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A discretizing method for reliability computation in complex stress-strength models

Alessandro Barbiero

Abstract—This paper proposes, implements and evaluates an original discretization method for continuous random variables, in order to estimate the reliability of systems for which stress and strength are defined as complex functions, and whose reliability is not derivable through analytic techniques. This method is compared to other two discretizing approaches appeared in literature, also through a comparative study involving four engineering applications. The results show that the proposal is very efficient in terms of closeness of the estimates to the true (simulated) reliability. In the study we analyzed both a normal and a non-normal distribution for the random variables: this method is theoretically suitable for each parametric family.

Keywords—approximation, asymmetry, experimental design, interference theory, Monte Carlo simulations.

I. Introduction

WHEN dealing with stress-strength models, we usually refer to a component (or system) that encounters a random stress during its functioning, and has got an intrinsic random strength which makes it work only when the strength is greater than the stress. The probability that it happens is denoted as reliability of the component.

A lot of literature has been written on reliability, its computation and its estimation under many statistical parametric assumptions on stress and strength. If the distributions of both strength and stress are known, the distribution of their difference can be determined using ordinary transformation techniques, like interference theory (see [3], [6]). Yet, these analytic approaches are cumbersome, if not impossible to apply, even for apparently simple problems. Moreover, the distributions of stress and strength are often unknown, or such are their parameters. A particular case occurs when stress and strength depend upon several stochastic factors through a known functional relationship: unless this relationship and the distribution of the factors are elementary (e.g. the ratio of two independent normal r.v., see [7]), almost never the exact continuous distributions of stress and strength can be analytically derived. The availability of samples by which making inference depends on the existence of a testing system, which is not likely for design problems. Therefore, reliability can be estimated merely by Monte Carlo simulation, which unluckily can require a lot of computation time in the model development phase. A continuous approximation exploiting Taylor expansion has been proposed for the normal case [9]. An approximating/discretizing approach may represent a pragmatic solution to these issues.

A. Barbiero is with the Department of Business, Economics and Statistics, Universit degli Studi di Milano, Milan, 20122, Italy, e-mail: alessandro.barbiero@unimi.it.

In this paper a straightforward discretization procedure is proposed for the evaluation of reliability in stress-strength models. The procedure can be applied to any continuous distribution, which is approximated through a number (chosen by the user) of points, whose calculation is strictly related to the typical standard normal intervals. Their probability are easily calculated using the original probability density function. The application of this novel procedure involves four engineering problems in the structural and electrotechnical fields. The proposed procedure is implemented and compared to other methods available in literature, and reveals very efficient. Some advice is given about the number of points of the discretized r.v.

The paper is structured as follows: in Section II some of the available discretization methods are described, from the stress-strength model perspective; Section III introduces and illustrates the proposal and its properties, exploring its wide applicability; Section IV describes the simulation study performed on four practical engineering cases, focusing on the performance of the new and existing approaches; Section V gives some final remarks and comments.

II. The methods

As said in the previous section, approximation through a discrete variable represents an alternative solution to numerical integration methods and Monte Carlo simulation for the evaluation of reliability in stress-strength models, where stress and strength are two functions of several stress and strength r.v. components. Giving the details, let the system strength be modeled by a r.v. X, that is a function of n sub-factors of strength: $X = f_1(X_1, X_2, \ldots, X_n)$. Let the stress be modeled by a r.v. Y, that is a function of m sub-factors of stress: $Y = f_2(Y_1, Y_2, \ldots, Y_m)$. The reliability is defined as $R = P(X > Y)$.

The main contribute to the solution of this problem through a discretization approach has been given by Taguchi [14], who suggested a factorial experiment approach, based on a discretization of the continuous r.v. modeling stress and strength. The discretization consists in synthesizing the continuous r.v. modeling stress, function of m stochastic factors, into 3-point discrete distributions, with equal probabilities, and in constructing a table of 3^m combinations on which an approximate value of reliability can be easily computed.

D'Errico and Zaino [5] provided a modification to Taguchi's approach, by suggesting a 3-point discrete distribution which retains the first five raw moments of a normal r.v., by using unequal probabilities; this modification is shown to be uniformly more efficient for any function of normal stochastic

factors. D'Errico and Zaino present a discretization approach according to which each point (or node) is placed α_i standard deviations from the mean, whereas the probability that the discrete r.v. assumes a given value $\mu+\alpha_i\sigma$ is some weigth w_i. They present an improved methodology based upon a Gaussian quadrature rule to determine these values, which depend upon the number of points. The method can be applied to any r.v. for which the first two moments are finite, even if it is conceived for the normal case.

English et al. [4] implemented this method to solve four engineering problems, performing a simulation study which shew that departures from the normal hypothesis slightly influence the results, at least when dealing with high values of reliability; yet the errors introduced with the approximation are reasonably small.

Roy and Dasgupta [10] further adjusted the previous proposals trying to better reproduce the original continuous r.v. through the discrete distribution, by considering not only the moments as a synthesis of a distribution, but also other aspects, specifically the form of the density function or the survival function. With this in mind, they proposed a discretization procedure particularly suited for symmetrical unimodal r.v., characterized by a flexible choice of the number of points.

