The Balanced Hamiltonian Cycle on the Toroidal Mesh Graphs

Wen-Fang Peng, Justie Su-Tzu Juan

***Abstract*—**The balanced Hamiltonian cycle problem is a quiet new topic of graph theorem. Given a graph $G = (V, E)$, whose edge set can be partitioned into k dimensions, for positive integer k and a Hamiltonian cycle C on G. The set of all i-dimensional edge of C, which is a subset by $E(C)$, is denoted as $E_i(C)$. If $||E_i(C)| - |E_j(C)|| \le 1$ for $1 \le i < j \le k$, C is called a balanced Hamiltonian cycle. In this paper, the proposed result shows that there exists a balanced Hamiltonian cycle for any Toroidal Mesh graph $T_{m,n}$ if and only if $m, n \ge 3$ and Toroidal Mesh graph $nm \ne 2 \pmod 4$, and how to find a balanced Hamiltonian cycle on $T_{m,n}$, for $n, m \ge 3$ and $mn \ne 2 \pmod 4$.

***Keywords*—**Hamiltonian cycle; balanced; Cartesian product

I. Introduction

THE research of optimal encode uses gray-code encode to signify the information of n-bit about the application of 3D scanning, which has been mentioned in the references [1], [2], [3], [5] and [8]. The utility of gray-code will decrease the consumption of resource and increase the precision. Nevertheless, there would be some problem when deal with those information of transforming between 1 and 0, such as it will spend much more cost in identification. How to decrease the cost in dealing with such problems is important. Hence, in this paper, it discusses a method to decrease the number of transformation between 0 and 1 in the some dimension.

Balanced Hamiltonian cycle (BHC) problems are widely discussed in recent years. Several issues about BHC have been proposed by other researchers [2]. Wang et al proposed the BHC on $C_n \times C_n$ for any positive integer n ≥ 3. This paper proposes an extended research about the BHC on $C_m \times C_n$, also called $T_{m,n}$, for any positive integer $m \ge 3$, $n \ge 3$.

Next section introduces some background knowledge about the Hamiltonian cycle (HC) problem, Cartesian product, and some related definitions. Section 3 describes the main results, the research about the BHC problem on $T_{m,n}$, for $m, n \ge 3$, proposed by this paper. Finally, the last section makes a conclusion and lists the future work.

II. Definition and Notation

This paper denotes the symbols below by referring to [4], [6], [7]and [9]. Define a walk W, which is in a graph $G = (V, E)$, is a sequence $w = x_1e_1x_2e_2\ldots x_ke_ky$ for $x_1, x_2, \ldots, x_k, y \in V(G)$ and $e_1, e_2, \ldots, e_k \in E(G)$. And let x be the *origin vertex* of W, y be the *terminus vertex* of W. If all of vertices in this walk are different, a walk W is denoted a *path*. When the origin vertex and the terminus vertex are the same vertex, then this path is denoted a *cycle*.

Su-Tzu Juan is with Department of Computer Science and Information Engineering National Chi Nan University Puli, Taiwan *Correspoding. e-mail: jsjuan@ncnu.edu.tw

A *Hamiltonian path* of graph $G = (V, E)$ is a path that contains all vertices. A *Hamiltonian cycle* of G is a cycle that contains all vertices.

Given a Hamiltonian cycle C on a graph $G = (V, E)$, whose edge set can be partition into k dimensions, for positive integer k. And let $E_i(C)$ represents the set of all i-dimensional edge of C which is a subset by $E(C)$. If $||E_i(C)| - |E_j(C)|| \le 1$ for $1 \le i \le j \le k$, C is called a balanced Hamiltonian cycle.

Let C_n denote a cycle with n vertices, given two graph G_1, G_2, the Cartesian product $G_1 \times G_2$ of G_1 and G_2 is a graph with vertex set $V(G_1 \times G_2) = \{(x, y) \mid x \in V(G_1), y \in V(G_2)\}$ and the edge set $\{(u, v), (u', v') \mid u = u' \in V(G_1)$ and $(v, v') \in E(G_2)$ or $v = v' \in V(G_2)$ and $(u, u') \in E(G_1)\}$. The *toroidal mesh graph*, $T_{m,n}$ is the graph $C_m \times C_n$.

The dimension of $T_{m,n}$ is 2. Given an Hamiltonian cycle C of $T_{m,n}$, let $E_1(C) = \{(x_i, y_j)(x_{i+1}, y_j) \mid 1 \le i \le m, 1 \le j \le n\}$ and $E_2(C) = \{(x_i, y_j)(x_i, y_{j+1}) \mid 1 \le i \le m, 1 \le j \le n\}$ mean the *1-dimension edge set* and *2-dimension edge set* of a Hamiltonian cycle C, respectively. Thus, the relation between vertex number and edge number of Hamiltonian cycle C is $|V(C)| = |V(C_n \times C_m)| = |E_1(C)| + |E_2(C)| = mn$. If C satisfied that $||E_1(C)| - |E_2(C)|| \le 1$, it presents that C is balanced.

In this paper, when we draw a figure of $T_{m,n}$, m is denoted the number of vertices on x-axis, and n is the number of vertices on y-axis, respectively. Besides, for any vertices (x, y) of $T_{m,n}$, x is called the 1st-dimension and y is called the 2nd-dimension. Furthermore, we define the lower-left vertex of $T_{m,n}$ to be the *origin vertex* and set it as (1, 1). Fig. 1 shows an example of $T_{3,4}$.

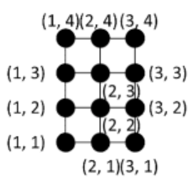

Fig. 1 $T_{3,4}$

The next section discusses the methods for getting the balanced Hamiltonian cycle on $T_{m,n}$ for $n, m \ge 3$.

