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Application of micro-continuum approach in the estimation of snow drift density, velocity and mass transport in hilly bound cold regions

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Abstract—We estimate snow velocity and snow drift density on hilly terrain under the assumption that the drifting snow mass can be represented using a micro-continuum approach (i.e. using a non-classical mechanics approach assuming a class of fluids for which basic equations of mass, momentum and energy have been derived). In our model, the theory of coupled stress fluids proposed by Stokes [1] has been employed for the computation of flow parameters. Analyses of bulk drift velocity, drift density, drift transport and mass transport of snow particles have been carried out and computations made, considering various parametric effects. Results are compared with those of classical mechanics (logarithmic wind profile). The results indicate that particle size affects the flow characteristics significantly.

Keywords—Snow velocity, Snow drift density, Mass transport of snow particles, Snow Avalanche.

I. Introduction

DRIFTING of snow is a commonly observed phenomenon in hilly bound cold regions. Generally, drifting is observed 6-8 h after the cessation of snow fall, a sufficient time gap to allow 'pick up' of snow from the surface. The wind generally erodes snow on the windward side and deposits on the leeward side of any surface undulation. If the leeward side has already accumulated sufficient snow as a result of snow fall, this additional mass at times triggers avalanches or adds to metamorphic processes within the snow pack. Thus, drift studies have direct relevance to avalanche forecasting.

Many researchers have proposed mathematical models for the estimation of velocity and drift density using the concepts of momentum transfer and assuming a logarithmic wind velocity profile ([2], [3], [4], [5], [6], [7] and [8]). Fohn [9] proposed a model based on the concept of Bernoulli's principle and boundary layer theory and obtained a relation for drift density with height above the ground, (z).

Kobayashi [10] proposed a mathematical model based on the principle of saltation and estimated the profiles of saltating particles, total transportation of snow, Q and mass flow rate. Most studies reported earlier have been based on the concept of continuum mechanics [11]. However, it is interesting to note that the flow of snow particles has a different velocity than the air velocity. Hence, the bulk velocity cannot be the same as the velocity of the particles which are being measured.

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In order to account for this bulk velocity (which includes the effects of particles also), one needs to examine micro-continuum theories ([1], [12] and [13]). It is assumed in these micro-continuum theories, that particles can have independent rotation velocities in addition to the bulk flow of the fluid. The theories put forward by Eringan [13] and Cowin [12] can even account for particle deformation and twist properties, in addition to rotation. Additional constitutive relations have also been proposed for coupled stresses due to rotation. We include the micro-continuum theory proposed by Stokes [1] in our model development. Stokes's [1] constitutive equation does not include independent equations for the particle rotation however it does include parameters [$\bar{\alpha}$, $\bar{\eta}$] which account for the effects of size and shape of the particles.

II. Analysis

The basic constitutive equations for a couple stress fluid developed by Stokes [1] are given by:

$$\tau_{ij} = -p\ \delta_{ij} + 2\mu d_{ij} \tag{1}$$

$$\mu_{ij} = 4\eta w_{j,i} + 4\acute{\eta} w_{i,j} \tag{2}$$

Where τ_{ij} is the symmetric part of the stress tensor, T_{ij} and μ_{ij} is the deviatoric part of the couple stress, M_{ij}. Also P denotes the hydrostatic pressure and μ is coefficient of viscosity. w_i is the rate of rotation vector η and $\acute{\eta}$ are constants associated with couple stress. The equations of motion are:

$$T_{ij,i} + \rho f_i = \rho \frac{Dv_i}{Dt} \tag{3}$$

$$e_{ijk} T^A_{jk} + M_{ji,j} + \rho c_i = \rho K^2 \frac{Dv_i}{Dt} \tag{4}$$

Where f_i is the body force vector per unit mass, c_i is the body moment vector per unit mass, v_i is the velocity vector and K is the radius of gyration of a unit cuboid with its sides normal to the spatial axes. Finally T^A_{jk} is the anti symmetric part of the stress tensor. Then

$$d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \quad , \quad w_i = \frac{1}{2} e_{ijk} v_{k,j} \tag{5}$$

As a result of equations (1) and (2), the equation of motion in the presence of a body force and body couples, can be written as:

$$\mu\nabla^2 V - \nabla P - \eta\nabla^2(\nabla\times\nabla\times V) + \rho f_x = \rho\frac{DV}{Dt} \quad (6)$$

The force gravity is shown in Fig. 1.

