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# Globally exponential stability for Hopfield neural networks with delays and impulsive perturbations

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*Abstract*—In this paper, we consider the global exponential stability of the equilibrium point of Hopfield neural networks with delays and impulsive perturbation. Some new exponential stability criteria of the system are derived by using the Lyapunov functional method and the linear matrix inequality approach for estimating the upper bound of the derivative of Lyapunov functional. Finally, we illustrate two numerical examples showing the effectiveness of our theoretical results.

*Keywords*—Hopfield Neural Networks, Exponential stability.

## I. INTRODUCTION

SINCE the American physicist Hopfield brought forward the Hopfield neural network (HNN) in 1982. It has been extensively studied and developed in recent years, and it has attracted much attention in the literature on Hopfield neural networks with time delays, (see, e.g., [1-4]). They are now recognized as candidates for information processing systems and have been successfully applied to associative memory, pattern recognition, automatic control, model identification, optimization problems, etc. (we refer to reader [5-12]). Therefore, the study of stability of HNN has caught many researchers attention. HNN with time delays has been extensively investigated over the years, and various sufficient conditions for the stability of the equilibrium point of such neural networks have been presented via different approaches. In [7], [15], some sufficient conditions of stability by utilizing the Lyapunov functional method, and linear matrix inequality approach for delayed continuous HNN are derived. In [16], G.Zong and J.Liu established a novel delay-dependent condition to guarantee the existence of HNN and its global asymptotic stability by resorting to the integral inequality and constructing a Lyapunov-Krasovskii functional. In [18], S.Long and D.Xu got the sufficient conditions for global exponential stability and global asymptotic stability by using Lyapunov-Krasovskii-type functionaly of negative definite matrix and Cauchy criterion.

In this paper, we consider a class of HNN with delays and impulsive perturbations. Some new sufficient conditions for the global exponential stability of the equilibrium point for such system are obtained by means of using a Lyapunov functional. The effects of impulses and delays on the solutions are stressed here. The conditions on global exponential stability are simpler and less restrictive versions of some recent results.

This paper is organized as follows: In section II, an impulsive continuous Hopfield neural network with delays model is described. In addition, we present some basic definitions and lemmas. New stability criteria for continuous Hopfield neural network are derived in section III. Two examples are given in section IV, to illustrate the advantage of the results obtained. Finally, some conclusions are drawn in section V.

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## II. PRELIMINARIES

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{Z}_+$ denote the positive integers and $\mathbb{R}^n$ denote the $n$-dimensional real space equipped with the Euclidean norm $\|.\|$.

Consider the following delayed HNN model with impulses

$$\begin{cases} \dot{x}_i(t) = -c_i x_i(t) + \sum\limits_{j=1}^{n} a_{ij} f_j(x_j(t)) \\ + \sum\limits_{j=1}^{n} b_{ij} g_j(x_j(t-\tau(t))) + I_i \ si\ t \neq t_k \\ \triangle x_i \backslash_{t=t_k} = x_i(t_k) - x_i(t_k^-) \quad i = 1, ..., n, \quad n, k \in \mathbb{Z}_+, \end{cases} \tag{1}$$

where $n \geq 2$ corresponds to the number of units in a neural network; the impulsive times $t_k$ satisfy $0 \leq t_0 < t_1 < ... < t_k < ...,\ \lim\limits_{k \longrightarrow +\infty} t_k = +\infty$; $x_i$ corresponds to the state of the unit $i$ at time $t$; $c_i$ is positive constant; $f_j, g_j$, denote respectively, the measures of response or activation to its incoming potentials of the unit $j$ at time $t$ and $t - \tau(t)$; constant $a_{ij}$ denotes the synaptic connection weight of the unit $j$ on the unit $i$ at time $t$; constant $b_{ij}$ denotes the synaptic connection weight of the unit $j$ on the unit $i$ at time $t - \tau(t)$; $I_i$ is the input of the unit $i$; $\tau(t)$ is the transmission delay such that $0 < \tau(t) \leq \tau$ and $\dot{\tau}(t) \leq \rho < 1$; $t \geq t_0$; $\tau, \rho$ are constants.