III. Main Results

This section gives theorem1 for prove ∄, and gives some cases to prove Theorem 3 that there exists a BHC on $T_{m,n}$ for positive integers $n, m \ge 3$, except for the situation on mn mod 4 = 2.

Theorem 1: For mn mod 4 = 2,there is no any balanced Hamiltonian cycle C exists on $T_{m,n}$.*

Proof.

When mn mod 4 = 2, one of the following case will hold. (i) n

mod 4 = 2 and m is odd; (ii) m mod 4 = 2 and n is odd. Without loss of generality, we say n mod 4 = 2 and m is odd. Furthermore, let $n = 4k_1 + 2$ and $m = 2k_2 + 1$ for some positive k_1 and k_2. Assume that there exists a balanced Hamiltonian cycle C^* on $T_{m,n}$. Since $|V(C^*)| = mn = (4k_1 + 2)(2k_2 + 1) = 8k_1k_2 + 4k_2 + 4k_1 + 2 = 2(4k_1k_2 + 2k_1 + 2k_2 + 1)$, $|E_1(C^*)| = |E_2(C^*)| = mn / 2 = 2(2k_1k_2 + k_1 + k_2) + 1$ is an odd integer.

We call a vertex u in $V(C^*)$ is black if $u \in \{(x, y) \mid 1 \le x \le m, 1 \le y \le n$ and y is odd$\}$; white if $u \in \{(x, y) \mid 1 \le x \le m, 1 \le y \le n$ and y is even$\}$. Hence the origin vertex is black. According to the definition of $E_2(C^*)$, $E_2(C^*)$ should trace black point to white point or white point to black point. After tracing all edges of C^*, find the terminate vertex of C^* is white due to $|E_2(C^*)|$ is odd. Obviously, the origin vertex and the terminate vertex of C^* are different. That is a contradiction. So, there is no BHC on $T_{m,n}$ when mn mod 4 = 2.

Lemma 2: For n = 3, m ≥ 3 and m is odd, there is a balanced Hamiltonian cycle on $T_{m,n}$.

Proof.

The proof is divided into two cases. Case 1 discusses the condition on m mod 4 = 1 and n = 3; Case 2 discusses the state on m mod 4 = 3 and n = 3.

Case 1. m mod 4 = 1 and n = 3

In this section, $T_{m,3}$ consists of the BHC on $T_{4,3}$ and the BHC on $T_{5,3}$, as shown in Fig. 2 and Fig. 3, respectively. Besides, Fig. 4 indicates how to connect all figures. First of all, let $x = (m - 5) / 4$, and inset Fig. 2 for x times on right side of Fig. 3 when $m > 5$ and $n = 3$. Then, delete edge set $E_1 = \{(6 + 4i, 3)(9 + 4i, 3) \mid 0 \le i \le (m - 9) / 4\} \cup (1, 3)(5, 3)$, and add edge set $E_2 = \{(5 + 4i, 3)(6 + 4i, 3) \mid 0 \le i \le (m - 9) / 4\} \cup (1, 3)(m, 3)$. After these steps, a Hamiltonian cycle C on $T_{m,3}$ is generated, whose $|E_1(C)| = 7 + 6x$ and $|E_2(C)| = 8 + 6x$. Consequently, $||E_1(C)| - |E_2(C)|| = 1$, C satisfies the definition of BHC.

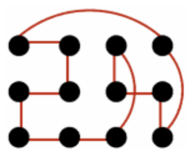

Fig. 2 The BHC on $T_{4,3}$

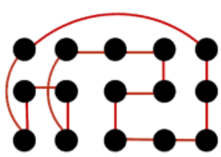

Fig. 3 The BHC on $T_{5,3}$

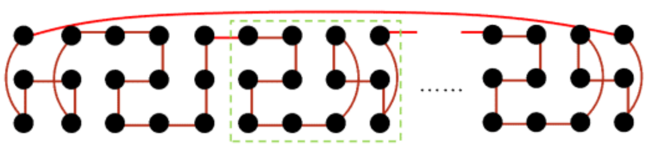

Fig. 4 The BHC on $T_{m,3}$

Case 2. m mod 4 = 3 and n = 3

Compare Fig. 4 with Fig. 6, which indicates how to construct the BHC on $T_{m,3}$, there is only one difference at the beginning. As a result, refer to Case 3.1, replace Fig. 3 with Fig. 5, which is one of possible BHCs on $T_{3,3}$, and revise $x = (m - 3) / 4$. Then correct the edge set $E_3 = \{(4 + 4i, 3)(7 + 4i, 3) \mid 1 \le i \le (m - 7) / 4\} \cup (1, 1)(3, 3)$ and $E_4 = \{(3 + 4i, 3)(4 + 4i, 3) \mid 0 \le i \le (m - 7) / 4\} \cup (1, 3)(m, 3)$, respectively. In the end, a Hamiltonian cycle C on $T_{m,3}$ is built, which $|E_1(C)| = 5 + 6x$ and $|E_2(C)| = 4 + 6x$. Obviously, C satisfies the definition of BHC as a result of $||E_1(C)| - |E_2(C)|| = 1$.

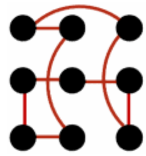

Fig. 5 The BHC on $T_{3,3}$

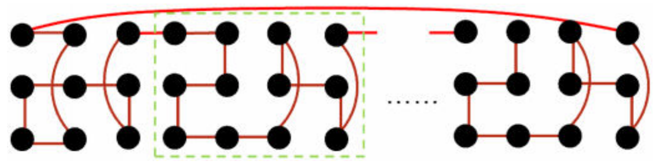

Fig. 6 The BHC $T_{m,3}$

Theorem 3: For n, m ≥ 3, there is a balanced Hamiltonian cycle on $T_{m,n}$, except for the state on mn mod 4 = 2.