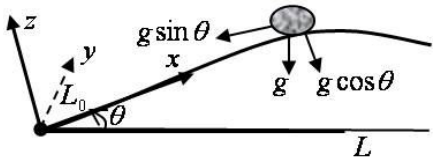


Fig. 1. Body force representation

Assuming that the flow is steady, laminar and one dimensional and $\frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial z}$ Equation (6) simplifies to

$$\frac{\partial p}{\partial x} = \mu\frac{\partial^2 u}{\partial z^2} + \rho f_x - \eta\nabla^4 u \quad (7)$$

Here u is the axial velocity, ρ is the density of snow, μ is the viscosity of snow, p is the pressure, η is the couple stresses parameter and f_x is the body force. This force, f_x is assumed of the form:

$$f_x = g\ (\sin\theta - \mu\cos\theta) \quad (8)$$

The general solution of Equation (4) can be written as

$$\begin{aligned} u &= c_3 e^{\sqrt{\frac{\mu}{\eta}}z} + c_4 e^{-\sqrt{\frac{\mu}{\eta}}z} + \frac{1}{2\mu}\left(\frac{\partial p}{\partial x} + \rho g\sin\theta - \mu\rho g\cos\theta\right) \\ &\quad \times (z^2 + \frac{2\eta}{\mu}) + c_1 z + c_2 \end{aligned} \quad (9)$$

$$\begin{aligned} u &= c_3 e^{\bar{\alpha}\frac{z}{z^*}} + c_4 e^{-\bar{\alpha}\frac{z}{z^*}} + \frac{1}{2\mu}\left(\frac{\partial p}{\partial x} + \rho g\sin\theta - \mu\rho g\cos\theta\right) \\ &\quad \times \left(z^2 + 2\left(\frac{z^*}{\bar{\alpha}}\right)^2\right) + c_1 z + c_2 \end{aligned} \quad (10)$$

In order to evaluate four constants, we need four conditions (initial / boundary). For the present model we have assumed:

1. $u|_{z=z_0} = 0$ 2. $\frac{du}{dz}|_{z=z_1} = 0$

3. $u|_{z=z_1} = U_0$ 4. $\frac{d^2u}{dz^2}|_{z=z_1} = \frac{\omega_0}{L_0(1+\bar{\eta})}$

Here w_0 is the particle (snow) rotational velocity, L_0 is the length over which drift is observed, $\bar{\eta}$ $(\frac{\acute{\eta}}{\eta})$ is the couple stresses parameter, z_0 is the surface roughness parameter, U_0 is the free air stream velocity and $\bar{\alpha} = z^*\sqrt{\frac{\mu}{\eta}}$ is the effects of size.

In order to compute equations (9) and (10), we need to have information about the pressure gradient $\frac{dp}{dx}$ over the hill top. This pressure gradient is obtained from the concept of Bernoulli's Equation [14] and the empirical relation for velocity gradient [9].

$$\frac{dp}{dx} = -\frac{0.17}{x}\rho U_0^2\left(\frac{3x}{L\sin^2\theta}\right)^{0.34} \quad (11)$$

It is intended to compare the present results with those of the logarithmic relation of Fohn [9].

$$u_z = \frac{u_*}{k}\ln\left(\frac{z}{z_0}\right) \quad (12)$$

Where u_* is the frictional velocity taken to be $u_* = 0.05U_0$ and k is Von-Karman's constant, assumed to be 0.41.