The initial conditions associated with system (1) are of the form:

$$x(s) = \phi(s),\ s \in [t_0 - \tau, t_0], \tag{2}$$

where $x(s) = (x_1(s), x_2(s), ..., x_n(s))^T$,
$\phi(s) = (\phi_1(s), \phi_2(s), ..., \phi_n(s))^T \in PC([-\tau, 0], \mathbb{R}^n) = \{\psi : [-\tau, 0] \longrightarrow \mathbb{R}^n$, is continuous everywhere except at finite number of points $t_k$, at which $\psi(t_k^+)$ and $\psi(t_k^-)$ exist and $\psi(t_k^+) = \psi(t_k)\}$. For $\psi \in PC([-\tau, 0], \mathbb{R}^n)$, the norm of $\psi$ is defined by $\|\psi\|_\tau = \sup\limits_{-\tau \leq \theta \leq 0} \|\psi(\theta)\|$. For any $t_0 \geq 0$, let $PC_\delta(t_0) = \{\psi \in PC([-\tau, 0], \mathbb{R}^n) : \|\psi\|_\tau < \delta\}$.

In this paper, we assume that some conditions are satisfied so that the equilibrium point of system (1) does exist, see ([7], [13]). Assume that $\bar{x} = (\bar{x_1}, \bar{x_2}, ..., \bar{x_n})$ is an equilibrium point of system (1). Impulsive operator is viewed as perturbation of the equilibrium point $\bar{x}$ of such system without impulsive effects. We assume that $\Delta x_i \backslash_{t=t_k} = x_i(t_k) - x_i(t_k^-) = d_k^{(i)}(x_i(t_k^-) - \bar{x_i})$, $d_k^{(i)} \in \mathbb{R}$, $i = 1, 2, ..., n$, $k = 1, 2, ...$.

Since $\bar{x}$ is an equilibrium point of system (1), one can derive from system (1) that the transformation $y_i = x_i - \bar{x}_i$, $i = 1, 2, ..n$ transforms such system into the following system:

$$\begin{cases} \dot{y}_i(t) = -c_i y_i(t) + \sum\limits_{j=1}^{n} a_{ij} F_j(y_j(t)) \\ + \sum\limits_{j=1}^{n} b_{ij} G_j(y_j(t - \tau(t))) \ if \ t \neq t_k \\ y_i(t_k) = (1 + d_k^{(i)}) y_i(t_k^-) \quad i = 1, ..., n, \quad n, k \in \mathbb{Z}_+ \end{cases}, \tag{3}$$

where

$$F_j(y_j(t)) = f_j(\bar{x}_j + y_j(t)) - f_j(\bar{x}_j)$$

$$G_j(y_j(t - \tau(t))) = g_j(\bar{x}_j + y_j(t - \tau(t))) - g_j(\bar{x}_j).$$

To prove the stability of $\bar{x}$ of system (1), it is sufficient to prove the stability of the zero solution of system (3).
In this paper, we assume that there exist constants $L_i$, $M_i \geq 0$ such as $\mid F_i(y) \mid \leq L_i \mid y \mid$, $\mid G_i(y) \mid \leq M_i \mid y \mid$, $i \in \Lambda = \{1, 2, ...n\}$
$L_{max} = \max\limits_{i \in \Lambda} L_i$, $M_{max} = \max\limits_{i \in \Lambda} M_i$
$c_{max} = \max\limits_{i \in \Lambda} c_i$, $c_{min} = \min\limits_{i \in \Lambda} c_i$
$D_k = diag(1 + d_k^{(1)}, 1 + d_k^{(2)}, ..., 1 + d_k^{(n)})$

Some definitions and lemma of stability for system (1) at its equilibrium point are introduced as follows:

### A. Definition

Assume $y(t) = y(t_0, \varphi)(t)$ be the solution of (3) through $(t_0, \varphi)$. then the zero solution of (3) is said to be [14]

- P1 stable, if for any $\epsilon > 0$ and $t_0 \geq 0$, there exists some $\delta(\epsilon, t_0) > 0$ such as $\varphi \in PC_\delta(t_0)$ implies $\|y(t_0, \varphi)(t)\| < \epsilon, \ t \geq t_0$.
- P2 globally exponentially stable, if there exists constant $\alpha > 0$, $\beta \geq 1$ such that for any initial value $\varphi$, $\|y(t_0, \varphi)(t)\| \leq \beta \|\varphi\|_\tau e^{-\alpha(t-t_0)}$.

We now need the following basic lemma used in our work.