Proof.

According to the condition of even or odd on n, m, the proof is divided into three cases. Case 1 proposes the condition on n, m both are even; Case 2 proposes the condition one of m, n is even and the other is odd; Case 3 discusses the condition on n, m both are odd.

TABLE I
THE RESULT OF THIS THEOREM

	m mod 4 = 0	m mod 4 = 2	m mod 8 = 1	m mod 8 = 5	m mod 8 = 3	m mod 8 = 7
n mod 4 = 0	Case 1.1		Case 2.1			
n mo 4 = 2	Case 1.2	Case 1.3	Case 2.2			
n mod 4 = 1	Case 2.1	Case 2.2	Case 3.1	Case 3.3	Case 3.2	Case 3.4
n mod 4 = 3, $n \ge 7$			Case 3.5	Case 3.7	Case 3.6	Case 3.8
n = 3			Lemma 2 Case 1		Lemma 2 Case 2	

Case 1. n, m both are even

This case is separated into three subcases for discussion. Case 1.1, Case 1.2 and Case 1.3 consider the states on m is even and n mod 4 = 0, m mod 4 = 0 and n mod 4 = 2, m mod 4 = 2 and n mod 4 = 2, respectively.

Case 1.1. m is even and n mod 4 = 0

Fig. 7 and Fig. 8 show one of the possible HCs on $T_{2,4}$ and one of the possible BHCs on $T_{4,4}$, respectively. When $m > 4$ and

$n = 4$, let $x = (m - 4) / 2$, and then duplicate Fig. 7 for x times. Next, inset them on the right side of Fig. 8 mentioned above. Then delete edge set $E_5 = \{(i, 2)(i, 3) \mid 4 \leq i \leq m - 1\}$, and insert edge set $E_6 = \{(i, 2)(i + 1, 2) \cup (i, 3)(i + 1, 3) \mid 4 \leq i \leq m - 2$ and i is even.$\}$.

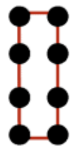

Fig. 7 The HC on $T_{2, 4}$

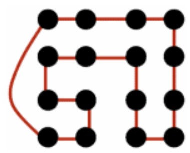

Fig. 8 The BHC on $T_{4, 4}$

After these steps, a Hamiltonian cycle C could be derived as shown in Fig. 9, whose $|E_1(C)| = 8 + (2 + 2)x = 8 + 4x$, and $|E_2(C)| = 7 + 4x + 1 = 8 + 4x$. Due to $||E_1(C)| - |E_2(C)|| = 0$, C is a BHC of $T_{m, 4}$.

When $n > 4$, Fig. 10 illustrates the way of constructing a BHC on $T_{m, n}$. Let $y = (n - 4) / 4$, stack $y + 1$ Cs on the top of each other. Next, delete edge set $E_7 = \{(1, 1 + 4i)(1, 4 + 4i) \mid 0 \leq i \leq y\}$, and add edge set $E_8 = \{(1, 4i)(1, 4i + 1) \mid 1 \leq i \leq y\} \cup (1, 1)(1, n)$. A BHC on $T_{m, n}$ for m is even and n mod $4 = 0$ is generated.

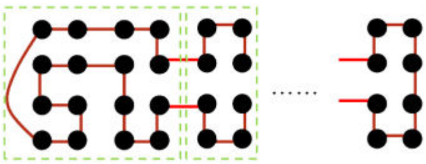

Fig. 9 The BHC on $T_{m, 4}$

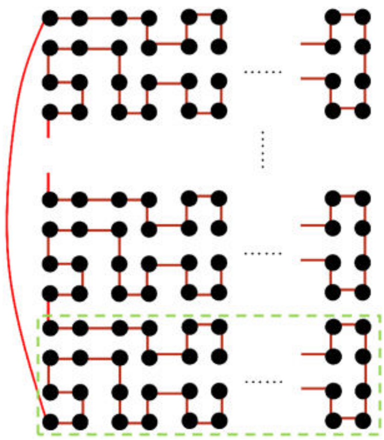

Fig. 10 The BHC on $T_{m, n}$

Case 1.2 m mod 4 = 0 and n mod 4 = 2

Fig. 11 and Fig. 12 are isomorphic BHC on $T_{4, 6}$. The following steps indirect how to find a BHC on $T_{m, 6}$ when $m > 4$. First, inset Fig. 12 on the right side of Fig. 11 for x times, where $x = (m - 4) / 4$. Second, delete edge set $E_9 = \{(1 + 4i, 6)(4 + 4i, 6) \mid 0 \leq i \leq x\}$. Third, add edge set $E_{10} = \{(4 + 4i, 6)(5 + 4i, 6) \mid 0 \leq i \leq (n - 8) / 4\}$. By implementing these steps above, a Hamiltonian cycle C is produced, whose $|E_1(C)| = 12 + 12x$ and $|E_2(C)| = 12 + 12x$, as shown in Fig. 13. Because of $||E_1(C)| - |E_2(C)|| = 0$, C satisfies the definition of BHC.

