A Cell-centered Diffusion Finite Volume Scheme and its Application to Magnetic Flux Compression Generators

Qiang ZHAO, Yina SHI, Guangwei YUAN, and Zhiwei DONG

Abstract—A cell-centered finite volume scheme for discretizing diffusion operators on distorted quadrilateral meshes has recently been designed and added to APMFCG to enable that code to be used as a tool for studying explosive magnetic flux compression generators. This paper describes this scheme. Comparisons with analytic results for 2-D test cases are presented, as well as 2-D results from a test of a “realistic” generator configuration.

Keywords—cell-centered FVM, distorted meshes, diffusion scheme, MFCG.

I. INTRODUCTION

Investigating the numerical schemes with high accuracy for diffusion equation on distorted meshes is very important in Lagrangian hydrodynamics and magnetohydrodynamics. Lagrangian method is widely used for the reason that it has a simple structure and keeps clear track of the material interface. Since the computational mesh in Lagrangian method is embedded in the fluid, it will follow the movement of the fluid and consequently, even being orthogonal initially, will become highly skewed or highly distorted, which will reduce the accuracy of the corresponding discrete schemes or even worse, stop the whole computation. If the discrete schemes have a high adaptability to the Lagrangian mesh, then the number of the mesh reconstruction will be much reduced or even better, the whole computation will proceed without any mesh reconstruction.

Magnetic flux compression generators (MFCGs) power supplies provide compact inexpensive generators of high current (megamperes), high energy (megajoules), and short current risetime (microseconds). Applications of these generators in plasma, solid-state, particle, and optical physics research have been reviewed by Knoepfel [1] and Fowler [2].

Helically-wound generators, described by Shearer, et al. [3] and Crawford and Damerow [4], have the advantage of a high initial inductance, permitting a potentially high amplification factor. Figure 1 shows a sketch of a typical generator. An initial pulsed current in the outer helical coil winding establishes magnetic flux between the coil and the inner metallic cylinder, referred to as the armature. The armature contains high explosive, which is detonated from one end, driving the armature towards the coil at high velocity. The inductance decrease in the helical winding is a result both of the decrease in radial separation between the armature and the coil and of the decrease in coil axial length as the armature short circuits successive turns of the coil. This decrease in inductance, combined with the coil’s tendency to conserve flux, leads to amplification of the electrical current which can be delivered to an external load. Chemical energy stored in the explosive is converted to kinetic energy of the armature. This energy in turn is transformed by the coil magnetic field into electrical energy delivered to the external load.

A two-dimensional magnetohydrodynamics (MHD) code which can be used to study at least isolated sections of the generator geometry has been constructed. The basic approach to be taken is to add a magnetic field solver, Joule heating, self-consistent magnetic forces, and an external circuit equation to the 2-D material response code.

This paper describes a cell-centered diffusion finite volume scheme. A two-dimensional finite difference magnetic field solver has been developed using the scheme to treat the flux compression and the nonlinear temperature-dependent resistive diffusion into both conductors.

II. CELL-CENTERED DIFFUSION FINITE VOLUME SCHEME

A. Problem and notations

Let \( \Omega \) be an open bounded polygonal subset of \( R^2 \) with boundary \( \partial \Omega \) and \( u = u(x) \) the solution of the following steady diffusion problem:

\[
- \nabla \cdot (\lambda \nabla u) = f, \quad \text{in} \quad \Omega \tag{1}
\]

where \( u \) is homogeneous to a temperature, \( \lambda \) is a scalar diffusion coefficient, and \( f \) denotes a source term.

The domain \( \Omega \) is paved with a collection of non-overlapping convex quadrangles: \( \{K, L, \ldots\} \) and with each cell \( K \) we associate one point (the so-called collocation point or cell-centered) denoted also by \( K \); the barycenters is a qualified candidate but other points can be chosen. Denote the cell vertex by \( A, B, \ldots, \) and the cell side by \( \sigma \). If the cell side \( \sigma \) is a common edge of cells \( K \) and \( L \), and its vertices are \( A \) and \( B \), then we denote \( \sigma = K \mid L = BA \) and its length is denoted by \( |A - B| \).