### B. Lemma

For any $a, b \in \mathbb{R}^n$, the inequality

$$\pm 2a^T b \leq a^T X a + b^T X^{-1} b$$

holds, where $X$ is any $n \times n$ matrix with $X > 0$ [17].

## III. Main results

Now, we shall establish an theorem which provide sufficient conditions for global exponential stability of system (1).

### A. Theorem

Assumes there are constants $\bar{\epsilon} > 0$ , $\sigma > 0$ and $n \times n$ definite positive matrix $Q$ satisfy:

$$\max_{i,j} q_{ij} < \frac{e^{-\bar{\epsilon}\tau} \cdot \min\limits_{i,j} |b_{ij}|}{\sigma \cdot \max\limits_{i} c_i}, \quad \forall i, j \in \{1, 2, ..., n\}$$

and assume that the following conditions are satisfy:

(i) $\frac{\bar{\epsilon}}{c_{min}} + \max\limits_{1 \leq i \leq n} \{\frac{1}{c_i} \sum\limits_{j=1}^{n} a_{ij}\} + \max\limits_{1 \leq j \leq n} \{L_j^2 \sum\limits_{i=1}^{n} \frac{a_{ij}}{c_i}\}$ $+ \frac{\lambda_{max}(C^{-1}BQ^{-1}B^TC^{-1})}{\sigma} + \max\limits_{1 \leq j \leq n} \{M_j^2 \sum\limits_{i=1}^{n} \frac{1}{c_i}|b_{ij}|\} < 2$

(ii) There exist constants $\nu \geq 0$, $\bar{\alpha} \in [0, \bar{\epsilon}[$ such that: $\sum\limits_{k=1}^{m} ln \ \max\{\xi_k.c_{max}, 1\} - \bar{\alpha}(t_m - t_0) < \nu$, $\forall m \in \mathbb{Z}_+$,

where $\xi_k$ is the largest eigenvalue of $D_k C^{-1} D_k$.

Then, the equilibrium point of system (1) is globally exponentially stable and approximate exponentially convergent rate is $\frac{(\bar{\epsilon} - \bar{\alpha})}{2}$.

If more $Q = I_n$ in this Theorem, then we have this corollary:

### B. Corollary

Assume that there exist constants $\bar{\epsilon} > 0$, $\sigma > 0$ such as: $\sigma < \frac{e^{-\bar{\epsilon}\tau} \cdot \min\limits_{i,j} |b_{ij}|}{\max\limits_{i} c_i}$ and

(i) $\frac{\bar{\epsilon}}{c_{min}} + \max\limits_{1 \leq i \leq n} \{\frac{1}{c_i} \sum\limits_{j=1}^{n} a_{ij}\} + \max\limits_{1 \leq j \leq n} \{L_j^2 \sum\limits_{i=1}^{n} \frac{a_{ij}}{c_i}\}$ $+ \frac{\lambda_{max}(C^{-1}BB^TC^{-1})}{\sigma} + \max\limits_{1 \leq j \leq n} \{M_j^2 \sum\limits_{i=1}^{n} \frac{1}{c_i}|b_{ij}|\} < 2$

(ii) There are constants $\nu \geq 0$, $\bar{\alpha} \in [0, \bar{\epsilon}]$ such as

$$\sum_{k=1}^{m} \ln \max\{c_{max} \max_{i=1,2,...,n} (1 + d_k^{(i)})^2, 1\} - \bar{\alpha}(t_m - t_0) < \nu$$

for all $m \in \mathbb{Z}_+$ holds.

Then, the equilibrium point of system (1) is globally exponentially stable and the approximate exponentially convergent rate is $\frac{(\bar{\epsilon} - \bar{\alpha})}{2}$.

## IV. Numerical applications

In this section, we present two numerical examples to illustrate that our conditions are more feasible than that given in earlier reference ([19],[25]).