A. Drift density

Mellor [6] has obtained a relation for drifting snow from conservation of the momentum concept given by:

$$\rho(z) = \rho_{z_0}(\frac{z_0}{z})^{\frac{w}{ku_*}} \quad (13)$$

Where ρ_{z_0} is the initial density of snow and w terminal velocity. Equations (9) and (10) are solved for the density relation and can be written as

$$\rho(z) = \frac{2\mu(u - c_3 e^{\sqrt{\frac{\mu}{\eta}}z} - c_4 e^{-\sqrt{\frac{\mu}{\eta}}z} - c_1 z - c_2) - \frac{\partial p}{\partial x}(z^2 + \frac{2\eta}{\mu})}{g(\sin\theta - \mu\cos\theta)(z^2 + \frac{2\eta}{\mu})} \quad (14)$$

$$\begin{aligned} \rho(z) &= \frac{2\mu(u - c_3 e^{\bar{\alpha}\frac{z}{z^*}} - c_4 e^{-\bar{\alpha}\frac{z}{z^*}} - c_1 z - c_2)}{g(\sin\theta - \mu\cos\theta)(z^2 + 2(\frac{z^*}{\bar{\alpha}})^2)} \\ &\quad - \frac{\frac{\partial p}{\partial x}}{g(\sin\theta - \mu\cos\theta)} \end{aligned} \quad (15)$$

In order to compute Equations (9) and (10) for drift velocity and Equations (14) and (15) for drift density, data on various parameters are required. These data of different parameters have been taken from ([1], [6], [9], [14] and [15]) are shown in Table 1.

TABLE I
DATA OF FLOW PARAMETERS

Symbol	Description	Range	Typical Value
θ	Angle	$0-60$	30^o
η	Couple stress	$-1-1$	$0.5\ kg/m.s$
$\bar{\eta}$	Couple stress	$-1-1$	1
ρ	Snow density	$0.0002-0.0003$	$0.0002\ kg/m^3$
μ	Viscosity	$0.16-0.3$	$0.2\ kg/m.s$

Stokes [1], in his classical theory of coupled stress fluids has introduced the parameters, $\bar{\alpha}$ and $\bar{\eta}$ to account for the effects of size and shape respectively, and for viscosity. In the present model the effects $\bar{\alpha}$ and $\bar{\eta}$ on the flow parameters have been observed. The units of $\eta, \acute{\eta}$ and μ are assumed to be same in our studies. The value of $\bar{\eta}$ in the classical theory [1] is restrict to -1 to +1 ($|\bar{\eta}| \leq 1$) only. We vary $\bar{\eta}$ from -1 to +1 in the present investigation. The value of $\bar{\alpha}$ has been varied from 2 to 8. Also we have assumed ([2],[3], [4], [5], [6], [7], [8] and [9])

$z_0 = 0.002\ m, L_0 = 1\ m, L = (x_1 - x_0)\cos\theta = 12\cos\theta\ m$
$U_0 = 10m/sec, w_0 = 0.0191\ 1/sec, w = 0.5\ m/sec$

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III. Drift Transport

Drift Transport is computed by using the relation

$$Q = Q(z)|_{z_0}^{z} = \int_{z_0}^{z} \rho(z) u(x,z) dz \qquad (16)$$

Here Q $(kg/m.s)$ is computed by both Fohn's [9] method and as well as by present approach. The computed values are shown in Table-2.

TABLE II
Drift Transport values at various heights

z_0 *to* z m	0.002 t0 1 m	0.002 t0 3 m	0.002 t0 5 m
Present method	0.0009	0.0061	0.0126
Fohn's [9] method	0.0012	0.0041	0.0073

IV. Mass (snow) Transport

Snow mass transport (M) is computed by using the relation

$$M = (y_1 - y_0) \int_{x_0}^{x_1} \int_{z_0}^{z_1} \int_{z_0}^{z} \rho(t) u(x,t) dt \qquad (17)$$

Here y_0 and y_1 are the distance of fetch available for snow transport on ridge crest. In the present analysis, they are taken as $y_0 = 0\ m$ and $y_1 = 8\ m$ Similarly $x_1 = 12\ m$ and $z_1 = 5\ m$ [$x_0 = 0\ m$, $z_0 = 0.002\ m$]. The computed results are shown in Figs. 15-17.