Given a continuous r.v. Z with c.d.f. $F_z(z)$, the method discretizes it by the r.v. Z_d; Z_d takes integer values such that for each integer i

$$P(Z_d = i) = F_z(i + \delta) - F_z(i - [1 - \delta]) \tag{1}$$

where $0 < \delta < 1$ is a parameter we will discuss later. To reduce the range of Z_d, a standard form of Equation 1 can be first considered. For example, if $Z^* \sim N(\mu, \sigma^2)$, the discretized form of Z^* is

$$Z_d^* = \mu + \sigma \cdot Z_d \tag{2}$$

where Z_d is the discretized version of $Z \sim N(0, 1)$. For the k-point discretization, with k odd, i takes the integer values from $-(k-1)/2$ to $(k-1)/2$. The optimal value of δ in Equation 1 can be derived imposing that the discrete r.v. Z_d yields the first two raw moments of the original continuous variable. For the normal case, and also for some other symmetrical unimodal distributions, the optimal value of δ is 0.5, so that

$$P(Z_d = i) = F_z(i + 0.5) - F_z(i - 0.5) \tag{3}$$

The choice of the number of points by which discretize Z should derive from a compromise between the accuracy of the results, which would suggest a high value, and the computational load, which would instead suggest a low value. The values 5 and 7 are suggested. In the following Table, the probability mass function of the discrete variable Z_d is reported for $k = 3, \ldots, 10$.

Whatever discretizing approach is applied, the reliability of the system can be approximately computed as

$$R \approx \sum \cdots \sum \left[\prod_{l=1}^{n} P(X_{d,l} = x_{d,l})\right] \cdot \left[\prod_{j=1}^{m} P(Y_{d,j} = y_{d,j})\right] \cdot I\left[f_1(x_{d,1}, \ldots, x_{d,n}) > f_2(y_{d,1}, \ldots, y_{d,m})\right] \tag{4}$$

TABLE I
PMF OF THE DISCRETE R.V. Z_d (ROY-DASGUPTA ALGORITHM)

k	z_i	p_i	k	z_i	p_i
3	∓ 1	0.308538	4	∓ 1.5	0.158655
	0	0.382925		∓ 0.5	0.341345
5	∓ 2	0.308538	6	∓ 2.5	0.022750
	∓ 1	0.241730		∓ 1.5	0.135905
	0	0.382925		∓ 0.5	0.341345
7	∓ 3	0.006210	8	∓ 3.5	0.001350
	∓ 2	0.060598		∓ 2.5	0.021400
	∓ 1	0.241730		∓ 1.5	0.135905
	0	0.382925		∓ 0.5	0.341345
9	∓ 4	0.000233	10	∓ 4.5	0.000032
	∓ 3	0.005977		∓ 3.5	0.001318
	∓ 2	0.060598		∓ 2.5	0.021400
	∓ 1	0.241730		∓ 1.5	0.135905
	0	0.382925		∓ 0.5	0.341345

where $I[E]$ is the indicator function: 1 if E is true, $=$ if E is false, and the sums extend over all possible $x_{d,l}$ and $y_{d,j}$ which are the values that the discretized versions of X_l ($X_{d,l}$) and Y_j ($Y_{d,j}$), $l = 1, \ldots, n$, $j = 1, \ldots, m$, can assume.

The methods here described suffer from some drawbacks. English and Sargent approach, that considers points symmetrically disposed around the mean, looks suitable for symmetric r.v. only. Roy and Dasgupta's discretization is suggested for the normal case, since it is based on the concept of standardized variable, which is strictly connected to the normal case. In fact, it is well known that a normal variable is still normal after standardization, but this is no longer true for other r.v. If extended to other families, like the Weibull [11], this concept has to be properly modified and adapted; a set of parameters which makes the Weibull almost symmetrical has to be first searched, a discretization (through the first 9 integers) is proposed for this standard case; for all the other sets of parameters, the discretization changes only the support, using a transformation connected with the cumulative density function, and retains the probability mass function for the standard case. Yet, there is no advice for other parametric families. Roy [12] and Roy and Ghosh [13] consider the discretization of a Rayleigh r.v., based on failure rate considerations; the same do Krishna and Pundir [8], deriving a discrete Maxwell distribution: their proposal is based on assigning the probability $S(x) - S(x + 1)$ to the integer values $x = 0, 1, \ldots$ of the discrete r.v., where $S(x) = 1 - F(x)$ is the survival function of the continuous r.v. Yet, here the authors are more interested on assuring to the discrete r.v. so obtained some features related to the failure rate, e.g. IFR or IFRA properties (see for example [1], chap. 2).

The next Section is devoted to the introduction of a new proposal for discretization.

III. THE PROPOSAL

Here a procedure for discretizing continuous r.v. from the perspective of assessing reliability in complex stress-strength models will be presented and discussed. The new discretization approach is as straightforward as the just examined procedures. Yet, its advantage stands in the applicability to other distribution than the normal, without any modification, thus

overcoming an issue that the other techniques seem not able to deal with.