### A. Example1

Consider the two-neuron delayed neural network with impulses [19] as follows:

$$\begin{cases} \dot{x_1}(t) = -x_1(t) + \frac{1}{8} f_1(x_1(t)) + \frac{1}{4} f_2(x_2(t)) \\ + \frac{1}{3} g_1(x_1(t - \tau)) - \frac{1}{6} g_2(x_2(t - \tau)) \\ \\ \dot{x_2}(t) = -x_2(t) + \frac{1}{4} f_1(x_1(t)) + \frac{1}{8} f_2(x_2(t)) \\ - \frac{1}{6} g_1(x_1(t - \tau)) + \frac{1}{4} g_2(x_2(t - \tau)) \\ \\ x(t_k) = \gamma_k x(t_k^-), \qquad k = 1, 2, ... \end{cases} \tag{4}$$

where $t_k - t_{k-1} = 1$, $\gamma_k = (-1)^k\sqrt{\frac{e^{0.224}+4}{5}}$, $k \in \mathbb{Z}_+$. Here we consider $\tau = 0.7$.

$$f_1(x) = f_2(x) = g_1(x) = g_2(x) = 0.5(|x+1| - |x-1|)$$

$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $A = \begin{pmatrix} \frac{1}{8} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{8} \end{pmatrix}$; $B = \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{4} \end{pmatrix}$

Next we show that the equilibrium point of system (4) is globally exponentially stable with $\tau \leq 0.7$.

It is easy to calculate that $L = M = 1$, $d_k^{(i)} = (-1)^k\sqrt{\frac{e^{0.224}+4}{5}} - 1$. Then, we may choose $\bar{\epsilon} = 0.0695$, $\bar{\alpha} = 0.0495$, $\sigma = e^{-1.15}$, $\nu = 0$, $Q = Id$.

It is clear that:

$$\bar{\alpha} \in [0, \bar{\epsilon}]$$

and

$$\sigma < \frac{|b| \cdot e^{-\bar{\epsilon}\tau}}{c}$$

Furthermore, we compute

$\frac{\bar{\epsilon}}{c_{min}} + \max_{1\leq i\leq 2}\{\frac{1}{c_i}\sum_{j=1}^{2}|a_{ij}|\} + \max_{1\leq j\leq 2}\{L_j^2\sum_{i=1}^{2}\frac{1}{c_i}|a_{ij}|\}$
$+ \frac{\lambda_{max}(C^{-1}BB^TC^{-1})}{\sigma} + \max_{1\leq j\leq 2}\{M_j^2\sum_{i=1}^{2}\frac{1}{c_i}|b_{ij}|\} < 2$

We also get, for any $m \in \mathbb{Z}_+$,

$\sum_{k=1}^{m}\ln\max\{\max_{i=1,2,...n}(1+d_k^{(i)})^2, 1\} - \bar{\alpha}(t_m - t_0)$

$$= m\ln\frac{e^{0.224}+4}{5} - 0.0495m \simeq -m0.0005 < 0 = \nu$$

Then, from Corollary III-B, the equilibrium point of system (4) $(0,0)^T$ is globally exponentially stable with approximate exponential convergence rate $0.01$. But for any $\alpha$, $A + A^T + \alpha I$ is not negative definite. Hence, the result in [20] cannot applied in this case.

***Remark:*** Meantime, the matrices $-(A + A^T)$ is obtained as

$$-(A + A^T) = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{4} \end{pmatrix}$$

It is obvious that $-(A + A^T)$ is not a positive definite. Therefore, the condition in ([21]-[24]) does not hold.

### B. Example2

Consider the two-neuron delayed neural network with impulses [25] as follows:

$$\begin{cases} \dot{x_1}(t) = -x_1(t) - 0.1f_1(x_1(t)) + 0.1f_2(x_2(t)) \\ -0.1g_1(x_1(t-\tau)) + 0.2g_2(x_2(t-\tau)) \\ \\ \dot{x_2}(t) = -x_2(t) + 0.1f_1(x_1(t)) - 0.1f_2(x_2(t)) \\ +0.2g_1(x_1(t-\tau)) + 0.1g_2(x_2(t-\tau)) \\ \\ x(t_k) = \gamma_k x(t_k^-),\ k = 1, 2, ... \end{cases} \tag{5}$$

where $t_k - t_{k-1} = 1$, $\gamma_k = (-1)^k\sqrt{\frac{e^{0.224}+4}{5}}$, $k \in \mathbb{Z}_+$. Here we consider $\tau = 1$.

$$f_1(x) = f_2(x) = g_1(x) = g_2(x) = 0.5(|x+1| - |x-1|)$$

$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $A = \begin{pmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{pmatrix}$

$B = \begin{pmatrix} -0.1 & 0.2 \\ 0.2 & 0.1 \end{pmatrix}$

Next we show that the equilibrium point of system (5) is globally exponentially stable with $\tau \leq 1$.