V. Results

The flow parameters such as velocity, drifting snow, drift transport and mass transport are computed in this model. The velocity variation with height has been computed for different stream velocity (U_0) [Speed of the upper almost frictionless flow], couple stress parameters $\bar{\alpha}$, $\bar{\eta}$ and different densities (ρ) and for viscosity (μ). The results have also been compared with classical Log law relation. These computed results have been shown in Fig. (2 to 7). The results indicate that, the velocity will be increased with increase in drift density (ρ), stream velocity (U_0), Couple stress parameters ($\bar{\eta}$) and it will be decreased with increase in viscosity (μ). The results are found to be in agreement with qualitative observation of physics of flow, since velocity has to increase with height in atmosphere [increase with $\bar{\eta}$ means size of particles (η) is small which implies more tending toward higher velocity]. Also, decreases in velocity with increase in μ indicate that higher velocity increases higher drag which reduces over all velocity in bulk. The results on Log - law velocity indicate that the present computed results are lower in comparison to logarithmic relation [6]. This is also justified since flow with micro - structure (snow particles) always yield to low velocity in comparison to clear flow. Also, variation of velocity with $\bar{\alpha}$ indicates that, as the parameter $\bar{\alpha}$ increases, velocity increases and this trend observed to yield to logarithmic wind velocity (Fig. 7) ([2],[3], [4], [6], [7] and [8]). This observation of yielding of couple stress theory to classical Newtonian results for higher values of $\bar{\alpha}$ is in agreement with the theory proposed by Stokes [1]. From physical justification point of view, it could be stated that, for a fixed μ and, η the only variable in $\bar{\alpha}$ is z, that is the height. As it could be seen that drift content decreases with the increase in height. Hence, the results must yield to classical logarithmic wind profile for higher values of $\bar{\alpha}$. The present computed results are adhering to this observation. It is of interest to note that the present velocity saturates at higher rates of flow in comparison to logarithmic law.

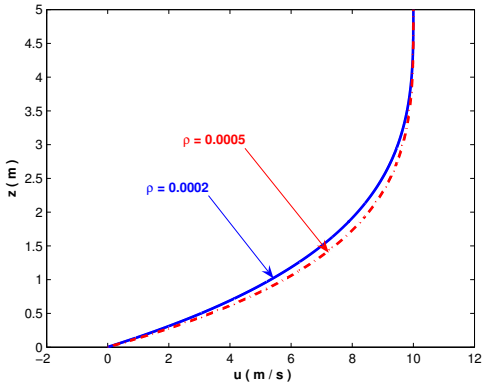


Fig. 2. Variation of velocity with z for several of drift density

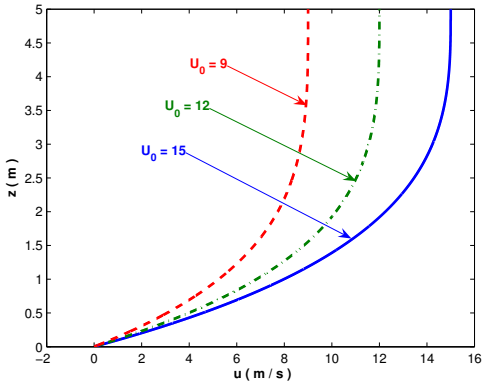


Fig. 3. Variation of velocity with z for several of maximum velocity

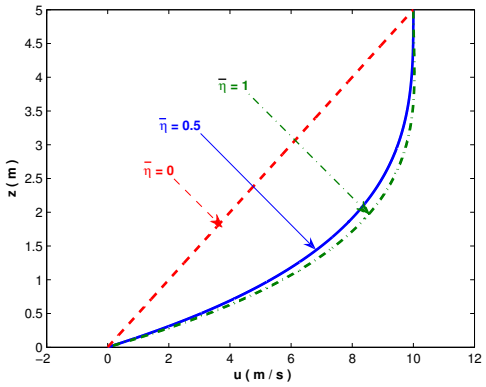


Fig. 4. Variation of velocity with z for several of couple stress parameter

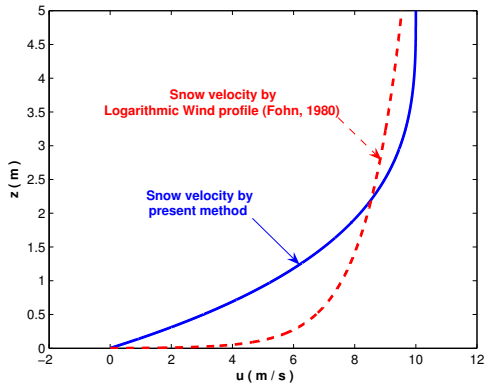


Fig. 5. Variation of velocity with z [Comparison of Logarithmic Wind profile with the present approach]