Let us consider the problem of synthesizing a continuous r.v. X with pdf f_x through a discrete random r.v. X_d whose support consists of k points. Let first consider for simplicity the particular case of a standard normal Z. Define $g = (k-1)/2$, and the points z_{di} as follows: $z_{di} = g - 1 + i$, $i = 1, 2, \ldots, k$. Let be $d'_i = \phi(z_{di})$ where $\phi(\cdot)$ is the probability density function of the standard normal r.v.; denote with $d'_i = d_i / \sum d_i$ the normalized version of the d_i's summing up to one. The variable Z_d, defined by the couple (z_{di}, d'_i), $i = 1, \ldots, k$, can be an intuitive discretization of the standard normal: it can be easily shown that $E(Z_d) = 0$ whereas $Var(Z_d) \approx 1$, i.e. the discretization retains the first moment and approximately the second one. In Table II, the results of its implementation for some values of k are reported:

TABLE II
PROBABILITY MASS FUNCTION OF Z_d FOR SOME VALUES OF k (STANDARD NORMAL CASE).

k	z_{di}	p_i	k	z_{di}	p_i
3	∓ 1	0.274069	4	∓ 1.5	0.134471
	0	0.451863		∓ 0.5	0.365529
5	∓ 2	0.054489	6	∓ 2.5	0.01756
	∓ 1	0.244201		∓ 1.5	0.129748
	0	0.40262		∓ 0.5	0.352692
7	∓ 3	0.004433	8	∓ 3.5	0.000873
	∓ 2	0.054006		∓ 2.5	0.017529
	∓ 1	0.242036		∓ 1.5	0.129522
	0	0.39905		∓ 0.5	0.352077
9	∓ 4	0.000134	10	∓ 4.5	0.000016
	∓ 3	0.004432		∓ 3.5	0.000873
	∓ 2	0.053991		∓ 2.5	0.017528
	∓ 1	0.241971		∓ 1.5	0.129518
	0	0.398943		∓ 0.5	0.352065

How can this discretization approach be extended to a generic (non-normal, specifically asymmetrical) r.v. variable X? Let us consider the points $x_{di} = z_{di}$, and calculate the corresponding cumulate probabilities $c_i = \Phi(z_{di})$, where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal. Compute the corresponding quantiles for the study r.v.: $q_i = F_x^{-1}(c_i)$; then calculate the density function on the q_i's: $f_i = f_x(q_i)$ and normalize them: $f'_i = f_i / \sum f_i$. The r.v. X_d defined by the couple (x_{di}, f'_i) represent a discretization of the generic r.v. X.

A. Example: Gamma variable

Let us consider the following example: we need to discretize a Gamma r.v. with parameters $\alpha = 8$ and $\beta = 2$. In Table III, the details of the discretization with 5 points. The expected value of X_d is 3.68 (against 4 for the original continuous variable X) and the variance 1.70 (against 2 for X).

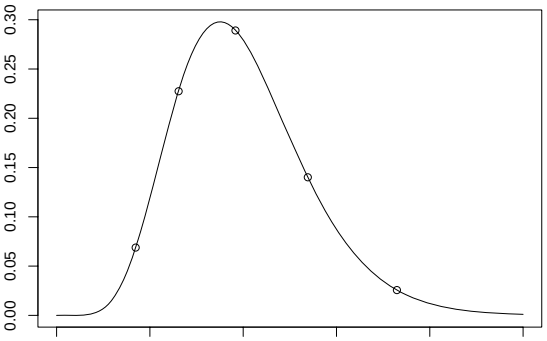
Figure 1 graphically displays the discretization procedure.

B. Example: Weibull variable

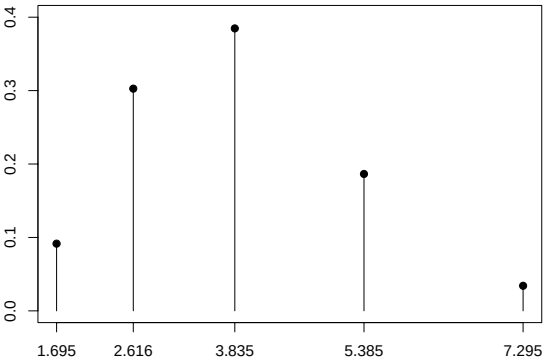
This example displays the discretization of a Weibull r.v., whose density function is given by $f_x(x; \alpha, \beta) = (\alpha/\beta)(x/\beta)^{(\alpha-1)} \exp(-(x/\beta)^\alpha)$, with shape parameter $\alpha = 3.5$ and scale parameter $\beta = 5$. A synthesis of the discretization is given in Table IV.

TABLE III
EXAMPLE OF DISCRETIZATION (GAMMA VARIABLE)

z_{di}	c_i	q_i	d_i	d'_i
-2	0.02275	1.69532	0.06887	0.09165
-1	0.15866	2.61581	0.22750	0.30274
0	0.5	3.83462	0.28917	0.38480
$+1$	0.84134	5.38514	0.14019	0.18655
$+2$	0.97725	7.29500	0.02574	0.03425



(a) Probability density function of the original r.v.