It is easy to calculate that $L = M = 1$, $d_k^{(i)} = (-1)^k\sqrt{\frac{e^{0.224}+4}{5}} - 1$. Then, we may choose $\bar{\epsilon} = 0.8695$, $\bar{\alpha} = 0.0495$, $\sigma = e^{-3.18}$, $\nu = 0$, $Q = Id$.

It is clear that:

$$\bar{\alpha} \in [0, \bar{\epsilon}]$$

and

$$\sigma < \frac{e^{-\bar{\epsilon}\tau}.\min_{i,j}|b_{ij}|}{\max_i c_i}$$

Furthermore, we compute

$\frac{\bar{\epsilon}}{c_{min}} + \max_{1\leq i\leq 2}\{\frac{1}{c_i}\sum_{j=1}^{2}|a_{ij}|\} + \max_{1\leq j\leq 2}\{L_j^2\sum_{i=1}^{2}\frac{1}{c_i}|a_{ij}|\}$
$+ \frac{\lambda_{max}(C^{-1}BB^TC^{-1})}{\sigma} + \max_{1\leq j\leq 2}\{M_j^2\sum_{i=1}^{2}\frac{1}{c_i}|b_{ij}|\} < 2$

We also get, for any $m \in \mathbb{Z}_+$,

$\sum_{k=1}^{m}\ln\max\{\max_{i=1,2,...n}(1+d_k^{(i)})^2, 1\} - \bar{\alpha}(t_m - t_0)$

$$= m\ln\frac{e^{0.224}+4}{5} - 0.0495m \simeq -m0.0005 < 0 = \nu$$

Then, from Corollary III-B, the equilibrium point of system (5) is globally exponentially stable with approximate exponential convergence rate $0.41$.

***Remark:*** In the work [25], authors proved that equilibrium point of system (5) is globally exponentially stable. According to their works, the maximum allowable bound $\tau$ for guaranteeing the exponential stability of system (5) is $0.5$ and the convergence rate is $0.19$. On the other hand, our delay-dependent exponential stability criterion in Corollary III-B presents $\tau = 1$ and the convergence rate is $0.41$. It is clear that for this example our criterion is less conservative than the existing delay-dependent criteria [25].

## V. Conclusion

In this paper, a class of HNN with delays and impulsive perturbations is considered. The problems of exponential stability and exponential convergence rate for neural networks with time-varying delays have been studied. We obtain some new criteria ensuring the global exponential stability of the equilibrium point for such system by using the Lyapunov method and linear matrix inequality. Our results show the effects of delay and impulsive to the stability of HNN. The results here are discussed from the point of view to its comparaison with earlier results. In comparison with some recent results reported in the literature, the present results provide new stability criteria for delayed neural networks. As well as these results can be applied to the case uncovered in some earlier references. Our criterias are more simpler to verify. An examples is given to illustrate the feasibility and efficiency of the results. Has the continuation of this work, we can refine and generalize our results for high-order Hopfield type neural networks.

## Appendix A
## Proof of the TheoremIII-A

Consider the Lyapunov functional as follows:

$$V(y)(t)=\sum_{i=1}^{n}\frac{1}{c_i}e^{\bar{\epsilon}t}y_i^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|b_{ij}|}{c_i}\int_{t-\tau(t)}^{t}e^{\bar{\epsilon}s}G_j^2(y_j(s))ds$$

It is clear that : $V(y)(t)>0,\ \forall y\neq 0$

We have:

$$\begin{aligned}&\tfrac{1}{c_{max}}e^{\bar{\epsilon}t}\|y(t)\|^2<V(y(t))\\ &\le \tfrac{1}{c_{min}}e^{\bar{\epsilon}t}\|y(t)\|^2+\sum_{i=1}^{n}\sum_{j=1}^{n}\tfrac{|b_{ij}|}{c_i}M_j^2\int_{t-\tau(t)}^{t}e^{\bar{\epsilon}s}|y_j(s)|^2ds\\ &\le \tfrac{1}{c_{min}}e^{\bar{\epsilon}t}\|y(t)\|^2+\tfrac{M_{max}^2}{c_{min}}\sup_{1\le j\le n}\sum_{i=1}^{n}|b_{ij}|\|y(t)\|^2\int_{t-\tau(t)}^{t}e^{\bar{\epsilon}s}ds\\ &\le \tfrac{1}{c_{min}}e^{\bar{\epsilon}t}\|y(t)\|^2+\tfrac{M_{max}^2}{c_{min}}\|B\|\|y(t)\|^2\tfrac{1}{\bar{\epsilon}}e^{\bar{\epsilon}t}(1-e^{-\bar{\epsilon}\tau(t)})\\ &\le [\tfrac{1}{c_{min}}+\tfrac{M_{max}^2}{c_{min}}\|B\|(\tfrac{1}{\bar{\epsilon}}(1-e^{-\bar{\epsilon}\tau(t)}))]e^{\bar{\epsilon}t}\|y(t)\|^2\end{aligned}$$