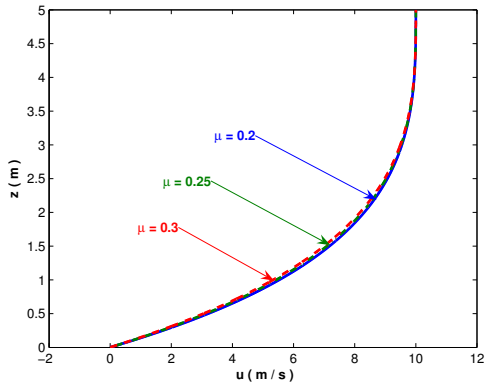


Fig. 6. Variation of velocity with z for several of drift viscosity

The results on drift density (Fig. 8 to 14) indicate that the drift increases with increase in viscosity (μ), decreases in slope angle (θ), increases in pressure gradient (dp/dx) and couple stress parameter (η). These results again can be quantified with physical observation. For instance ρ decreasing with increase in slope angle (θ) justifies that more is the slope angle less is the drifting, and more is pressure gradient implies more is drifting, and more is size effects (η), less is the drift density. Similarly higher the drift content implies it has higher friction coefficient (μ). Variation in drift density (ρ) with $\bar{\alpha}$ (Fig. 14) have been found to increase with increase of $\bar{\alpha}$. This is again found to be in agreement with the physical observation that, higher value of implies less of size effects and higher is the drift velocity and drift transport capacity. The results of the present method have been compared to that of [9] and found that (Fig. 9), the present values are higher than Fohn [9] values. This is justified since the model accounts the size effects in the flow of wind (snow particles [1]) as a result, the model must yield higher values of drift density as compared to that of Newtonian approach [9].

The results on drift transport [Table 2] indicate the present values are higher than the one computed by Fohn [9] except in the range of $0.002 - 1$ meters. This is accepted since bulk of snow transport in this range is by saltation approach [10] and

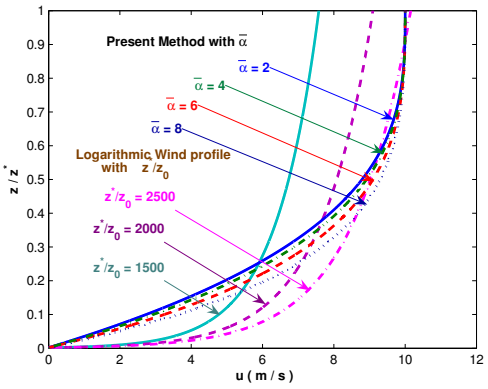


Fig. 7. Variation of velocity with $\frac{z}{z^*}$ [Comparison of Logarithmic Wind profile with different $\frac{z^*}{z_0}$ and the present approach with different $\bar{\alpha}$]

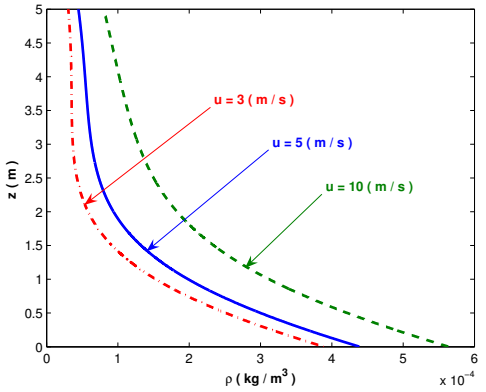


Fig. 8. Variation of snow drift density with z for several of velocity (with present method)

the present model accounts for turbulent diffusion approach [3]. The results on mass transport indicate that, (Fig. 15) the present computed values are higher than Fohn's [9] values and effect of $\bar{\alpha}$ and $\bar{\eta}$ have significant variations on mass transport values (Figs. 16 and 17).