Let \( J \) be the set of all cells, \( E \) be the set of all cell side, and \( E_K \) be the set of all cell side of cell \( K \). Denote \( E_{int} = E \cap \Omega \), \( E_{ext} = \{ \sigma \in E_{int} : \sigma \cap \partial \Omega = \emptyset \} \), \( E_{ext} = E \cap \partial \Omega \). Denote \( h = \left( \sup_{K \in J \cap m(K)} \right)^{1/2} \), where \( m(K) \) is the area of cell \( K \). The distance from the cell-center \( K \) (resp. \( L \)) to the edge \( \sigma \) is...
denoted by \( d_{K,\sigma} \) (resp. \( d_{L,\sigma} \)), see Figure 1. \( I \) denotes a point on the line \( AB \) which is usually the intersection point of line segment \( KL \) with \( \sigma \) or the midpoint of the edge \( \sigma \).

We adopt the following notations (see Figure 1). \( \vec{n}_{K,\sigma} \) (resp. \( \vec{n}_{L,\sigma} \)) is the outward unit normal to the edge \( \sigma \) of cell \( K \) (resp. \( L \)). There holds \( \vec{n}_{K,\sigma} = -\vec{n}_{L,\sigma} \) for \( \sigma = K \mid L \). \( \vec{t}_{K,\sigma} \) denotes the unit vector tangential to \( \sigma \), see Figure 2.

**Fig. 1.** Stencil and notation.

\[ \text{Fig. 2.} \text{ } \text{Outward unit vector normal to} \sigma \text{ and unit vector tangent to} \sigma. \]

**B. Construction of the scheme**

By integrating (1) over the cell \( K \), and using Green formula, one obtains

\[
\sum_{\sigma \in E_K} F_{K,\sigma} = \int_K f(x) dx, \tag{2}
\]

where the continuous diffusion flux on edge \( \sigma \) is

\[
F_{K,\sigma} = -\int_{\sigma} \lambda(x) \nabla u(x) \cdot \vec{n}_{K,\sigma} dl. \tag{3}
\]

Note that

\[
\begin{align*}
L - I &= \vec{I}_L = d_{L,\sigma} \vec{n}_{K,\sigma} + r_1 \vec{t}_{K,\sigma}, \\
A - B &= \vec{B}A = |A - B| \vec{t}_{K,\sigma}, \\
I - K &= \vec{K}I = d_K, \vec{n}_{K,\sigma} + r_2 \vec{t}_{K,\sigma},
\end{align*}
\]

where

\[
\begin{align*}
r_1 &= \frac{(L - I, A - B)}{|A - B|}, \quad r_2 = \frac{(I - K, A - B)}{|A - B|}. \tag{5}
\end{align*}
\]

It is straightforward to express the outward unit vector normal to \( \sigma \) using the third equation of (4) and the expression \( I - K = \vec{K}I = |I - K| \vec{t}_{K,\sigma} \), where \( \vec{t}_{K,\sigma} \) is the unit vector tangential to \( I - K \); \( \vec{n}_{K,\sigma} = \frac{\cos \theta_{K,\sigma}}{|\vec{K}I|} \vec{t}_{K,\sigma} - \tan \theta_{K,\sigma} \vec{t}_{K,\sigma} \), where \( \theta_{K,\sigma} \) denotes the angle between \( \vec{n}_{K,\sigma} \) and \( I - K \).