Therefore,

$$V(y)(t)\le[\frac{1}{c_{min}}+\frac{M_{max}^2}{c_{min}}\|B\|(\frac{1}{\bar{\epsilon}}(1-e^{-\bar{\epsilon}\tau}))]e^{\bar{\epsilon}t}\|y(t)\|^2 \quad (6)$$

Besides we have:

$\forall k\ge 1$

$$\begin{aligned}&V(y)(t_k)=\sum_{i=1}^{n}\tfrac{1}{c_i}e^{\bar{\epsilon}t_k}y_i^2(t_k)\\ &+\sum_{i=1}^{n}\sum_{j=1}^{n}\tfrac{|b_{ij}|}{c_i}\int_{t_k-\tau(t_k)}^{t_k}e^{\bar{\epsilon}s}G_j^2(y_j(s))ds\\ &=e^{\bar{\epsilon}t_k}y^T(t_k)C^{-1}y(t_k)+\sum_{i=1}^{n}\sum_{j=1}^{n}\tfrac{|b_{ij}|}{c_i}\int_{t_k-\tau(t_k)}^{t_k}e^{\bar{\epsilon}s}G_j^2(y_j(s))ds\\ &=e^{\bar{\epsilon}t_k}y^T(t_k^-)D_kC^{-1}D_ky(t_k^-)\\ &+\sum_{i=1}^{n}\sum_{j=1}^{n}\tfrac{|b_{ij}|}{c_i}\int_{t_k^- -\tau(t_k^-)}^{t_k^-}e^{\bar{\epsilon}s}G_j^2(y_j(s))ds\\ &\le e^{\bar{\epsilon}t_k}\xi_k y^T(t_k^-)y(t_k^-)+\sum_{i=1}^{n}\sum_{j=1}^{n}\tfrac{|b_{ij}|}{c_i}\int_{t_k^- -\tau(t_k^-)}^{t_k^-}e^{\bar{\epsilon}s}G_j^2(y_j(s))ds\\ &\le e^{\bar{\epsilon}t_k}\tfrac{\xi_k}{\lambda_{min}(C^{-1})}y^T(t_k^-)C^{-1}y(t_k^-)\\ &+\sum_{i=1}^{n}\sum_{j=1}^{n}\tfrac{|b_{ij}|}{c_i}\int_{t_k^- -\tau(t_k^-)}^{t_k^-}e^{\bar{\epsilon}s}G_j^2(y_j(s))ds\end{aligned}$$

Therefore,

$$V(y(t_k))\le\max\{\xi_k.c_{max},1\}V(t_k^-) \quad (7)$$

On the other hand, from (7), we have:

$$\frac{1}{c_{max}}e^{\bar{\epsilon}t}\|y(t)\|^2\le V(t)\le V(t_0)\prod_{t_0<t_k\le t}\max\{\xi_k.c_{max},1\} \quad (8)$$

And from (6) we have:

$$V(t_0)\le[\frac{1}{c_{min}}+\frac{M_{max}^2}{c_{min}}\|B\|(\frac{1}{\bar{\epsilon}}(1-e^{-\bar{\epsilon}\tau}))]e^{\bar{\epsilon}t_0}\|\varphi\|^2$$

Therefore,

$$\|y(t)\|^2\le[\tfrac{c_{max}}{c_{min}}+\tfrac{c_{max}.M_{max}^2}{c_{min}}\|B\|(\tfrac{1}{\bar{\epsilon}}(1-e^{-\bar{\epsilon}\tau}))]$$
$$e^{-\bar{\epsilon}(t-t_0)}\|\varphi\|^2\times\prod_{t_0\le t_k\le t}\max\{\xi_k.c_{max},1\}$$