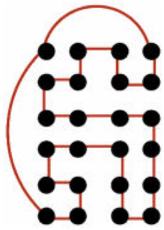

Fig. 11 The BHC on $T_{4, 6}$

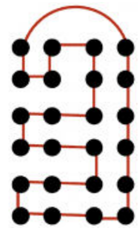

Fig. 12 The BHC on $T_{4, 6}$

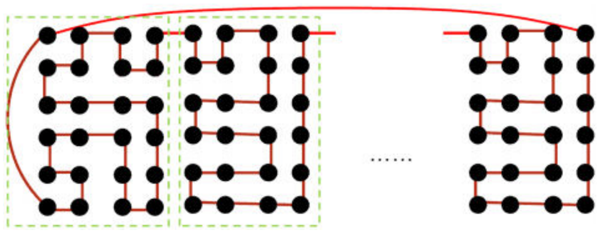

Fig. 13 The BHC on $T_{m, 6}$

When $n > 6$, let $y = (n - 6) / 4$. Stack Fig. 9 for y times on top of Fig. 13. Subsequently, delete edge set $E_{11} = \{(1, 7 + 4i)(1, 10 + 4i) \mid 0 \leq i \leq (n - 10) / 4\} \cup (1, 1)(1, 6)$, and put edge set $E_{12} = \{(1, 6 + 4i)(1, 7 + 4i) \mid 0 \leq i \leq (n - 10) / 4\} \cup (1, 1)(1, n)$. As a result, a BHC on $T_{m, n}$ can be built, which connect the BHC on $T_{m, 6}$, shown as Fig. 13, and the y BHCs on $T_{m, 4}$, shown as Fig. 9.

Case 1.3. m mod 4 = 2 and n mod 4 = 2

Fig. 14 shows a BHC on $T_{6, 6}$ which can be used to build a BHC on $T_{m, 6}$. Consider $m > 6$, make $x = (m - 6) / 4$. Use Fig. 14 as the beginning, and inset Fig. 12 on the right side for x times. After that, remove edge set $E_{13} = \{(7 + 4i, 6)(10 + 4i, 6) \mid 0 \leq i \leq (m - 10) / 4\} \cup (1, 6)(6, 6)$, and add edge set $E_{14} = \{(6 + 4i, 6)(7 + 4i, 6) \mid 0 \leq i \leq (m - 10) / 4\} \cup (1, 6)(m, 6)$. Finally, a Hamiltonian cycle C on $T_{m, 6}$ is built as shown in Fig. 15, whose $|E_1(C)| = 18 + 12x$ and $|E_2(C)| = 18 + 12x$. Obviously, C satisfies the definition of BHC owing to $||E_1(C)| - |E_2(C)|| = 0$.

When $n > 6$, the way of constructing the BHC on $T_{m, n}$ is similar to Case 1.2. Only difference is to replace Fig. 13 with Fig. 15, else parts are the same.

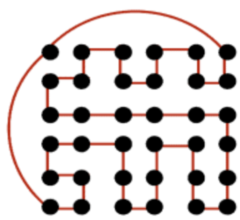

Fig. 14 The BHC on $T_{6, 6}$

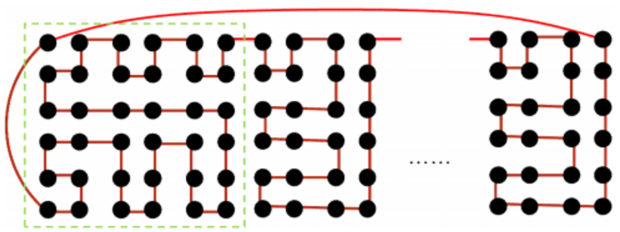

Fig. 15 The BHC on $T_{m,6}$

Case 2. One of m, n is even, and the other is odd

For any positive integer n, m, $T_{m,n}$ and $T_{n,m}$ are isomorphic. Hence, if one of m, n is even, and the other is odd, without lost of generality, set that n is even and m is odd.

This case can be also divided into two subcases for discussion. Case 2.1 discusses the condition on m is odd and n mod 4 = 0; Case 2.2 discusses the state on m is odd and n mod 4 = 2.

Case 2.1. m is odd and n mod 4 = 0

Fig. 16 shows a possible BHC on $T_{3,4}$. When $m > 3$, the BHC on $T_{m,4}$ consists of Fig. 7 and Fig. 16. Fig. 17 illustrates the way of connecting. First, for $x = (m - 3) / 4$, inset x duplicate BHC, which has been shown in Fig. 7, on the right side of Fig. 16. Second, eliminate edge set $E_{15} = \{(3, 3)\ (i, 3)(i, 4) \mid 4 \le i \le m\} \cup (1, 4)(3, 4) \cup (1, 3)$. Third, put edge set $E_{16} = \{(3 + 2i, 3)(4 + 2i, 3) \cup (3 + 2i, 4)(4 + 2i, 4) \mid 0 \le i \le (m - 4) / 2\} \cup (1, 3)(m, 3) \cup (1, 4)(m, 4)$ on the graph produced by previous steps. After that, a Hamiltonian cycle C is established, whose $|E_1(C)| = 6 + (2 + 2)x = 6 + 4x$ and $|E_2(C)| = 6 + 4x$. Due to $||E_1(C)| - |E_2(C)|| = 0$, C satisfies the definition of BHC.

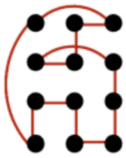

Fig. 16 The BHC on $T_{3,4}$

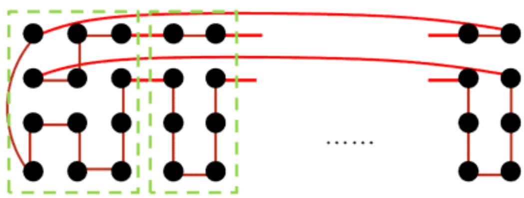

Fig. 17 The BHC on $T_{m,3}$

When $n > 4$, for $y = (n - 4) / 4$, stack $y + 1$ Cs. Next, remove edge set $E_{17} = \{(1, 1 + 4i)(1, 4 + 4i) \mid 0 \le i \le y\}$, and add edge set $E_{18} = \{(1, 4i)(1, 4i + 1) \mid 1 \le i \le y\} \cup (1, 1)(1, n)$. Therefore, a BHC on $T_{m,n}$ is built, as shown in Fig. 18.