The continuous diffusion flux can be written as

\[
F = \tan \theta_{K,\sigma} \int_{\sigma} \lambda(x) \nabla u(x) \cdot \vec{t}_{K,\sigma} dl - \frac{1}{\cos \theta_{K,\sigma}} \int_{\sigma} \lambda(x) \nabla u(x) \cdot \vec{n}_{K,\sigma} dl. \tag{6}
\]

In order to discretize the above continuous diffusion flux, we use the Taylor expansion for the function \( u \), and then obtain

\[
\begin{align*}
u(I) - u(K) &= \nabla u(x) \cdot (I - K) + \int_0^1 (H_I - H_K) ds, \tag{7}
\end{align*}
\]

where \( H_K = (\nabla^2 u(sz + (1 - s)K)(K - x), K - x) \), \( \nabla^2 u(sz + (1 - s)K) \) denotes the Hessian matrix of \( u \) at the point \( sz + (1 - s)K \). The notations \( H_I, H_A, H_B \) have the similar meaning.

Note that \( I - K = \vec{K}I = |I - K| \vec{t}_{K,\sigma} \) and \( A - B = \vec{B}A = |A - B| \vec{t}_{K,\sigma} \). By multiplying the above two equations by \( \lambda(x) \) and integrating on the edge \( \sigma \), one obtains

\[
\begin{align*}
\int_{\sigma} \lambda(x) (u(I) - u(K)) dl &= \int_{\sigma} \lambda(x) \nabla u(x) \cdot \vec{t}_{K,\sigma} dl + \int_{\sigma} \left( \lambda(x) \int_0^1 (H_I - H_K) ds \right) dl, \tag{8}
\end{align*}
\]

So we have

\[
\begin{align*}
\int_{\sigma} \lambda(x) \nabla u(x) \cdot \vec{t}_{K,\sigma} dl &= \frac{|A - B|}{|I - K|} \lambda_{K,\sigma} (u(I) - u(K)) \\
&- \frac{1}{|I - K|} \int_{\sigma} \left( \lambda(x) \int_0^1 (H_I - H_K) ds \right) dl, \tag{9}
\end{align*}
\]

and similarly

\[
\begin{align*}
\int_{\sigma} \lambda(x) \nabla u(x) \cdot \vec{t}_{K,\sigma} dl &= \lambda_{K,\sigma} (u(A) - u(B)) \\
&- \frac{1}{|A - B|} \int_{\sigma} \left( \lambda(x) \int_0^1 (H_A - H_B) ds \right) dl, \tag{10}
\end{align*}
\]

where \( \lambda_{K,\sigma} = \frac{1}{|A - B|} \int_{\sigma} \lambda(x) dl \) is the average value of \( \lambda(x) \) along \( \sigma \) in the cell \( K \). The second terms of right hand side in the above two equations are \( O(h^2) \). Substitute (9) and (10)
with (6) to obtain
\[ F_{K,\sigma} = \tan \theta_{K,\sigma} \left[ \lambda_{K,\sigma}(u(A) - u(B)) \right] - \frac{1}{|A - B|} \int_{\sigma} \left[ \lambda(x) \int_{0}^{1} (H_{A} - H_{B}) \, ds \right] \, dl \]
\[ - \frac{1}{\cos \theta_{K,\sigma}} \left[ \frac{|A - B|}{|I - K|} \lambda_{K,\sigma}(u(I) - u(K)) \right] - \frac{1}{|I - K|} \int_{\sigma} \left[ \lambda(x) \int_{0}^{1} (H_{I} - H_{K}) \, ds \right] \, dl \]
\[ = -\tau_{K,\sigma} [u(I) - u(K) - D_{K,\sigma}(u(A) - u(B))] + R_{K,\sigma}, \tag{11} \]
where
\[ \tau_{K,\sigma} = \frac{|A - B| \lambda_{K,\sigma}}{|I - K| \cos \theta_{K,\sigma}} = \frac{|A - B| \lambda_{K,\sigma}}{d_{K,\sigma}}, \]
\[ D_{K,\sigma} = \frac{\sin \theta_{K,\sigma}|I - K|}{|A - B|} = \frac{r_{2}}{r_{1}}, \]
\[ R_{K,\sigma} = \int_{\sigma} \frac{\lambda(x)}{d_{L,\sigma}} [H_{I} - H_{K} - D_{K,\sigma}(H_{A} - H_{B})] \, ds \, dl = O(h^{2}). \tag{12} \]