From the condition (ii) we will have:

$$\|y(t)\|^2\le[\tfrac{c_{max}}{c_{min}}+\tfrac{c_{max}.M_{max}^2}{c_{min}}\|B\|(\tfrac{1}{\bar{\epsilon}}(1-e^{-\bar{\epsilon}\tau}))]$$
$$e^{-(\bar{\epsilon}-\bar{\alpha})(t-t_0)}e^{\nu}\|\varphi\|^2$$

So,

$$\|y(t)\|^2\le M^{'}\|\varphi\|^2e^{-\frac{1}{2}(\bar{\epsilon}-\bar{\alpha})(t-t_0)},\ \forall t\ge t_0$$

Where

$$M^{'}=\sqrt{[\tfrac{c_{max}}{c_{min}}+\tfrac{c_{max}.M_{max}^2}{c_{min}}\|B\|(\tfrac{1}{\bar{\epsilon}}(1-e^{-\bar{\epsilon}\tau}))]e^{\nu}}\ge 1$$

Hence, the zero solution of (1) is globally exponentially stable.

Verify now is function $V(t)$ is Lyapunov function. To ensure that, it is sufficient to show that:

$$\frac{\partial V(y)(t)}{\partial t}<0$$

We have:

$$\begin{aligned}\tfrac{\partial V(y)(t)}{\partial t}&=\bar{\epsilon}e^{\bar{\epsilon}t}\sum_{i=1}^{n}\tfrac{1}{c_i}y_i^2(t)+e^{\bar{\epsilon}t}\sum_{i=1}^{n}\tfrac{1}{c_i}2y_i(t)\dot{y}_i(t)+\\ &\sum_{i=1}^{n}\sum_{j=1}^{n}\tfrac{|b_{ij}|}{c_i}[e^{\bar{\epsilon}t}G_j^2(y_j(t))\\ &-e^{\bar{\epsilon}(t-\tau(t))}G_j^2(y_j(t-\tau(t)))(1-\dot{\tau}(t))]\end{aligned}$$

We know that:

$$\dot{y}_i(t)=-c_iy_i(t)+\sum_{j=1}^{n}a_{ij}F_j(y_j(t))+\sum_{j=1}^{n}b_{ij}G_j(y_j(t-\tau(t))$$

Therefore,

$$\begin{aligned}&\tfrac{\partial V(y)(t)}{\partial t}=\bar{\epsilon}e^{\bar{\epsilon}t}\sum_{i=1}^{n}\tfrac{1}{c_i}y_i^2(t)\\ &+e^{\bar{\epsilon}t}\sum_{i=1}^{n}\tfrac{1}{c_i}2y_i(t)(-c_iy_i(t)\end{aligned}$$

$+\sum\limits_{j=1}^{n} a_{ij}F_j(y_j(t)) + \sum\limits_{j=1}^{n} b_{ij}G_j(y_j(t-\tau(t)))$
$+\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\frac{|b_{ij}|}{c_i}[e^{\bar{\epsilon}t}G_j^2(y_j(t))$
$-e^{\bar{\epsilon}(t-\tau(t))}G_j^2(y_j(t-\tau(t)))(1-\dot{\tau}(t))]$

Therefore,

$$\begin{aligned}
\frac{\partial V(y)(t)}{\partial t} &\le \bar{\epsilon}e^{\bar{\epsilon}t}\sum_{i=1}^{n}\frac{1}{c_i}y_i^2(t)\\
&- 2e^{\bar{\epsilon}t}\sum_{i=1}^{n}y_i^2(t)\\
&+ 2e^{\bar{\epsilon}t}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{c_i}a_{ij}y_i(t)F_j(y_j(t))\\
&+ 2e^{\bar{\epsilon}t}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{c_i}b_{ij}y_i(t)G_j(y_j(t-\tau(t)))\\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|b_{ij}|}{c_i}e^{\bar{\epsilon}t}G_j^2(y_j(t))\\
&- e^{\bar{\epsilon}(t-\tau(t))}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|b_{ij}|}{c_i}G_j^2(y_j(t-\tau(t)))
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial V(y)(t)}{\partial t} &\le \bar{\epsilon}e^{\bar{\epsilon}t}\sum_{i=1}^{n}\frac{1}{c_i}y_i^2(t)\\
&- 2e^{\bar{\epsilon}t}\sum_{i=1}^{n}y_i^2(t)\\
&+ e^{\bar{\epsilon}t}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{c_i}a_{ij}y_i^2(t) + e^{\bar{\epsilon}t}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{c_i}a_{ij}F_j^2(y_j(t))\\
&+ 2e^{\bar{\epsilon}t}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{c_i}b_{ij}y_i(t)G_j(y_j(t-\tau(t)))\\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|b_{ij}|}{c_i}e^{\bar{\epsilon}t}G_j^2(y_j(t))\\
&-e^{\bar{\epsilon}(t-\tau(t))}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|b_{ij}|}{c_i}G_j^2(y_j(t-\tau(t))) \qquad (9)
\end{aligned}$$