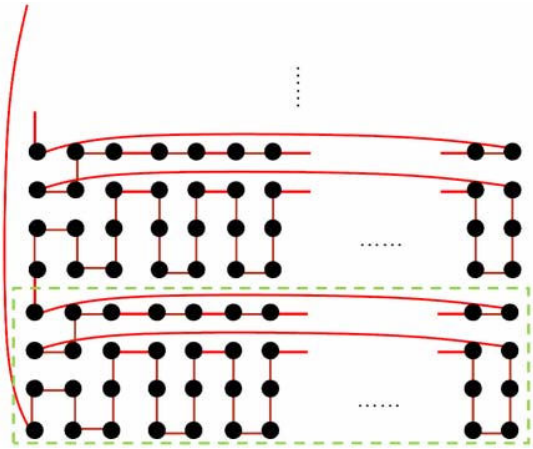

Fig. 18 The BHC on $T_{m,n}$

Case 2.2: n mod 4 = 2 and m is odd

According to theorem 1, there is no balanced Hamiltonian cycle on $T_{m,n}$ for m mod is odd and n mod 4 = 2.

Case 3. m, n both are odd

This case is also separated into eight subcases for discussion. Case 3.1 discusses the state on n mod 4 = 1 and m mod 8 = 1; Case 3.2 proposes the state on n mod 4 = 3 and m mod 8 = 3; Case 3.3 consults the operations when n mod 4 = 1 and m mod 8 = 5; Case 3.4 concerns the details when n mod 4 = 1 and m mod 8 = 7.

The other four remaining cases propose the method under the condition of $n > 3$. Case 3.5 discusses the state on n mod 4 = 3 and m mod 8 = 1; Case 3.6 considers the state on n mod 4 = 3 and m mod 8 = 3; Case 3.7 concerns the operations when n mod 4 = 3 and m mod 8 = 5; Case 3.8 consults the details when n mod 4 = 3 and m mod 8 = 7.

Case 3.1. m mod 8 = 1 and n mod 4 = 1

Fig. 19 and Fig. 20 show one of the possible HCs on $T_{8,5}$ and one of possible BHCs on $T_{9,5}$, respectively. When $m > 9$ and $n = 5$, make $x = (m - 9) / 8$. First, use Fig. 20 as the beginning, and inset Fig. 19 for x times on its right side. Then eliminate edge set $E_{19} = \{(10 + 8i, 4)(10 + 8i, 5) \cup (17 + 8i, 4)(17 + 8i, 5) \mid 0 \le i \le (m - 17) / 8\} \cup (1, 4)(9, 4) \cup (1, 5)(9, 5)$, and add edge set $E_{20} = \{(9 + 8i, 4)(10 + 8i, 4) \cup (9 + 8i, 5)(10 + 8i, 5) \mid 0 \le i \le (m - 17) / 8\} \cup (1, 4)(m, 4) \cup (1, 5)(1, m)$. Finally, a Hamiltonian cycle C on $T_{m,5}$ is produced, which is shown in Fig. 21. For $|E_1(C)| = 22 + (18 + 2)x = 22 + 20x$ and $|E_2(C)| = 23 + 20x$, C satisfies that $||E_1(C)| - |E_2(C)|| = 1$. Undoubtedly, C is a BHC of $T_{m,5}$.

When $n > 5$, let $y = (n - 5) / 4$. Use Fig. 21 as base, then stack y BHCs, which is shown in Fig.12. Next, remove edge set $E_{21} = \{(1, 6 + 4i)(1, 9 + 4i) \mid 0 \le i \le (n - 9) / 4\} \cup (1, 1)(1, 5)$, and insert edge set $E_{22} = \{(1, 1)(1, n) \mid 0 \le i \le (n - 9) / 4\} \cup (1, 5 + 4i)(1, 6 + 4i)$. After complete all of the steps, a BHC on $T_{m,n}$ for m mod 8 = 1 and n mod 4 = 1 is established.

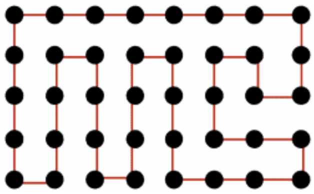

Fig. 19 The HC on $T_{8,5}$

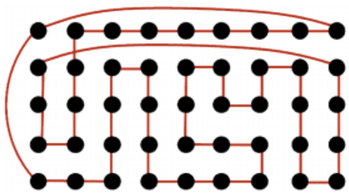

Fig. 20 The BHC on $T_{9,5}$

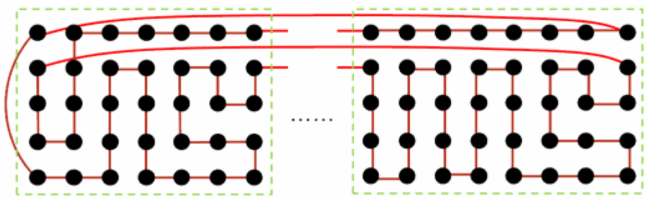

Fig. 21 The BHC on $T_{m,5}$

Case 3.2. m mod 8 = 3 and n mod 4 = 1

Fig. 23 represents the way of constructing a BHC on $T_{m,5}$. When $m > 3$, let $x = (m - 3) / 8$, and then duplicate Fig. 19 for x times. Next, inset them on the right side of the BHC on $T_{3,5}$, which is shown in Fig. 22. In order to connect every figure, delete edge set $E_{23} = \{(4 + 8i, 4)(4 + 8i, 5) \cup (11 + 8i, 4)(11 + 8i, 5) \mid 0 \le i \le (m - 11) / 8\} \cup (1, 4)(3, 4) \cup (1, 5)(3, 5)$, and add edge set $E_{24} = \{(3 + 8i, 4)(4 + 8i, 4) \cup (3 + 8i, 5)(4 + 8i, 5) \mid 0 \le i \le (m - 11) / 8\} \cup (1, 4)(m, 4) \cup (1, 5)(m, 5)$. By implementing the steps above, a Hamiltonian cycle C is generated, whose $|E_1(C)| = 8 + (18 + 2)x = 8 + 20x$ and $|E_2(C)| = 7 + 20x$. Due to $||E_1(C)| - |E_2(C)|| = 1$, C satisfies the definition of BHC.