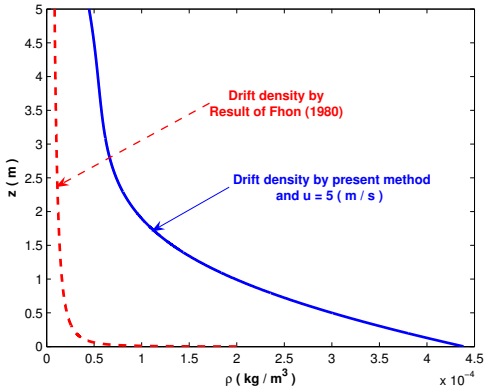


Fig. 9. Variation of snow drift density with z [Comparison of result of Fohn [9] with the present approach]

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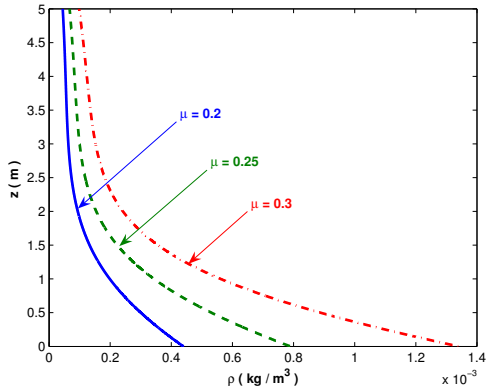


Fig. 10. Variation of snow drift density with z for several of drift viscosity (with present method)

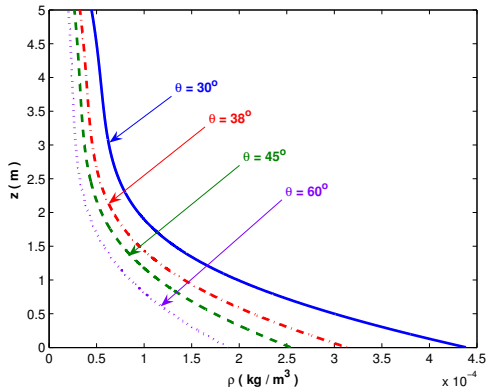


Fig. 11. Variation of snow drift density with z for several of Angle (with present method)

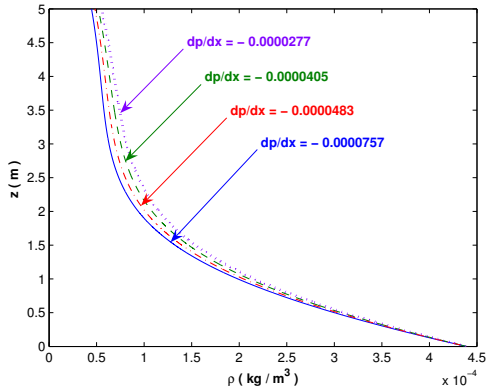


Fig. 12. Variation of snow drift density with z for several of pressure gradient (with present method)

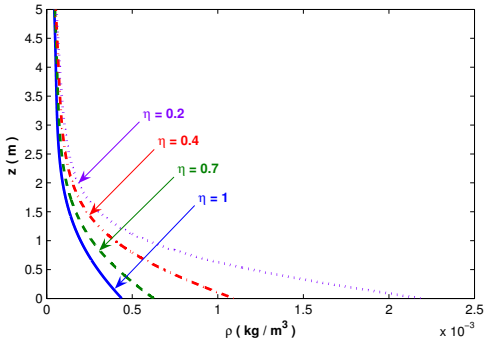


Fig. 13. Variation of snow drift density with z for several of couple stress parameter (with present method)

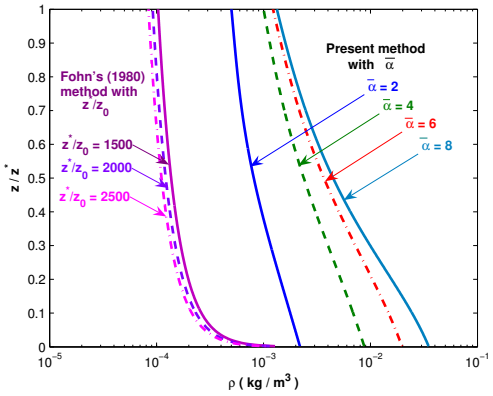


Fig. 14. Variation of snow drift density with $\frac{z}{z^*}$ [Comparison of Logarithmic Wind profile with different $\frac{z^*}{z_0}$ and the present approach with different $\bar{\alpha}$]