(b) Probability mass function of the discretized r.v.

Fig. 1. Discretization of a Gamma variable

TABLE IV
EXAMPLE OF DISCRETIZATION (WEIBULL VARIABLE)

z_{di}	c_i	q_i	d_i	d'_i
-4	0.00003	0.25909	0.00043	0.00061
-3	0.00135	0.75708	0.00624	0.00882
-2	0.02275	1.70201	0.04625	0.06541
-1	0.15866	3.02756	0.16803	0.23766
0	0.50000	4.50289	0.26938	0.38101
$+1$	0.84134	5.95252	0.17174	0.24291
$+2$	0.97725	7.31261	0.04119	0.05826
$+3$	0.99865	8.57572	0.00364	0.00515
$+4$	0.99997	9.75156	0.00012	0.00017

One can compare the support and probability mass function for the discrete r.v. identified by the proposed procedure to the one provided in [11]: they differ in a significative way.

Generally, the proposed approach does not retain the expected value of the original continuous variable (unless it is symmetrical), yet it tries to retain its characteristic in terms of density function, which is preferred to the cumulative density function as it does not require further adjustments than the normalization, whereas, as we noted, a proper shift is needed for the method by Roy and Dasgupta in Equation 1 to get the probability mass function of the discrete r.v. The approach can be easily applied to every kind of continuous r.v, also to r.v. defined on a finite interval (like the Beta family); thus overcoming typical problems connected with previous approaches. When applied to the most common normal case, the difference with other specific methods is not so significant, at least when the number of points is high: yet, the proposed method tends to privilege in terms of probability mass function the central points rather then the extremes, thus leading to an underrated variance, whereas the expected value is unchanged.

The simplicity of the procedure makes it suitable for software automation and implementation. The next section is devoted to the illustration of some practical engineering problems, where the discretization methods presented in Sections II and III can be usefully applied.

IV. Examples of application

In this section we will consider several engineering stress-strength problems, whose reliability can be computed either recalling Monte Carlo simulations or the discretizing techniques described in the previous sections.

A. Hollow cylinder

The maximum shear stress that a shaft with a hollow cylindric section can transmit is given by:

$$Y = \frac{16M \cdot a}{\pi(a^4 - b^4)}, \tag{5}$$

where a and b are the outer and inner diameters, and M the applied torque. The stress that the shaft has to withstand is denoted as X.

1) Normal case: Let suppose a, b, M and X are r.v. following a normal distribution with the parameters reported in Table V (the mean of X is let vary from 600 to 1100).

TABLE V
Strength parameters for application A

component	μ	σ
a	2.4	0.02
b	2	0.02
M	1200	60
X	$600 \div 1100$	55

Following the approaches described in the previous sections, we are going to compute the approximate value of $P(Y > X)$. Specifically, we will consider i) English and Sargent method (abbreviated as ES), with 6 points (the value suggested by the authors) ii) Roy and Dasgupta method (RD), with 5, 7 and 9 points iii) the new proposal (AB), with the same choice of points as ii). As a true (estimated) value of R, we will consider the value given by a Monte Carlo simulation: we will simulate a huge number of independent samples from the distributions of a, b, M and X; we will compute on each sample the corresponding value of Y. The overall rate of samples for which results $Y > X$ will be considered as the Monte Carlo estimate of R. An index for evaluating the performance of the method is the absolute error, that is calculated under each scenario (i.e. for each value of μ_x) and is an easy measure of closeness. As an overall performance index, following Roy-Dasgupta, we will take into account the Mean Absolute Error (MAE), that is the average absolute error over all the simulated scenarios.

A particular attention should be devoted when simulating the value of reliability through the usual Monte Carlo approach. In fact, even a number of runs like $10,000$, which has been employed in [10] and which is commonly considered quite huge, can instead provide unstable results. We suggest, then, and we employ a far greater number of Monte Carlo simulations (e.g. 1,000,000), that we expect to provide more reliable estimates.

In Table VI the results are reported. They show that the proposed method with 5 points perform better than English and Sargent's approach; if we consider 7 or 9 points, the performance improves and gets better than Roy and Dasgupta's method with the corresponding number of points. The proposed method here gets better increasing the number of points, whereas Roy and Dasgupta's performs best when employing 5 points.

TABLE VI
Comparative closeness study for hollow cylinder and normal r.v.