Similarly, we have
\[ F = -\tau_{L,\sigma} [u(I) - u(L) - D_{L,\sigma}(u(B) - u(A))] + R_{L,\sigma}, \tag{13} \]
where
\[ \tau_{L,\sigma} = \frac{|A - B| \lambda_{L,\sigma}}{d_{L,\sigma}}, \]
\[ D_{L,\sigma} = \frac{r_{1}}{|A - B|}, \]
\[ R_{L,\sigma} = \int_{\sigma} \frac{\lambda(x)}{d_{L,\sigma}} [H_{I} - H_{L} - D_{L,\sigma}(H_{B} - H_{A})] \, ds \, dl = O(h^{2}). \tag{14} \]

From the continuity of the normal flux
\[ F_{K,\sigma} = -F_{L,\sigma}, \tag{15} \]
we get
\[ u(I) = \frac{1}{\tau_{K,\sigma} + \tau_{L,\sigma}} [\tau_{K,\sigma} u(K) + \tau_{L,\sigma} u(L) + (\tau_{K,\sigma} D_{K,\sigma} - \tau_{L,\sigma} D_{L,\sigma})(u(A) - u(A))] + \frac{1}{\tau_{K,\sigma} + \tau_{L,\sigma}} (R_{K,\sigma} + R_{L,\sigma}). \tag{16} \]

Substitute (16) into (11) to obtain
\[ F_{K,\sigma} = -\tau_{\sigma} [u(L) - u(K) - D_{\sigma}(u(A) - u(B))] + \tilde{R}_{K,\sigma}, \tag{17} \]
where
\[ \tau_{\sigma} = \frac{\tau_{K,\sigma} D_{K,\sigma} + D_{L,\sigma}}{\tau_{K,\sigma} + \tau_{L,\sigma}} \left( \frac{L - K, A - B}{|A - B|^{2}} \right), \]
\[ \tilde{R}_{K,\sigma} = \frac{\tau_{L,\sigma} R_{K,\sigma} - \tau_{\kappa,\sigma} R_{L,\sigma}}{\tau_{K,\sigma} + \tau_{L,\sigma}}. \tag{18} \]
Let $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \Gamma$, where $\Gamma$ is the and discontinuous line.

Now we distinguish two cases.

**Case 1.** $K_A$, $A$ and $L_A$ are collinear or $K$, $A$ and $L$ are collinear.

Firstly, assume $K$, $A$ and $L$ are collinear. There is

$$
\frac{\partial u}{\partial AL}|_{K(A)} = \frac{u(A) - u(L)}{|L - A|} + O(h),
$$

where $K'A$ denotes the unit vector tangent to $KA$ and $AL$ denotes the unit vector tangent to $AL$. Note that

$$
\lambda|_{K(A)} = \lambda|_{L(A)}, \lambda(K) - \lambda(L) + O(h),
$$

Similarly, when $K_A$, $A$ and $L_A$ are collinear, we set

$$
\frac{\partial u}{\partial AL}|_{K(A)} = \frac{u(L) - u(A)}{|L - A|} + O(h),
$$

Suppose

$$
\lambda(K)^{u_A - u_K} = \lambda(L)^{u_L - u_A}_{|L - A|}.
$$

A simple calculation shows

$$
u_A = \frac{1}{\lambda(K)} + \frac{\lambda(L)}{|A - K|} u_K + \frac{\lambda(L)}{|L - A|} u_L.
$$

Similarly, when $K_A$, $A$ and $L_A$ are collinear, we set

$$
u_A = \frac{1}{\lambda(K)} + \frac{\lambda(L)}{|A - K|} u_K + \frac{\lambda(L)}{|L - A|} u_L.
$$