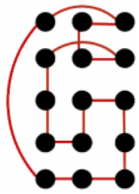

Fig. 22: The BHC on $T_{3,5}$

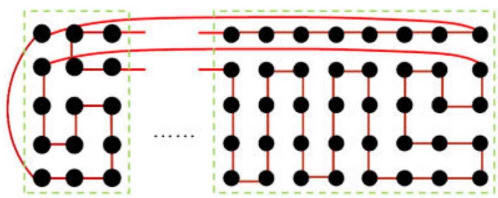

Fig. 23: The BHC on $T_{m,5}$

When $n > 5$, the way of constructing the BHC on $T_{m,n}$ is similar to Case 3.1. Only one difference is to replace Fig. 21 with Fig. 23.

Case 3.3. m mod 8 = 5 and n mod 4 = 1

Fig. 24 represents a BHC on $T_{5,5}$, which is used to construct the BHC on $T_{m,5}$. When $m > 5$, let $x = (m - 5) / 8$. To begin with, inset Fig. 19 for x times on the right side of Fig. 24. Next, delete edge set $E_{25} = \{(6 + 8i, 4)(6 + 8i, 5) \cup (13 + 8i, 4)(13 + 8i, 5) \mid 0 \le i \le (m - 13) / 8\} \cup (1, 4)(5, 4) \cup (1, 5)(5, 5)$, and put edge set $E_{26} = \{(5 + 8i, 4)(6 + 8i, 4) \cup (5 + 8i, 5)(6 + 8i, 5) \mid 0 \le i \le (m - 13) / 8\} \cup (1, 4)(m, 4) \cup (1, 5)(m, 5)$. Thus, a Hamiltonian cycle C is established, which is shown in Fig. 24. For $|E_1(C)| = 12 + (18 + 2)x = 12 + 20x$ and $|E_2(C)| = 13 + 20x$, C obviously satisfies $||E_1(C)| - |E_2(C)|| = 1$ that make it be a BHC of $T_{m,5}$ for m mod 8 = 5 and n mod 4 = 1.

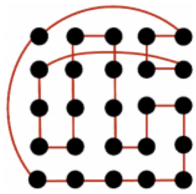

Fig. 24 The BHC on $T_{5,5}$.

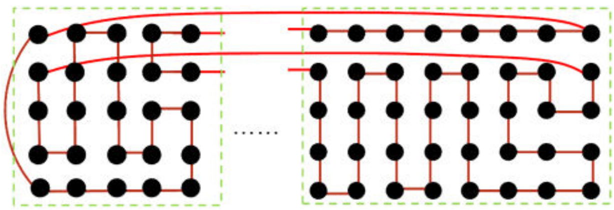

Fig. 24 The BHC on $T_{m,5}$

Case 3.1 and Case 3.3 operate alike during $n > 5$. Only difference is to replace Fig. 21 with Fig. 24.

Case 3.4. m mod 8 = 7 and n mod 4 = 1

The following steps indirect how to construct a BHC on $T_{m,5}$. A BHC on $T_{7,5}$ is shown in Fig. 25. When $m > 7$, make $x = (m - 7) / 8$. Then use Fig. 25 as the beginning, and inset Fig. 19 for x times on the right side. In order to connect all figures, eliminate edge set $E_{27} = \{(8 + 8i, 4)(8 + 8i, 5) \cup (15 + 8i, 4)(15 + 8i, 5) \mid 0 \le i \le (m - 15) / 8\} \cup (1, 4)(7, 4) \cup (1, 5)(7, 5)$, and insert edge set $E_{28} = \{(7 + 8i, 4)(8 + 8i, 4) \cup (7 + 8i, 5)(8 + 8i, 5) \mid 0 \le i \le (m - 15) / 8\} \cup (1, 4)(m, 4) \cup (1, 5)(m, 5)$. Therefore, a Hamiltonian cycle C is built, which as shown in Fig. 26. For $|E_1(C)| = 18 + (18 + 2)x = 18 + 20x$ and $|E_2(C)| = 17 + 20x$, C satisfies $||E_1(C)| - |E_2(C)|| = 1$. Without a doubt, C is a BHC of $T_{m,5}$ for m mod 8 = 7 and n mod 4 = 1.

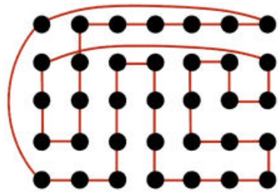

Fig. 25 The BHC on $T_{7,5}$

Refer to Case 3.1 when $n > 5$. Replace Fig. 21 with Fig. 26, else parts are similar to Case 3.1. Finally, a BHC on $T_{m,n}$ is produced.