#	μ_x	MC	ES6	RD5	RD7	RD9	AB5	AB7	AB9
1	600	0.0012	0.0014	0.0009	0.0018	0.0018	0.0006	0.0012	0.0013
2	650	0.0080	0.0082	0.0079	0.0104	0.0106	0.0056	0.0081	0.0082
3	700	0.0366	0.0426	0.0363	0.0409	0.0411	0.0297	0.0350	0.0352
4	710	0.0471	0.0445	0.0457	0.0502	0.0504	0.0378	0.0434	0.0437
5	720	0.0601	0.0626	0.0632	0.0677	0.0679	0.0541	0.0601	0.0604
6	730	0.0753	0.0754	0.0848	0.0893	0.0895	0.0749	0.0812	0.0814
7	740	0.0935	0.0909	0.0913	0.0965	0.0966	0.0803	0.0874	0.0877
8	750	0.1147	0.1345	0.1157	0.1203	0.1205	0.1038	0.1107	0.1110
9	760	0.1389	0.1370	0.1471	0.1515	0.1516	0.1354	0.1421	0.1424
10	770	0.1665	0.1712	0.1807	0.1847	0.1848	0.1698	0.1761	0.1763
11	780	0.1969	0.1850	0.1935	0.1975	0.1976	0.1811	0.1877	0.1880
12	790	0.2306	0.2351	0.2337	0.2376	0.2377	0.2227	0.2287	0.2289
13	800	0.2665	0.2785	0.2803	0.2830	0.2830	0.2706	0.2754	0.2756
14	810	0.3051	0.3088	0.3123	0.3150	0.3151	0.3046	0.3088	0.3090
15	820	0.3461	0.3606	0.3412	0.3436	0.3437	0.3346	0.3385	0.3386
16	830	0.3884	0.3712	0.3885	0.3903	0.3904	0.3836	0.3864	0.3865
17	840	0.4322	0.4244	0.4322	0.4328	0.4328	0.4280	0.4296	0.4296
18	850	0.4761	0.4673	0.4759	0.4762	0.4762	0.4751	0.4753	0.4754
19	860	0.5200	0.5340	0.5258	0.5255	0.5255	0.5292	0.5281	0.5281
20	870	0.5632	0.5741	0.5630	0.5622	0.5622	0.5676	0.5658	0.5658
21	880	0.6057	0.6047	0.6068	0.6051	0.6051	0.6126	0.6097	0.6096
22	890	0.6467	0.6372	0.6409	0.6383	0.6383	0.6479	0.6440	0.6438
23	900	0.6860	0.6522	0.6851	0.6822	0.6821	0.6947	0.6896	0.6894
24	910	0.7233	0.7148	0.7282	0.7248	0.7247	0.7412	0.7350	0.7348
25	920	0.7578	0.7627	0.7552	0.7517	0.7516	0.7681	0.7617	0.7615
26	930	0.7894	0.8054	0.7753	0.7715	0.7714	0.7869	0.7808	0.7805
27	940	0.8185	0.8214	0.8078	0.8028	0.8027	0.8187	0.8117	0.8115
28	950	0.8447	0.8357	0.8408	0.8356	0.8355	0.8535	0.8459	0.8456
29	1000	0.9364	0.9411	0.9354	0.9296	0.9294	0.9448	0.9377	0.9374
30	1100	0.9936	0.9923	0.9936	0.9915	0.9913	0.9956	0.9935	0.9934
	MAE		0.0079	0.0041	0.0063	0.0064	0.0062	0.0042	0.0041

In Figure 2, the absolute errors for ES method with 6 points, RD method with 5 points and the proposed approach with 9

points are plotted, in the range $700 - 950$ for μ_x (scenarios $\#3 \div 28$).

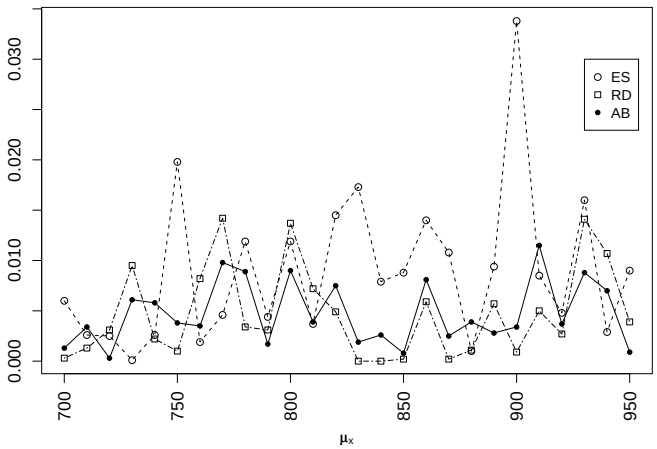


Fig. 2. Plot of absolute errors for the three methods considered in the comparative closeness study for hollow cylinder and normal r.v. (ES6, RD5, AB9)

2) Gamma case: Almost all the methods described in Section 2 for the assessment of reliability in complex stress-strength systems assume that the distribution of stress and strength factors are normal, or at least unimodal and symmetric. Yet, in practical applications the normal set-up may sometimes fail, especially because of the tails of its distribution; other parametric families can be used for modeling stress and strength components: here we will refer to the two-parameter Gamma family, which along with other families, like Weibull or Beta, is able to model a great variety of density functions by varying its parameters.

For a Gamma r.v. with parameters (α, β) the expected value is α/β, the variance is α/β^2, the skewness is given by $2/\sqrt{\alpha}$ and kurtosis excess $6/\alpha$. This means that the Gamma variable is always positively skewed, its skewness asymptotically goes to zero as α tends to $+\infty$; it has got heavier tails than the normal, and its leptokurtosis decreases as α increases.