**Case 2.** $K_A$, $A$ and $L_A$ are not collinear, and $K$, $A$ and $L$ are not collinear.

Let $\vec{n} = \vec{n} + \vec{n}'$. Then there exists $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$, such that

$$
\vec{n} = \alpha_1 A\vec{K}_A + \alpha_2 A\vec{L}_A = \beta_1 K\vec{A} + \beta_2 L\vec{A}.
$$

Let $\vec{t}$ denotes the unit vector normal to $\vec{n}$. The dot products of (30) with $\vec{n}$ and $\vec{t}$, respectively, give

$$
|\vec{n}|^2 = \alpha_1 A\vec{K}_A \cdot \vec{n} + \alpha_2 A\vec{L}_A \cdot \vec{n},
\text{and}
|\vec{n}'|^2 = \beta_1 K\vec{A} \cdot \vec{n} + \beta_2 L\vec{A} \cdot \vec{n},
$$

where

$$
\alpha_1 = \frac{|\vec{n}'|^2 A\vec{L}_A \cdot \vec{t}}{A\vec{K}_A \cdot \vec{n} \cdot A\vec{L}_A \cdot \vec{n}}, \quad \alpha_2 = -\frac{|\vec{n}'|^2 A\vec{K}_A \cdot \vec{t}}{A\vec{K}_A \cdot \vec{n} \cdot A\vec{L}_A \cdot \vec{n}},
\beta_1 = \frac{|\vec{n}'|^2 K\vec{A} \cdot \vec{t}}{K\vec{A} \cdot \vec{n} \cdot K\vec{A} \cdot \vec{t}}, \quad \beta_2 = -\frac{|\vec{n}'|^2 K\vec{A} \cdot \vec{t}}{K\vec{A} \cdot \vec{n} \cdot K\vec{A} \cdot \vec{t}}.
$$

By using expression (30), we have

$$
\frac{\partial u}{\partial K} |_{K(A)} + \frac{\partial u}{\partial L} |_{L(A)} = \alpha_1 \frac{\partial u}{\partial K} |_{K - A} + \alpha_2 \frac{\partial u}{\partial L} |_{L - A} + O(h),
$$

Similarly, $\omega_{B_1}(i = 1, 2, 3, 4)$ can be defined.
It follows that
\[ F_{K,\sigma} = \begin{cases} 
  -\tau_\sigma [u_L - u_K] \\
  -D_\sigma (\omega_{A_1} u_{K_A} + \omega_{A_2} u_K + \omega_{A_3} u_L + \omega_{A_4} u_{L_A}) \\
  - (\omega_{B_1} u_{K_B} + \omega_{B_2} u_{K_B} + \omega_{B_3} u_{L_B} + \omega_{B_4} u_L) \bigg|_{\sigma \in \mathcal{E}_0^{\text{int}},} \\
  -\tau_\sigma [u_L - u_K] \\
  -D_\sigma (\omega_{B_1} u_{K_B} + \omega_{B_2} u_{K_B} + \omega_{B_3} u_{L_B} + \omega_{B_4} u_L) \bigg|_{\sigma \in \mathcal{E}_0^{\text{int}},} \\
  -\tau_\sigma u_K \bigg|_{\sigma \in \mathcal{E}_{\text{ext}}} \\
\end{cases} \]

The stability and convergence of the resulting scheme are proved in [6] as well as the numerical results obtained on different meshes for several test problems to show the accuracy and the efficient of the method.