We have by LemmaII-B:

$$\begin{aligned}
2\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{c_i}b_{ij}y_i(t)G_j(y_j(t-\tau(t)))&\\
&= 2y^T(t)C^{-1}BG(y(t-\tau(t)))\\
&= 2G^T(y(t-\tau(t)))B^TC^{-1}y(t)\\
&= 2[G(y(t-\tau(t)))\sqrt{\sigma}]^T(B^TC^{-1}y(t)\frac{1}{\sqrt{\sigma}})\\
&\le \sigma G^T(y(t-\tau(t)))QG(y(t-\tau(t)))\\
&+\frac{1}{\sigma}y^T(t)C^{-1}BQ^{-1}B^TC^{-1}y(t)\\
&\le \sigma\sum_{i=1}^{n}\sum_{j=1}^{n}q_{ij}G_j^2(y_j(t-\tau(t)))\\
&+\frac{1}{\sigma}\lambda_{max}(C^{-1}BQ^{-1}B^TC^{-1})\sum_{i=1}^{n}y_i^2(t) \qquad (10)
\end{aligned}$$

By substituting (10) in (9), we will have this result

$$\begin{aligned}
\frac{\partial V(y)(t)}{\partial t} &\le \bar{\epsilon}e^{\bar{\epsilon}t}\sum_{i=1}^{n}\frac{1}{c_i}y_i^2(t) - 2e^{\bar{\epsilon}t}\sum_{i=1}^{n}y_i^2(t)\\
&+ e^{\bar{\epsilon}t}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{c_i}a_{ij}y_i^2(t)\\
&+ e^{\bar{\epsilon}t}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{c_i}a_{ij}F_j^2(y_j(t))\\
&+ e^{\bar{\epsilon}t}[\sigma\sum_{i=1}^{n}\sum_{j=1}^{n}q_{ij}G_j^2(y_j(t-\tau(t)))\\
&+ \frac{1}{\sigma}\lambda_{max}(C^{-1}BQ^{-1}B^TC^{-1})\textstyle\sum_{i=1}^{n}y_i^2(t)]\\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|b_{ij}|}{c_i}e^{\bar{\epsilon}t}G_j^2(y_j(t))\\
&- e^{\bar{\epsilon}(t-\tau(t))}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|b_{ij}|}{c_i}G_j^2(y_j(t-\tau(t)))
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial V(y)(t)}{\partial t} &\le [\frac{\bar{\epsilon}}{c_{min}} - 2 + \max_{1\le i\le n}\{\frac{1}{c_i}\sum_{j=1}^{n}a_{ij}\}\\
&+ \max_{1\le j\le n}\{L_j^2\sum_{i=1}^{n}\frac{a_{ij}}{c_i}\} + \frac{1}{\sigma}\lambda_{max}(C^{-1}BQ^{-1}B^TC^{-1})\\
&+ \max_{1\le j\le n}\{M_j^2\sum_{i=1}^{n}\frac{1}{c_i}|b_{ij}|\}]e^{\bar{\epsilon}t}\sum_{i=1}^{n}y_i^2(t)\\
&+ \{e^{\bar{\epsilon}t}\sigma\sum_{i=1}^{n}\sum_{j=1}^{n}q_{ij}G_j^2(y_j(t-\tau(t)))\\
&- e^{\bar{\epsilon}(t-\tau(t))}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|b_{ij}|}{c_i}G_j^2(y_j(t-\tau(t)))\}
\end{aligned}$$

Therefore,

$$\frac{\partial V(y)(t)}{\partial t} < 0$$

Wich completes the proof.

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