In this study, we assume for each stress factor and for strength a two-parameter gamma distribution. We will retain the same values of mean and variance as for the normal case for all of them; the corresponding values of parameters α and β are reported in Table VII. Since the RD procedure is specifically addressed for the normal case (or, at least, for symmetrical variables) it is not suited for the gamma set-up, and then it is excluded from the comparative study; the ES approach can be applied, but it provides the same results as for the normal case, since the mean and variance of the stress and strength components are unchanged, and the procedure is based only on these values (and on the number of points). The proposal is instead expected to provide different results from the normal set-up, since it strongly relies upon the chosen parametric distribution. The results for ES and AB method (with 6 points the former and 5, 7, 9 points the latter) are displayed in Table VIII. They show the superiority of the new proposal even when using only 5 points; the performance improves passing to 7 and 9 points.

TABLE VII
STRENGTH PARAMETERS FOR APPLICATION A (GAMMA VARIABLES).

component	μ_x	σ_x
a	2.4	0.02
b	2	0.02
M	1200	60

TABLE VIII
COMPARATIVE CLOSENESS STUDY FOR HOLLOW CYLINDER AND GAMMA R.V.

#	μ_x	ES6	AB5	AB7	AB9
1	600	0.0014	0.0005	0.0013	0.0013
2	650	0.0082	0.0045	0.0069	0.0071
3	700	0.0426	0.0297	0.0347	0.0349
4	710	0.0445	0.0392	0.0447	0.0449
5	720	0.0626	0.0506	0.0566	0.0568
6	730	0.0754	0.0707	0.0768	0.0771
7	740	0.0909	0.0803	0.0870	0.0873
8	750	0.1345	0.1023	0.1089	0.1092
9	760	0.1370	0.1315	0.1378	0.1380
10	770	0.1712	0.1624	0.1685	0.1687
11	780	0.1850	0.1757	0.1823	0.1826
12	790	0.2351	0.2205	0.2263	0.2265
13	800	0.2785	0.2648	0.2695	0.2697
14	810	0.3088	0.2985	0.3027	0.3028
15	820	0.3606	0.3313	0.3351	0.3353
16	830	0.3712	0.3805	0.3832	0.3833
17	840	0.4244	0.4208	0.4224	0.4225
18	850	0.4673	0.4703	0.4705	0.4705
19	860	0.5340	0.5144	0.5136	0.5136
20	870	0.5741	0.5624	0.5606	0.5605
21	880	0.6047	0.6013	0.5987	0.5986
22	890	0.6372	0.6447	0.6407	0.6406
23	900	0.6522	0.6915	0.6863	0.6861
24	910	0.7148	0.7385	0.7322	0.7319
25	920	0.7627	0.7680	0.7613	0.7611
26	930	0.8054	0.7896	0.7830	0.7827
27	940	0.8214	0.8122	0.8055	0.8052
28	950	0.8357	0.8419	0.8343	0.8340
29	1000	0.9411	0.9441	0.9370	0.9367
30	1100	0.9923	0.9955	0.9935	0.9934
	MAE	0.0079	0.0066	0.0046	0.0045

Let us remember that a gamma variable, for $\alpha \to +\infty$, tends to approach the normal distribution; since the values of the α parameter used for the study are quite high, we would expect under each scenario a very slight difference among the simulated Monte Carlo values of reliability in the gamma and normal case. Yet, looking at the results, this difference is not negligible, and can be explained through the critical role played by the upper and lower tails of the stress/strength distributions, as well described in [2].

B. Solid shaft

The shear stress of a solid shaft can be expressed as a function of the torque M applied and the diameter d:

$$Y = \frac{16M}{\pi \cdot d^3}, \tag{6}$$

where M and d are r.v. with distributional parameters reported in Table IX and the strength X is a r.v.with standard deviation $\sigma_y = 55$ and mean varying from 450 to 550.

In Table X the results are reported. They show that the new method with 5 points performs better than the other two considered; there is a very slight difference in terms of MAE passing from 5 to 9 points for the proposal.

TABLE IX
STRENGTH PARAMETERS FOR APPLICATION B.

component	μ_x	σ_x
d	2.4	0.02
M	1200	60

TABLE X
COMPARATIVE CLOSENESS STUDY FOR SOLID SHAFT AND NORMAL R.V.