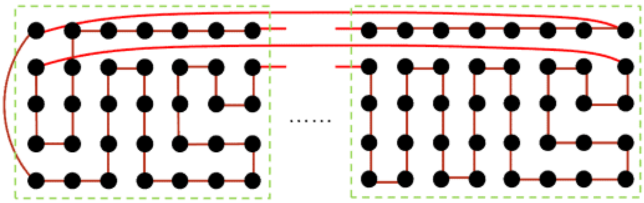

Fig. 27 The BHC on $T_{m,5}$

Case 3.5. m mod 8 = 1 and n mod 4 = 3

Fig. 27 and Fig. 28 show one of the possible HCs on $T_{8,7}$ and one of possible BHCs on $T_{9,7}$, respectively. Furthermore, Fig. 29 illustrates the way of constructing a BHC on $T_{m,7}$, which is described in the content below. When $m > 9$, let $x = (m - 9) / 8$. First, inset x HCs, which has been mentioned above, on the right side of Fig. 28. Second, remove edge set $E_{29} = \{(10 + 8i, 6)(10 + 8i, 7) \cup (17 + 8i, 6)(17 + 8i, 7) \mid 0 \le i \le (m - 17) / 8\} \cup (1, 6)(9, 6) \cup (1, 7)(9, 7)$. Third, add edge set $E_{30} = \{(9 + 8i, 6)(10 + 8i, 6) \cup (9 + 8i, 7)(10 + 8i, 7) \mid 0 \le i \le (m - 17) / 8\} \cup (1, 6)(m, 6) \cup (1, 7)(m, 7)$.Finally, a Hamiltonian cycle C is yielded, whose $|E_1(C)| = 32 + (26 + 2)x = 32 + 28x$ and $|E_2(C)| = 31 + 28x$. As a result of $||E_1(C)| - |E_2(C)|| = 1$, it verify that C is a BHC.

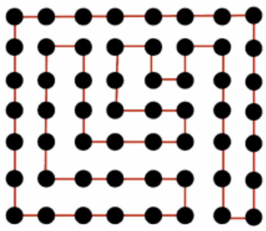

Fig. 27 The HC on $T_{8,7}$.

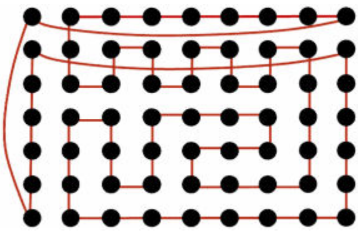

Fig. 28 The BHC on $T_{9,7}$

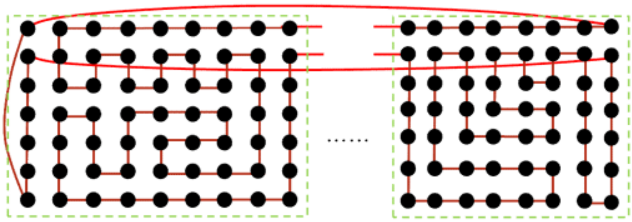

Fig. 29 The BHC on $T_{m,7}$

When $n > 7$, let $y = (n - 7) / 4$. Stack y BHCs, which is shown in Fig. 12, above Fig. 28. Then delete edge set $E_{31} = \{(1, 8 + 4i)(1, 11 + 4i) \mid 0 \le i \le (n - 11) / 4\} \cup (1, 1)(1, 7)$, and add edge set $E_{32} = \{(1, 7 + 4i)(1, 8 + 4i) \mid 0 \le i \le (n - 11) / 4\} \cup (1, 1)(1, n)$. After that, a BHC on $T_{m,n}$ is generated.

Case 3.6. m mod 8 = 3 and n mod 4 = 3

Fig. 30 represents a BHC on $T_{3,7}$, which is used to construct a BHC on $T_{m,7}$. Besides, Fig. 31 illustrates how to connect Fig. 27 and Fig. 30. Let $x = (m - 3) / 8$ for $m > 3$. Then, copy Fig. 27 for x times, and inset them on the right side of Fig. 30. Next, eliminate edge set $E_{33} = \{(4 + 8i, 6)(4 + 8i, 7) \cup (11 + 8i, 6)(11 + 8i, 7) \mid 0 \le i \le (m - 11) / 8\} \cup (1, 6)(3, 6) \cup (1, 7)(3, 7)$, and put edge set $E_{34} = \{(3 + 8i, 6)(4 + 8i, 6) \cup (3 + 8i, 7)(4 + 8i, 7) \mid 0 \le i \le (m - 11) / 8\} \cup (1, 6)(m, 6) \cup (1, 7)(m, 7)$. Therefore, a Hamiltonian cycle C is established, whose $|E_1(C)| = 10 + (26 + 2)x = 10 + 28x$ and $|E_2(C)| = 11 + 28x$. Because of $||E_1(C)| - |E_2(C)|| = 1$, C satisfies the definition of BHC.

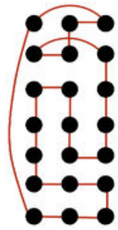

Fig. 30 The BHC on $T_{3,7}$

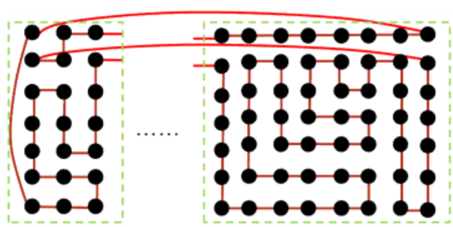

Fig. 31 The BHC on $T_{m,7}$

Refer to Case 3.5 when $n > 7$. Fig. 31 substitutes for Fig. 29, and the else parts are same as Case 3.5. Eventually, a BHC on $T_{m,n}$ is built.

Case 3.7. m mod 8 = 5 and n mod 4 = 3

In this section, $T_{m,7}$ consists of the HC on $T_{8,7}$, as shown in Fig. 27, and the BHC on $T_{5,7}$, as shown in Fig. 32. When $m > 5$, make $x = (m - 5) / 8$. First of all, inset Fig. 27 for x times on the right side of Fig. 32. So as to connect all figures, delete edge set $E_{35} = \{(6 + 8i, 6)(6 + 8i, 7) \cup (13 + 8i, 6)(13 + 8i, 7) \mid 0 \le i \le (m - 13) / 8\} \cup (1, 6)(5, 6) \cup (1, 7)(5, 7)$, and add edge set $E_{36} = \{(5 + 8i, 6)(6 + 8i, 6) \cup (5 + 8i, 7)(6 + 8i, 7) \mid 0 \le i \le (m - 13) / 8\} \cup (1, 6)(m, 6) \cup (1, 7)(m, 7)$. As a result, a Hamiltonian cycle C on $T_{m,7}$ is yielded, whose $|E_1(C)| = 18 + (26 + 2)x = 18 + 28x$ and $|E_2(C)| = 17 + 28x$, as shown in Fig. 33. Without a doubt, C is a BHC due to $||E_1(C)| - |E_2(C)|| = 1$.