VI. Conclusion

Mathematical modeling of snow drifting in snow bound hilly regions has been studied in the present investigations. One of the main thrust in studying this model is to introduce particle (snow) effects in the flow. The particle effects have been introduced by introducing micro-continuum approach [**?**] in the model. Earlier models ([3], [5], [6], [7], [8] and [9]) have been focusing on Logarithmic law based approach of the wind profile and estimating the parameters such as: Drift

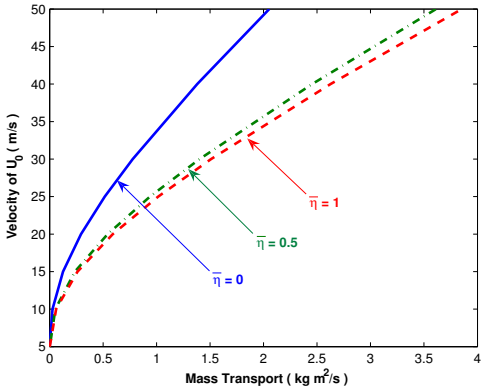


Fig. 15. Variation of mass transport with U_0 for several of couple stress parameter

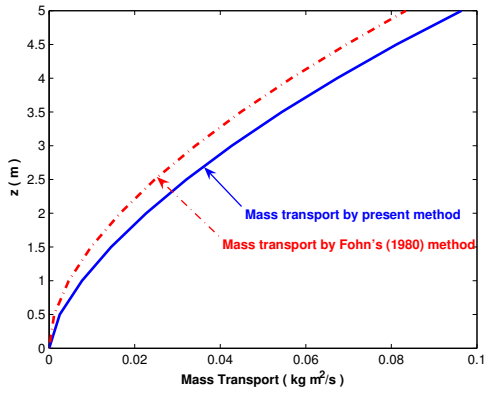


Fig. 16. Variation of mass transport with z [Comparison of u_z Wind profile With the present approach]

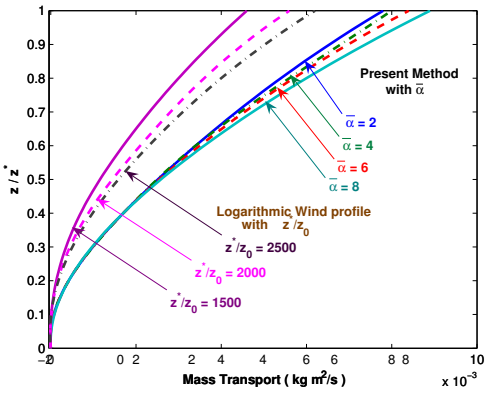


Fig. 17. Variation of mass transport with $\frac{z}{z^*}$ [Comparison of Logarithmic Wind profile with different $\frac{z^*}{z_0}$ and the present approach with different $\bar{\alpha}$]

density, Drift transport and Mass transport.

The results indicate that particle (snow) ($\bar{\alpha}$, $\bar{\eta}$) influences the flow characteristics significantly. For instance, velocity results computed from the present model yields lower than the one computed from classical approach [9], similarly, drift content (density) yields higher values to that of classical model [9]. These results are in agreement with the physics of flow observation since velocity should be lower for flow with particles when compared to clear flow [9]. Also it is observed that, when the model is given for cheek for the classical flow [that is relaxing parametric values ($\bar{\alpha} \to \infty$)], the model has been exhibiting the results of classical Logarithmic law.

Estimation of drift transport , drift density , and mass transport are the most important parameters in snow drift modeling since those parametric values will help in assessing how much snow mass is getting transported to leeward side. Zones which are prone to avalanches may get triggered as a result of additional mass (drifting snow) arriving onto the leeward zones. Hence accurate estimation of parameters will help in assessing for avalanche formation and forecasting purpose. Also flow parametric results could be used for the water management board for observing the water yield / loss from the zones getting deposited / eroded to. The results of the present investigations can also be used as a guide line for designing avalanche control structures.

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