III. THE MAGNETIC FIELD EQUATIONS AND 2-D SOLVER

Figure 4 shows a cross section of the computational region defined for the model. All three-dimensional effects are omitted, and a cylindrically symmetric region is considered. A further approximation made is to neglect coil motion in the field solver. No field calculations take place in the conductors in axial regions below the point of contact, primarily because these regions are decoupled by the turns shorting and do not affect the generator performance.

\[ \nabla \times \vec{B} = -\frac{\partial \vec{E}}{\partial t} \]

(40)

\[ \nabla \times \vec{B} = \mu_0 \vec{J} \]

(41)

where the displacement current has been dropped in Eq. (41), with Ohm’s law

\[ \vec{J} = \sigma \left[ \vec{E} + \vec{v} \times \vec{B} \right] \]

(42)

where \( \sigma \) is the electrical conductivity. Using the definition \( \nabla \times \vec{A} = \vec{B} \) and \( \vec{E} = -\frac{\partial \vec{A}}{\partial t} \), one obtains the vector potential equation

\[ \frac{\partial \vec{A}}{\partial t} = -\frac{1}{\mu_0 \sigma} \nabla \times (\nabla \times \vec{A}) + \vec{v} \times \nabla \times \vec{A} \]

(43)

If we write this in cylindrical coordinates in terms of a single component \( \Psi = rA_\theta \), the final result is

\[ \frac{\partial \Psi}{\partial t} = \frac{1}{\mu_0 \sigma} \left( \frac{\partial^2 \Psi}{\partial r^2} + \frac{\partial^2 \Psi}{\partial z^2} \right) - \left( \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{1}{r \sigma} \frac{\partial \Psi}{\partial z} \right) \]

(44)

In the vacuum region, the current density is zero, so that one has

\[ \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{1}{r \sigma} \frac{\partial \Psi}{\partial z} = 0 \]

(45)

The basic sequence of operations in the coupled hydro/magnetic field calculation begins with initialization of all hydro and field quantities. The hydro variables are advanced one time step and are used to define the new conductor configurations and velocities. These new velocities are used with Eq. (44) to update the stream function in the conductor regions to the new level. After the conductor regions solution is updated, the updated vacuum solution is obtained from Eq. (45). This completes the time step and the process repeats, again beginning with the hydro variables. The fully self-consistent version of the code will compute magnetic forces and Joule heating contributions at the end of the field solver calculation to be used in the next advancement of the hydro variables.
FIG. 5. Schemas of simple winded HMFCG.

TABLE I
PARAMETERS OF TTU-1 (SERIES TTU1)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter of coil core</td>
<td>2mm</td>
</tr>
<tr>
<td>Outer radius of coil</td>
<td>33mm</td>
</tr>
<tr>
<td>Inner radius of coil</td>
<td>31mm</td>
</tr>
<tr>
<td>Conductance of Cu</td>
<td>$5.8 \times 10^7$ (1/Ω·m)</td>
</tr>
<tr>
<td>Pitch of coil</td>
<td>3mm</td>
</tr>
<tr>
<td>Width of armature</td>
<td>3mm</td>
</tr>
<tr>
<td>Outer radius of Al-armature</td>
<td>19.05mm</td>
</tr>
<tr>
<td>Inner radius of Al-armature</td>
<td>16.05mm</td>
</tr>
<tr>
<td>Conductance of Al</td>
<td>$3.6 \times 10^7$ (1/Ω·m)</td>
</tr>
<tr>
<td>Turns</td>
<td>32</td>
</tr>
</tbody>
</table>

IV. NUMERICAL CALCULATION

In this section the results of APMFCG calculation are given. The basic parameters of MFCF are selected from a simply winded HMFCG in Texas Tech University [5] mainly.

Figures (6) compare the computed and experimental currents with two different seed current. It shows an acceptable agreement.

Figure (7) show the circuit characteristic of the device.

V. CONCLUSION

We have described the construction of a finite volume scheme for diffusion equations and a 2-D magnetic field solver designed specifically for the MFCG application.

There are many possible reason for the disagreement including turn-to-turn electrical breakdown in the coil, delayed electrical shorting of the armature and coil caused by the insulating material, instabilities or three-dimensional behavior in armature expansion, venting of explosive gases into the region between the armature and coil, residual inductances of the generator output and return connections, and three-dimensional effects at the end of the generator where the helical coil couples to the output connectors. These and other effects are under study.

REFERENCES