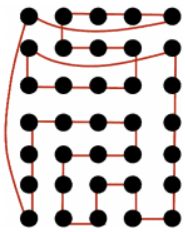

Fig. 32 The BHC on $T_{5,7}$

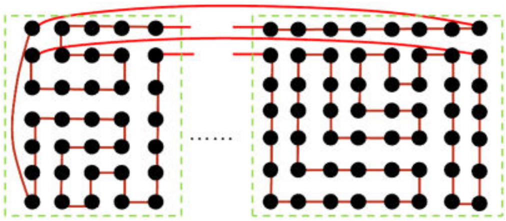

Fig. 33 The BHC on $T_{m,7}$

When $n > 7$, the way of constructing the BHC on $T_{m,n}$ is similar to Case 3.5. Only one difference is to replace Fig. 29 with Fig. 33.

Case 3.10. m mod 8 = 7 and n mod 4 = 3

There is a BHC on $T_{7,7}$ as shown in Fig. 34. When $m > 7$, use the BHC on $T_{7,7}$ mentioned above as the beginning. Make $x = (m - 7) / 8$, then inset x HCs, which is shown in Fig. 27, on the right side of Fig. 34. Then, remove edge set $E_{37} = \{(15 + 8i, 6)(15 + 8i, 7) \mid 0 \le i \le (m - 15) / 8\} \cup (1, 6)(7, 6) \cup (1, 7)(7, 7) \cup (8 + 8i, 6)(8 + 8i, 7)$, and add edge set $E_{38} = \{(7 + 8i, 6)(8 + 8i, 6) \cup (7 + 8i, 7)(8 + 8i, 7) \mid 0 \le i \le (m - 15) / 8\} \cup (1, 6)(m, 6) \cup (1, 7)(m, 7)$. Thus, a Hamiltonian cycle C on $T_{m,7}$ is built, which is shown in Fig. 35.

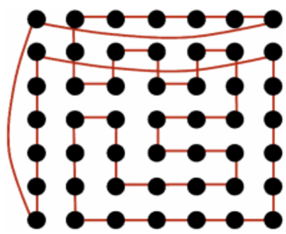

Fig. 34 The BHC on $T_{7,7}$

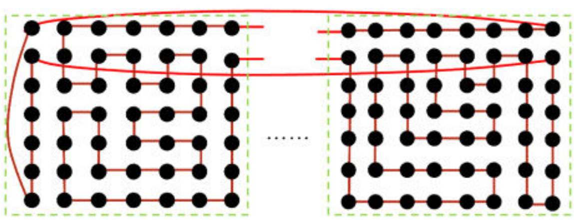

Fig. 35 The BHC on $T_{m,7}$

For $|E_1(C)| = 24 + (26 + 2)x = 24 + 28x$ and $|E_2(C)| = 25 + 28x$, C satisfies that $||E_1(C)| - |E_2(C)|| = 1$. Undoubtedly, C is a BHC of $T_{m,7}$.

Compare Case 3.5 with Case 3.8 for $n > 7$, the only difference is that Fig. 29 is replaced with Fig. 35, and the else parts of operating are all the same. Then a BHC on $T_{m,n}$ is constructed.

IV. Conclusion

By giving Theorem 1 and 2 in this paper, the main result below can be verified. In general cases, there exists a BHC on $T_{m,n}$ for positive integers n, m, except for the situation of $mn \equiv 2 \pmod 4$. How to find a balanced Hamiltonian cycle in the graph of k-dimension Cartesian product for any positive integer k, will be the future work.

References

[1] Y. Y. Chin, *Binary and M-ary Structure Light Patterns with Gray Coding Rule in Active Non-contact 3D Surface Scanning*. Master's Thesis of Department of Computer Science and Information Engineering, National Chi-Nan University, 2009.

[2] E. Horn and N. Kiryati, "Toward optimal structured light patterns," *Proceedings of the International Conference on Recent Advances in 3-D Digital Imaging and Modeling*, pp. 28–35, 1997.

[3] S. P. Koninckx, and L. V. Gool, "Real-time range acquisition by adaptive structured light," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 28, No. 3, pp. 432–445, 2006.

[4] S. Novotny, J. Ortiz, and D. A. Narayan, "Minimal k-rankings and the rank number of $p_2 \times p_n$," *Information Processing Letters*, vol. 109, pp. 193–198, 2009.

[5] J. Salvi, J. Pag′es, and J. Batlle, "Pattern codification strategies in structure light systems," *Pattern Recognition*, vol. 37, pp. 827□849, 2004.

[6] Hou-Ren Wang, *The Balanced Hamiltonian Cycle problem.* Master's Thesis of Department of Computer Science and Information Engineering, National Chi-Nan University, 2011.

[7] Hou-Ren Wang, Bo-Han Wu and Justie Su-Tzu Juan, "The Balanced Hamiltonian Cycle Problem", manuscript.

[8] S. H. Wang, *3-D Surface Acquisition with Non-Uniform Albedo Using Structured Light Range Sensor.* Master's Thesis of Department of Computer Science and Information Engineering, National Chi-Nan University, 2007.

[9] Tai-Lung Wu, *Edge Ranking Problem on Cartesian Product Graphs*. Master's Thesis of Department of Computer Science and Information Engineering, National Chi-Nan University, 2010.