	μ_x	MC sim	ES6	RD5	AB5	AB7	AB9
1	450	0.5516	0.5296	0.5382	0.5412	0.5407	0.5406
2	455	0.5846	0.5333	0.6061	0.6134	0.6120	0.6119
3	460	0.6164	0.6165	0.6071	0.6142	0.6127	0.6126
4	465	0.6478	0.6169	0.6721	0.6816	0.6783	0.6782
5	470	0.6776	0.7003	0.6721	0.6816	0.6791	0.6790
6	475	0.7066	0.7039	0.7018	0.7117	0.7084	0.7083
7	480	0.7344	0.7041	0.7303	0.7411	0.7378	0.7377
8	485	0.7607	0.7873	0.7303	0.7411	0.7380	0.7379
9	490	0.7856	0.7884	0.7831	0.7940	0.7899	0.7897
10	492	0.7951	0.7884	0.7831	0.7940	0.7899	0.7897
11	494	0.8043	0.7885	0.7831	0.7940	0.7899	0.7898
12	496	0.8135	0.7917	0.7831	0.7940	0.7904	0.7902
13	498	0.8222	0.8600	0.8264	0.8401	0.8352	0.8351
14	500	0.8307	0.8633	0.8336	0.8461	0.8412	0.8410
15	502	0.8390	0.8638	0.8336	0.8461	0.8416	0.8415
16	504	0.8470	0.8638	0.8336	0.8461	0.8416	0.8415
17	506	0.8547	0.8638	0.8336	0.8461	0.8417	0.8415
18	508	0.8622	0.8639	0.8495	0.8619	0.8571	0.8569
19	510	0.8693	0.8639	0.8729	0.8866	0.8821	0.8819
20	512	0.8762	0.8967	0.8746	0.8878	0.8833	0.8831
21	514	0.8828	0.8999	0.8746	0.8878	0.8833	0.8831
22	516	0.8891	0.8999	0.8746	0.8878	0.8833	0.8831
23	518	0.8952	0.8999	0.8757	0.8886	0.8840	0.8838
24	520	0.9011	0.8999	0.9092	0.9219	0.9164	0.9163
25	525	0.9149	0.9007	0.9092	0.9219	0.9171	0.9169
26	530	0.9269	0.9308	0.9196	0.9307	0.9256	0.9254
27	535	0.9376	0.9308	0.9355	0.9464	0.9415	0.9413
28	540	0.9470	0.9489	0.9355	0.9464	0.9415	0.9414
29	545	0.9553	0.9525	0.9556	0.9636	0.9584	0.9582
30	550	0.9625	0.9525	0.9556	0.9636	0.9584	0.9582
	MAE		0.0152	0.0109	0.0098	0.0090	0.0090

C. Hollow rectangular tube

The functional form of shear stress of a hollow rectangular tube is:

$$Y = \frac{M}{2t \cdot (W-t) \cdot (H-t)}, \tag{7}$$

where M is the applied torque, t is the thickness, W the width and H the height of the rectangular section for the tube. We assume all this quantities are independent r.v. with parameters reported in Table XI. The strength X is a r.v. with standard deviation $\sigma_x = 60$, and its mean varies from 750 to 1000. Table XII shows the reliability values from Monte Carlo simulations and applying the discretization methods by English and Sargent, Roy and Dasgupta, and the proposal.

TABLE XI
STRENGTH PARAMETERS FOR APPLICATION C.

component	μ_x	σ_x
M	1500	150
t	0.2	0.005
W	2	0.02
H	3	0.03

They show that the new method perform better than the other two. Passing from 5 to 7 and 9 points, here there is a significant improvement in terms of closeness to the true simulated values.

TABLE XII
COMPARATIVE CLOSENESS STUDY FOR HOLLOW RECTANGULAR TUBE AND NORMAL R.V.

	μ_x	MC sim	ES6	RD 5	AB5	AB7	AB9
1	750	0.5235	0.5315	0.5237	0.5264	0.5255	0.5255
2	760	0.5642	0.5859	0.5750	0.5827	0.5802	0.5802
3	770	0.6039	0.6281	0.6109	0.6196	0.6170	0.6169
4	780	0.6427	0.6569	0.6341	0.6423	0.6397	0.6396
5	790	0.6799	0.6766	0.6689	0.6774	0.6745	0.6744
6	800	0.7151	0.6972	0.6981	0.7074	0.7041	0.7040
7	805	0.7321	0.7050	0.7248	0.7362	0.7322	0.7320
8	810	0.7490	0.7271	0.7362	0.7485	0.7442	0.7441
9	815	0.7650	0.7363	0.7582	0.7719	0.7671	0.7669
10	820	0.7804	0.7671	0.7791	0.7952	0.7895	0.7893
11	825	0.7954	0.7771	0.7878	0.8036	0.7980	0.7979
12	830	0.8095	0.8114	0.8099	0.8272	0.8211	0.8209
13	835	0.8231	0.8166	0.8135	0.8303	0.8243	0.8241
14	840	0.8359	0.8531	0.8304	0.8476	0.8414	0.8412
15	845	0.8481	0.8550	0.8432	0.8602	0.8540	0.8538
16	850	0.8598	0.8828	0.8501	0.8662	0.8603	0.8601
17	855	0.8710	0.8841	0.8607	0.8764	0.8705	0.8703
18	860	0.8814	0.8998	0.8664	0.8816	0.8758	0.8756
19	865	0.8912	0.9023	0.8802	0.8949	0.8892	0.8890
20	870	0.9004	0.9026	0.8875	0.9024	0.8965	0.8963
21	875	0.9089	0.9137	0.8975	0.9119	0.9060	0.9058
22	880	0.9168	0.9149	0.9092	0.9241	0.9179	0.9177
23	885	0.9245	0.9226	0.9149	0.9294	0.9232	0.9230
24	890	0.9315	0.9253	0.9256	0.9403	0.9340	0.9338
25	895	0.9380	0.9318	0.9296	0.9440	0.9377	0.9375
26	900	0.9439	0.9370	0.9382	0.9524	0.9461	0.9459
27	925	0.9671	0.9697	0.9620	0.9730	0.9678	0.9676
28	950	0.9820	0.9852	0.9774	0.9856	0.9816	0.9813
29	975	0.9907	0.9902	0.9888	0.9948	0.9915	0.9913
30	1000	0.9954	0.9951	0.9937	0.9974	0.9954	0.9952
	MAE		0.0111	0.0077	0.0068	0.0039	0.0039

