, (2) - RRQR and (3) – MUSIC, where SNR = 16dB.

The CSM of uncorrelated noise $\mathbf{\Gamma}_b$ used in our first experiment is: $\mathbf{\Gamma}_b = 0.1691\,\mathbf{I}$, where $\mathbf{I}$ is an identity matrix. Fig. 1 shows the obtained localization results of the tow simulated sources. MUSIC and minimum-norm methods localized the two sources better , where the noise is white, than the QR and LU methods.

B. Experiment 2: correlated noise

The localization, where the noise is correlated from sensor to sensor, is the delicate problem for MUSIC, minimum norm and QR factorization methods. An example of the $\mathbf{\Gamma}_b$, used in this section, is:

$$\mathbf{\Gamma}_b(i,m) = \sigma^2 \rho^{|i-m|} e^{j\pi(i-m)/2} \qquad (11)$$
$$\mathbf{\Gamma}_b(i,m) = 0 \quad si \ |i-m| > K$$

where K (K<N) is the length of the spatial correlation. Our examples are established in cases where the noise is correlated ($\sigma^2 = 1$, $\rho = 0.7$ and $K = 2$). Figs. 2 shows the results of the localisation with MUSIC algorithm, QR factorization method and the proposed LU factorization method. The performances of the MUSIC and RRQR factorization method degrade. Indeed, the proposed method reduces the computational load.

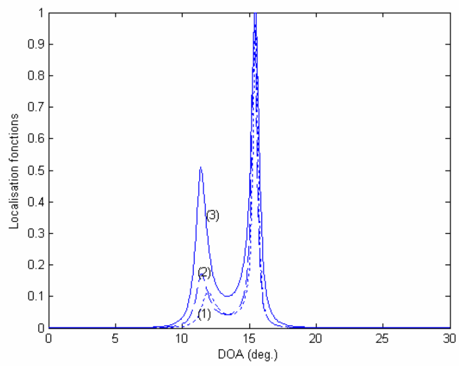

FIG. 2 Localisation methods with: (1) - RRLU, (2) - RRQR and (3) - MUSIC, with SNR = 3dB.

The MUSIC and RRQR factorization methods can distinguish between the two sources. On the other hand, the proposed method is realized by the projection on the signal-subspace to eliminate the components of the noise which are orthogonal in the signal sources. In order to point out the improvement of the localization of the sources based on the RRLU factorization of the CSM, our second experiment is carried out. With the same former signals, with the aim to study and to compare the performance of our algorithm to the classical minimum-norm method. Fig. 3 shows the obtained localization results of the two simulated sources. We can conclude that the RRLU factorization of the CSM of the received data have improved the spatial resolution. Indeed, the two sources are perfectly localized. The obtained results show the performance of our algorithm to localize the narrowband uncorrelated sources very faster than minimum norm method. To estimate the performances of our method of proposed location, we define the square root of the error quadratic mean that one notes SREQM:

$$SREQM = \frac{1}{P}\left[\sum_{p=1}^{P}\left|\theta_p - \hat{\theta}_p\right|^2\right]^{1/2} \quad (12)$$

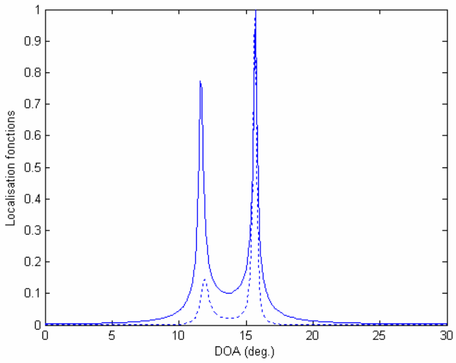

FIG. 3 Localisation with: (__) minimum-norm; (--) proposed RRLU factorization method. SNR= 4dB

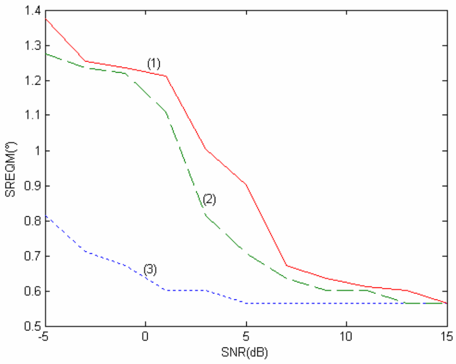

FIG. 4 Various techniques std location according to SNR: (1) MUSIC; (2) QR method; (3) proposed algorithm.

To compare the performances of the method which we propose to classic algorithms, we calculated standard deviation for every operator of location by using expression (12). The study is based on the analysis of synthetic signals. We were interested in the variations of standard deviation (std) according to the signal on noise ratio (SNR=- 5, …, 15dB).

Not that where Signal on Noise Ratio (SNR) is high, the standard deviation of estimation is even when one applies our algorithm which uses RRLU factorization by comparing it with the techniques of coherent signal subspace based on the MUSIC method and RRQR factorization. On the other hand, we confirm improvement due to the RRLU factorization included in our algorithm. We conclude that our method presents the weakest values of SREQM in every case studied.

V. Conclusion

We presented in this paper methods which use the subspaces definition to estimate the DOA. These methods put into action operations on the CSM of signals, received on sensors' array. Besides, their principle remains valid what ever the geometry of the array. We have improved the high resolution methods for locating the narrow-band sources. This improvement is based on the recent results of the linear algebra concerning the LU factorization of the CSM. A major advantage of our method, it does not require the eigendecomposition of the CSM. The upper triangular matrix, obtained by RRLU factorization, has been used to improve the RRQR factorization method complexities. The RRLU method is approximately two times faster than the QR factorization method .

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