D. Power dissipated by a circuit

Consider two resistors of resistance R_1 and R_2 connected in parallel, with a voltage V across both of them. The power dissipated from the resistors is

$$Y = V^2(1/R_1 + 1/R_2). \tag{8}$$

The resistors draw power from a power supply whose power X has a standard deviation $\sigma_y = 12$ and mean μ_y varying from 110 to 150. The distributional parameters of the system factors are reported in Table XIII.

TABLE XIII
STRENGTH PARAMETERS FOR APPLICATION D.

component	μ_x	σ_x
V	100	5
R_1	150	15
R_2	230	20

In Table XIV the results are reported. They show that the proposed method with 5 points performs better than ES with 6 points and RD with 5 points; yet, unlike the previous three applications, here increasing the number of points for the new approach leads to a worsening in terms of closeness to the simulated true values of reliability. In this case, for all the methods but ES, the fifth scenario is particularly penalizing.

TABLE XIV
COMPARATIVE CLOSENESS STUDY FOR POWER DISSIPATED IN A CIRCUIT AND NORMAL R.V.

	μ_x	MC sim	ES6	RD5	AB5	AB7	AB9
1	110	0.4996	0.4904	0.4759	0.4757	0.4754	0.4754
2	111	0.5214	0.4967	0.5068	0.5110	0.5097	0.5097
3	112	0.5434	0.5276	0.5240	0.5282	0.5266	0.5265
4	113	0.5648	0.5643	0.5319	0.5360	0.5343	0.5342
5	114	0.5862	0.5718	0.5422	0.5455	0.5441	0.5440
6	115	0.6073	0.5821	0.5954	0.6020	0.5988	0.5987
7	116	0.6282	0.5943	0.6059	0.6121	0.6087	0.6086
8	117	0.6487	0.6241	0.6152	0.6223	0.6188	0.6187
9	118	0.6683	0.6878	0.6474	0.6577	0.6531	0.6529
10	119	0.6878	0.6920	0.6664	0.6774	0.6722	0.6720
11	120	0.7065	0.6952	0.6814	0.6916	0.6865	0.6864
12	121	0.7250	0.7014	0.6904	0.7002	0.6951	0.6949
13	122	0.7426	0.7018	0.7228	0.7353	0.7292	0.7290
14	123	0.7594	0.7448	0.7457	0.7603	0.7536	0.7534
15	124	0.7757	0.7712	0.7552	0.7685	0.7618	0.7615
16	125	0.7911	0.7794	0.7590	0.7716	0.7648	0.7646
17	126	0.8059	0.7800	0.7702	0.7825	0.7761	0.7759
18	127	0.8201	0.8092	0.8121	0.8259	0.8185	0.8182
19	128	0.8337	0.8230	0.8174	0.8303	0.8228	0.8225
20	129	0.8466	0.8352	0.8198	0.8324	0.8251	0.8248
21	130	0.8585	0.8441	0.8451	0.8601	0.8521	0.8518
22	131	0.8698	0.8471	0.8557	0.8694	0.8612	0.8609
23	132	0.8805	0.8767	0.8652	0.8782	0.8703	0.8700
24	133	0.8904	0.8950	0.8691	0.8813	0.8736	0.8733
25	134	0.8997	0.9026	0.8892	0.9023	0.8943	0.8940
26	135	0.9085	0.9045	0.9010	0.9138	0.9056	0.9053
27	136	0.9168	0.9062	0.9063	0.9181	0.9103	0.9099
28	137	0.9244	0.9066	0.9075	0.9191	0.9114	0.9110
29	140	0.9378	0.9374	0.9379	0.9805	0.9401	0.9397
30	150	0.9792	0.9804	0.9775	0.9959	0.9774	0.9772
	MAE		0.0140	0.0196	0.0132	0.0154	0.0155

V. CONCLUSIONS

In this paper, we presented a novel discretization technique that can be usefully applied to complex stress-strength models for finding an approximate value for reliability. The discretization of continuous random variables is often the only accessible solution for assessing reliability when stress and strength are an explicit known function of stochastic components: classic interference theory may not be able to supply exact results and Monte Carlo approximation via simulations may require too much computational time. The proposal is a straightforward procedure based upon typical standard normal intervals and the corresponding quantiles; it can be applied to any continuous distribution. A comparative study involving other discretizing techniques, carried out on well-known engineering problems, provides encouraging results. The performance of the method in terms of closeness of its estimates to the real simulated values is overall better than other two existing procedures; its general nature further suggests its use.

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